

ON THE DISTRIBUTIVITY OF THE LATTICE OF FILTERS OF A GROUPOID

ORRIN FRINK AND ROBERT S. SMITH

In this note we present the results announced in the Notices of the American Mathematical Society, January, 1969. Algebraic lattices are interesting and important algebraic structures. They occur in many branches of algebra, e.g. the lattice of all subalgebras of a universal algebra, the lattice of all filters of a groupoid, and the lattice of all ideals of a ring are all algebraic lattices. Moreover there is a natural connection between algebraic lattices and groupoids, since every algebraic lattice is isomorphic to the lattice of all filters of some groupoid, and in particular of the groupoid of all compact elements of the lattice. If an algebraic lattice is distributive, it is relatively pseudo-complemented and is a complete Brouwerian lattice in the sense of Garrett Birkhoff [1]. Hence it is natural to look for simple conditions on a groupoid that will insure that the lattice of its filters is distributive.

We show that the lattice of all filters is distributive if it is a sublattice of the lattice of all subgroupoids, but this condition is not always necessary for distributivity. If the groupoid is a semilattice, this condition is both necessary and sufficient. We then derive some conditions that are both necessary and sufficient for distributivity for groupoids. One of these is a modification of a condition given by Grätzer and Schmidt for semilattices.

A groupoid is a pair (G, τ) where G is a set and τ is a binary operation defined on G . We will use the following conventions:

(1) The operation τ will be called multiplication, and we will write ab for $a\tau b$.

(2) We will write simply G for (G, τ)

(3) The symbol 1 will denote the identity element of G if there is one.

(4) G^1 will denote the groupoid G with 1 adjoined if G has no identity element; otherwise $G = G^1$.

The operation in a groupoid G need not be associative or commutative, hence the elements a_1, \dots, a_n have in general many products which can be distinguished from each other by means of parentheses. We find it convenient to denote one of the products of elements a_1, \dots, a_n by $P(a_1, \dots, a_n)$. We allow the elements to be used more than once and in any order. For example $P_1(a_1, a_2)$ might be $(a_1 a_2) a_1$, while $P_2(a_1, a_2)$ could be $a_2((a_2 a_1) a_1)$.

DEFINITION. An element a of G is called a *factor* of element b if there exist elements a_1, \dots, a_n of G such that $b = P(a_1, \dots, a_n, a)$.

DEFINITION. A subset F of a groupoid G is called a *filter* of G if F is a subgroupoid of G which contains all factors of each of its elements.

PROPOSITION. A subset F of a groupoid is a filter if and only if $ab \in F$ is equivalent to $a \in F$ and $b \in F$.

We note that in the theory of algebraic semigroups filters have been studied under the names of co-prime ideals and consistent sub-semigroups in the commutative case, and under the names of faces and p -semigroups in the general case [2, 5].

Let $\mathcal{F}(G)$ denote the collection of all filters of a groupoid G , and let $\mathcal{S}(G)$ denote the collection of all subgroupoids of G . When only one groupoid G is involved, we will write \mathcal{F} and \mathcal{S} for $\mathcal{F}(G)$ and $\mathcal{S}(G)$ respectively. Observe that \mathcal{F} and \mathcal{S} are algebraic lattices; that is, they are complete lattices in which every element is a join of compact elements. The empty set is the smallest element of both \mathcal{F} and \mathcal{S} .

The lattices \mathcal{F} and \mathcal{S} are families of sets, and their elements are subsets of G . When A and B are sets, we shall use the notation $A \cup B$ and $A \cap B$ to represent the set union and intersection of A and B respectively. In the lattices \mathcal{F} and \mathcal{S} the lattice meet of two elements A and B is always their set intersection; hence it will be represented by $A \cap B$. The lattice join operations in \mathcal{F} and \mathcal{S} are more complicated.

For the lattice of subgroupoids \mathcal{S} , we denote the join of two subgroupoids A and B by $A \vee B$; it is the intersection of all subgroupoids which contain both A and B . It consists of all elements of the union $A \cup B$ together with all products involving elements of $A \cup B$.

For the lattice $\mathcal{F}(G)$ of all filters of G , we denote the lattice join by $\bar{\vee}$ and define for $F, H \in \mathcal{F}$ the join $F \bar{\vee} H$ to be the intersection of all filters containing F and H . We later give a more constructive way of obtaining the filter join of two filters, and more generally of the filter generated by any subset of G .

An element a of a complete lattice L is said to be *compact* if $a \leq \mathbf{V} \{a_\gamma : \gamma \in \Gamma\}$ implies that there exists a finite subset F of Γ such that $a \leq \mathbf{V} \{a_\gamma : \gamma \in F\}$. A lattice is said to be *compactly generated* if every element is a join of compact elements. Lattices which are complete and compactly generated are called *algebraic lattices*. The lattice of all subalgebras of an algebra with finitary operations is

an algebraic lattice in which the compact elements are the finitely generated subalgebras. In particular the lattices $\mathcal{S}(G)$ and $\mathcal{F}(G)$ of all subgroupoids and all filters of a groupoid G are algebraic lattices, in which the compact elements are the finitely generated subgroupoids and filters respectively. In the lattice $\mathcal{F}(G)$ every nonempty finitely generated filter is a *principal filter* generated by a single element, as we shall show.

DEFINITION. A subset I of a groupoid G is an *ideal* of G if $a \in I$ implies that $ab, ba \in I$ for all $b \in G$. An ideal I of G is said to be *prime* if $ab \in I$ implies that $a \in I$ or $b \in I$.

The following facts about the lattice $\mathcal{S}(G)$ of all ideals of a groupoid G are easily verified:

- (i) $\mathcal{S}(G)$ is a complete ring of sets, closed under arbitrary union and arbitrary intersection.
- (ii) A subset F of G is a filter if and only if its complement $G \setminus F$ is a prime ideal.
- (iii) The lattice $\mathcal{F}(G)$ of all filters of a groupoid G is distributive if and only if the dual lattice of all prime ideals is distributive.

We find the following notation useful:

- F_a will denote the filter generated by $a \in G$.
- S_a will denote the subgroupoid generated by $a \in G$.
- I_a will denote the ideal generated by $a \in G$.
- F_A will denote the filter generated by $A \subseteq G$.
- S_A will denote the subgroupoid generated by $A \subseteq G$.

If A is any subset of a groupoid G , then S_A consists of the elements of A together with all products of elements of A . From this we see that if S and T are subgroupoids of G , then their subgroupoid join consists of all elements and all products of the elements of the union $S \cup T$.

We now give a representation of the filter F_A generated by a subset A of G . This generalizes a result of Petrich for semigroups [5], and will provide us with a representation for the filter join of two filters.

THEOREM 1. *If A is any subset of a groupoid G , then $F_A = \bigcup \{N_k : k \geq 1\}$, where $N_1 = A$, and N_{k+1} is the subgroupoid generated by $\{y \in G : I_y \cap N_k \neq \emptyset\}$ for $k \geq 1$.*

Proof. If $x \in N_n$ then $I_x \cap N_n \neq \emptyset$. Hence $x \in N_{n+1}$ and $N_n \subseteq N_{n+1}$. Each N_k is a subgroupoid for $k \neq 1$, and the chain union of subgroupoids is a subgroupoid. Therefore the union $\bigcup N_k$ is a subgroupoid.

We now show that this union is a filter. If $xy \in \bigcup N_k$ then $xy \in N_n$ for some n . But $xy \in I_x$ and $xy \in I_y$, whence $xy \in I_{xy} \subseteq I_x \cap I_y$.

Theorefore $\emptyset \neq N_n \cap I_x \cap I_y$, and hence $N_n \cap I_y \neq \emptyset \neq N_n \cap I_x$, and so $x, y \in N_{n+1} \cong \bigcup N_k$. Hence this union is a filter.

To complete the proof we show that $\bigcup N_k \subseteq F_A$. We do this by induction. Clearly $A = N_1 \subseteq F_A$. Assume that $N_n \subseteq F_A$ and show that $N_{n+1} \subseteq F_A$. If $z \in I_y \cap N_n$, then y is a factor of z and since $z \in N_n \subseteq F_A$, all factors of z are in F_A . Hence $y \in F_A$ and thus $\{w \in G: I_w \cap N_n \neq \emptyset\} \subseteq F_A$. Hence N_{n+1} , which is the subgroupoid generated by this set, is a subset of F_A , completing the induction. Therefore $\bigcup N_k \subseteq F_A$. This completes the proof.

Before giving a characterization of the filter join $F \bar{\vee} H$ of two filters, we call attention to some facts about the lattice of filters \mathcal{F} which are easily verified. First we note that $F_{xy} = F_x \bar{\vee} F_y$. It follows that the join of any finite number of principal filters is a principal filter. Hence every finitely generated filter is principal, with the exception of the empty filter.

Let $\mathcal{M}(G) = \{F_x: x \in G\}$ denote the set of all principal filters of G . Then $\mathcal{M}(G)$ is an upper semilattice with respect to the operation $\bar{\vee}$ of filter join. The mapping $x \rightarrow F_x$ is a homomorphism of G onto $\mathcal{M}(G)$, and $\mathcal{M}(G)$ is the maximal semilattice homomorphic image of G . The existence of such a maximal semilattice homomorph of a groupoid is well known; see [2] page 18, [5] page 70. We state these facts as a theorem.

THEOREM 2. *Every groupoid G has a maximal semilattice homomorphic image $\mathcal{M}(G)$, consisting of the principal filters of G .*

The semilattice $\mathcal{M}(G)$ may not have an identity element, since the empty filter is not a principal filter. If G has an identity element 1, then F_1 is the identity element of $\mathcal{M}(G)$. In an upper semilattice an identity element is usually denoted by 0, and a filter is called an ideal. Our definition of *ideal* is that used in the theory of semigroups and groupoids. Note that $\mathcal{M}(G)$ is not in general a lattice, since the intersection of two principal filters need not be principal.

We now define two important binary relations in a groupoid.

DEFINITION. In a groupoid G we define $a \subset b$ to mean $F_a \subseteq F_b$, and $a \sim b$ to mean $F_a = F_b$.

Note that the relation $a \subset b$ is reflexive and transitive and hence a quasi-ordering. In general it is not a relation of partial order, since from $a \subset b$ and $b \subset a$ follows only $a \sim b$ but not always $a = b$. We note that $a \subset b$ is equivalent to $a \in F_b$. We now give a characterization of the join of two filters.

THEOREM 3. *If F and H are two filters of G , then $F \bar{\vee} H = \{a \in G: a \subset P(b_1, \dots, b_n), \text{ where } b_1, \dots, b_n \in F \cup H\}$.*

Proof. Let $a \in F \bar{\vee} H$. Then $F_a \subseteq \bar{\vee} \{F_b : b \in F \cup H\}$. But F_a is a compact element of the lattice $\mathcal{F}(G)$. Hence there exist elements b_1, \dots, b_n in $F \cup H$ such that $F_a \subseteq F_{b_1} \vee \dots \vee F_{b_n}$. Hence $a \in F_c$ where $c = P(b_1, \dots, b_n)$, whence $a \subset c$.

Conversely, let $a \subset P(b_1, \dots, b_n)$, with $b_1, \dots, b_n \in F \cup H$. Then $a \in F_c$, where $c = P(b_1, \dots, b_n)$, $F_c \subseteq F \bar{\vee} H$, hence $a \in F \bar{\vee} H$. This, completes the proof.

We note that \sim is a congruence relation on G , and that the quotient G/\sim is the semilattice $\mathcal{M}(G)$. If $a \subset b$, it does not follow that a is a factor of b ; but the converse is true. However, a subgroupoid T is a filter of G if and only if $x \subset y$ and $y \in T$ implies that $x \in T$.

Now that we have characterized the lattice operations in the lattice of filters $\mathcal{F}(G)$, we look for conditions for the distributivity of this lattice. If G is a semilattice, it will be shown that $\mathcal{F}(G)$ is distributive if and only if it is a sublattice of $\mathcal{S}(G)$. For groupoids in general, this condition is sufficient but not necessary.

DEFINITION. An element a of a lattice (L, \wedge, \vee) is called *meet-distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all b, c in L .

THEOREM 4. *Every filter of a groupoid G is a meet-distributive element of the lattice $\mathcal{S}(G)$ of subgroupoids of G .*

Proof. Let F be a filter and S and T subgroupoids. It is sufficient to show that $F \cap (S \vee T) \subseteq (F \cap S) \vee (F \cap T)$. Suppose $x \in F \cap (S \vee T)$. Since $x \in S \vee T$, $x = P(a_1, \dots, a_n)$ where a_1, \dots, a_n are in $S \cup T$. Since $x \in F$ and F is a filter, each of the elements a_1, \dots, a_n is in F . It follows that $x \in (F \cap S) \vee (F \cap T)$. This ends the proof.

COROLLARY. *If the lattice of filters \mathcal{F} is a sublattice of the lattice of subgroupoids \mathcal{S} , then \mathcal{F} is distributive.*

However, the condition that \mathcal{F} is a sublattice of \mathcal{S} is not necessary for the distributivity of \mathcal{F} , as is shown by the following example of a groupoid $G = \{x, y, z, u\}$, with multiplication table:

	x	y	z	u	
x	x	z	z	u	
y	z	y	z	u	
z	z	z	z	u	
u	u	u	u	z	.

Here the lattice of filters $\mathcal{F}(G)$ is a well-known distributive lattice with the four elements \emptyset , F_x , F_y , and $F_z = F_u$. The subgroupoid join $F_x \vee F_y$ is the set $\{x, y, z\}$, while the filter join $F_x \bar{\vee} F_y$ is the set $\{x, y, z, u\}$. Hence the lattice $\mathcal{F}(G)$ is not a sublattice of $\mathcal{S}(G)$. Note that G is a commutative semigroup, but not a semilattice, since $u^2 \neq u$. For semilattices it will be shown that the sublattice condition is also necessary for distributivity.

We now give another characterization of the filter join of two filters.

THEOREM 5. *If A and B are filters of a groupoid G , then (1) $A \bar{\vee} B = A \cup B \cup \{x: x \subset ab, a \in A, b \in B\}$.*

Proof. The right side of (1) is a subset of the left side, since $A \bar{\vee} B$ is a filter containing $A \cup B$. To show the reverse inclusion, suppose $x \in A \bar{\vee} B$. By Theorem 3, we have $x \subset P(c_1, \dots, c_n)$, where $\{c_1, \dots, c_n\} \subseteq A \cup B$. Hence $x \in F_P = \bar{\vee} \{F_c: c \in \{c_1, \dots, c_n\}\}$. If all the elements c_i are in A or in B , then so is x , and x is a member of the right side of (1). If not, then $F_P = F_a \bar{\vee} F_b$, where a is a product of the elements c_i in A , and b is a product of the elements c_i in B . But $F_a \bar{\vee} F_b = F_{ab}$, hence $x \in F_{ab}$ and $x \subset ab$, where $a \in A, b \in B$. Hence x is in the right side of (1), which completes the proof.

We now give a necessary and sufficient condition for the distributivity of $\mathcal{F}(G)$ in terms of a relation between the filter join and the subgroupoid join of two filters.

THEOREM 6. *The filter lattice $\mathcal{F}(G)$ of a groupoid G is distributive if and only if to each element x of the filter join $A \bar{\vee} B$ of two filters there exists an element y of the subgroupoid join $A \vee B$ such that $x \sim y$.*

Proof. To prove sufficiency, let $x \in A \cap (B \bar{\vee} C)$, where A, B, C are filters of G . We must show that $x \in (A \cap B) \bar{\vee} (A \cap C)$. Now $x \in A$, and by the condition of the theorem there exists y in $B \vee C$ such that $x \sim y$. Note that $x \sim y$ holds if and only if x and y are in the same filters. Hence $y \in A$, and $y \in A \cap (B \vee C)$. By Theorem 4, A is a meet-distributive element of $\mathcal{S}(G)$. Hence $A \cap (B \vee C) = (A \cap B) \vee (A \cap C) \subseteq (A \cap B) \bar{\vee} (A \cap C)$. It follows that y , and hence x , is in the filter $(A \cap B) \bar{\vee} (A \cap C)$, which proves the sufficiency of the condition.

To prove the necessity, suppose $\mathcal{F}(G)$ is distributive and $x \in A \bar{\vee} B$, where A and B are filters. We must show that $x \sim y$, where $y \in A \vee B$. Since $x \in F_x \cap (A \bar{\vee} B)$, by distributivity $x \in (F_x \cap A) \bar{\vee}$

$(F_x \cap B)$. By Theorem 5, either $x \in A \cup B$, whence the conclusion would follow with $x = y$, or $x \subset ab$ with $a \in F_x \cap A$ and $b \in F_x \cap B$. Then $a \in A, b \in B$, hence $ab \in A \vee B$. Also $a \in F_x, b \in F_x$ hence $ab \in F_x$ and $ab \subset x$. But also $x \subset ab$. Hence $x \sim ab$. Let $y = ab$. Then $y \in A \vee B$ and $x \sim y$, which was to be proved.

We now show that the sublattice condition is both necessary and sufficient for distributivity of $\mathcal{F}(G)$ if G is a semilattice.

THEOREM 7. *The filter lattice $\mathcal{F}(G)$ of a semilattice G is distributive if and only if it is a sublattice of the lattice $\mathcal{S}(G)$ of all sub-semilattices of G .*

Proof. The sufficiency of the condition follows from the Corollary to Theorem 4. To prove the necessity, suppose $\mathcal{F}(G)$ is distributive and A and B are filters of G . We must show that $A \overline{\vee} B \cong A \vee B$. Suppose $x \in A \overline{\vee} B$. By Theorem 6 there exists an element y in $A \vee B$ such that $x \sim y$. It is well known that in a semilattice the relation $x \subset y$ is one of partial order rather than merely a quasi-order. In fact, $x \subset y$ if and only if $xy = y$, and $y \subset x$ if and only if $xy = x$. If $x \sim y$, then $x \subset y$ and $y \subset x$, hence $x = y$. It follows that $x \in A \vee B$, and hence $A \overline{\vee} B = A \vee B$. Then $\mathcal{F}(G)$ is a sublattice of $\mathcal{S}(G)$, which was to be proved.

We now derive a necessary and sufficient condition for the distributivity of $\mathcal{F}(G)$ which is somewhat similar to a condition of Grätzer and Schmidt for upper semilattices (S, \vee) . The original Grätzer-Schmidt condition is:

If $a \leq b \vee c$, then there exist in S elements b', c' such that $b' \leq b, c' \leq c$, and $a = b' \vee c'$.

This condition is referred to in the paper [6] by E. T. Schmidt, and is studied in the unpublished paper [3] by Grätzer and Schmidt. It is sufficient for the distributivity of $\mathcal{F}(S)$, which is called the lattice of ideals if S is a semilattice. If S has an identity element 0, the Grätzer-Schmidt condition is also necessary for distributivity. Unfortunately it is not always necessary if S has no zero element, as will be shown by an example.

For a groupoid G , the Grätzer-Schmidt condition takes the following form:

If $a \subset bc$, then there exist in G elements b', c' such that $b' \subset b, c' \subset c$, and $a \sim b'c'$.

Again this condition is sufficient for the distributivity of $\mathcal{F}(G)$, and is also necessary if G has an identity element. The following condition is more complicated, but is always both necessary and sufficient for distributivity of $\mathcal{F}(G)$:

If $a \subset bc$, then either $a \subset b$, or $a \subset c$, or there exist elements b' , c' of G such that $b' \subset b$, $c' \subset c$, and $a \sim b'c'$.

THEOREM 8. *The lattice of filters $\mathcal{F}(G)$ is distributive if and only if whenever $a \subset bc$, then either $a \subset b$, or $a \subset c$, or there exist elements b' , c' of G such that $b' \subset b$, $c' \subset c$, and $a \sim b'c'$.*

Proof. To prove sufficiency, suppose the condition holds, and $x \in A \bar{\vee} B$, where A and B are filters of G . By Theorem 6 we must show that $x \sim y$ for some element y in $A \vee B$. By Theorem 5, either x is in $A \cup B$, in which case $y = x$ is in $A \vee B$, or $x \subset ab$ with $a \in A$ and $b \in B$. In the latter case by the hypothesis of our theorem either $x \subset a$ or $x \subset b$, in which case $y = x$ is in $A \vee B$, or there exist elements $a' \subset a$ and $b' \subset b$ such that $x \sim a'b'$. Since A and B are filters, $a' \in A$ and $b' \in B$. Hence $a'b' \in A \vee B$. Let $y = a'b'$. Then $x \sim y$ and $y \in A \vee B$. This proves the sufficiency of the condition.

To prove necessity, suppose the lattice $\mathcal{F}(G)$ is distributive, and $a \subset bc$. Then $F_a \subseteq F_{bc} = F_b \bar{\vee} F_c$. It follows that $a \in F_a \cap (F_b \bar{\vee} F_c)$. By distributivity, we see that $a \in (F_a \cap F_b) \bar{\vee} (F_a \cap F_c)$. Then by Theorem 5 either $a \in F_b$, whence $a \subset b$, or $a \in F_c$, whence $a \subset c$, or $a \subset b'c'$, where $b' \in F_a \cap F_b$, and $c' \in F_a \cap F_c$. In the latter case it follows that $b' \subset b$, $c' \subset c$, and $b'c' \subset a$. Since also we have $a \subset b'c'$, it follows that $a \sim b'c'$. But this is just the condition of the theorem, and the proof ends.

The Grätzer-Schmidt condition is not always necessary for the distributivity of \mathcal{F} even for semilattices, as the following example shows. Consider the three-element semilattice $G = \{x, y, z\}$ in which $xy = xz = yz = z$; note that G has no identity element. The lattice $\mathcal{F}(G)$ consists of the four elements \emptyset, F_x, F_y, F_z and is distributive. Now $x \subset xy$, but there do not exist elements $x' \subset x$ and $y' \subset y$ such that $x \sim x'y'$. For if $x' \subset x$, then $x' = x$, and if $y' \subset y$, then $y' = y$. Hence $x'y' = z$, but $x \sim z$ is false, and condition fails.

However, for groupoids with an identity element, and more generally for groupoids which are *lower-directed*, the Grätzer-Schmidt condition is both necessary and sufficient for the distributivity of \mathcal{F} .

DEFINITION. A groupoid G is said to be *lower-directed* if given any two elements a and b of G , there exists an element x of G such that $x \subset a$ and $x \subset b$.

Note that G is lower-directed if and only if the filter lattice $\mathcal{F}(G)$ has the *finite intersection property*; this means that the intersection of any two, and hence of any finite number of nonempty filters is a nonempty filter. Note also that a groupoid with an identity

element 1 is lower-directed, since the element 1 is a factor of every element, hence $1 \subset x$ for all x in G .

THEOREM 9. *The filter lattice $\mathcal{F}(G)$ of a lower-directed groupoid G , and hence of a groupoid with identity element, is distributive if and only if the Grätzer-Schmidt condition holds, that is, if $a \subset bc$, then there exist elements $b' \subset b$ and $c' \subset c$ in G such that $a \sim b'c'$.*

Proof. The condition is sufficient, since the Grätzer-Schmidt condition is stronger than and implies the condition of Theorem 8. To show that it is also necessary in this case, suppose $\mathcal{F}(G)$ is distributive and $a \subset bc$. Then by Theorem 8 either elements b' and c' with the required properties exist, or $a \subset b$ or $a \subset c$. The argument is the same either way, so suppose $a \subset b$. Since G is lower-directed, there exists an element x in G such that $x \subset a$ and $x \subset c$. Then $ax \subset aa$. Let $b' = a$ and $c' = x$. Then $b' \subset b$, $c' \subset c$, and $b'c' = ax$. But $a \subset ax \subset aa \subset a$, hence $a \sim ax$ and $a \sim b'c'$. Hence the conclusion of the Grätzer-Schmidt condition holds, and the condition is also necessary, ending the proof.

Note that a groupoid with the Grätzer-Schmidt property is necessarily lower-directed. For if a and b are any two elements of G , then $a \subset ab$, hence elements $a' \subset a$ and $b' \subset b$ exist with $a \sim a'b'$. Then b' is a common lower bound of a and b , since $b' \subset b$, and $b' \subset ab \subset a$. Hence G is lower-directed.

We now apply Theorem 9 to obtain a necessary and sufficient condition for the distributivity of an arbitrary algebraic lattice. This result is not really new, since it was known some time ago to Grätzer and Schmidt.

THEOREM 10. *An algebraic lattice (L, \wedge, \vee) is distributive if and only if whenever a, b, c are compact elements of L , and $a \leq b \vee c$, then there exist compact elements b' and c' such that $b' \leq b$, $c' \leq c$, and $a = b' \vee c'$.*

Proof. Since L is complete, it has a least element 0 which is compact. The compact elements of L form an upper semilattice (G, \vee) with identity element 0. It is well known that L is isomorphic to $\mathcal{F}(G)$, which is usually called the lattice of ideals of G . By Theorem 9, L is distributive if and only if the Grätzer-Schmidt condition holds in G . The condition of the theorem is the form this condition takes in an upper semilattice. This ends the proof.

We now give another necessary and sufficient condition for the distributivity of the lattice $\mathcal{F}(G)$.

THEOREM 11. *The lattice $\mathcal{F}(G)$ of a groupoid G is distributive if and only if the Grätzer-Schmidt condition holds in G^1 .*

Proof. If G has an identity element 1, then $G = G^1$ and the result follows from Theorem 9. If not, then G^1 is formed by adjoining to G a single element 1, which is a factor of every element of G^1 and a member of every nonempty filter of G^1 . Hence $\mathcal{F}(G)$ is isomorphic to the lattice of all nonempty filters of G^1 , and is distributive if and only if $\mathcal{F}(G^1)$ is distributive. By Theorem 9, this is true if and only if the Grätzer-Schmidt condition holds in G^1 .

Since every algebraic lattice, and hence every finite lattice, is isomorphic to some filter lattice $\mathcal{F}(G)$, filter lattices in general are neither modular or distributive.

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MIAMI UNIVERSITY