PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

GENERALIZED QUASICENTER AND HYPERQUASICENTER OF A FINITE GROUP

J. B. DERR AND N. P. MUKHERJEE

The notion of quasicentral element is generalized to p-quasicentral element and the p-quasicenter and the p-hyperquasicenter are defined. It is shown that the p-quasicenter is p-supersolvable and the p-hyperquasicenter is p-solvable.

The quasicenter $Q(G)$ of a group G is the subgroup of G generated by all quasicentral elements of G, where an element x of G is called a quasicentral element (QC-element) when the cyclic subgroup $\langle x\rangle$ generated by x satisfies $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all elements y of G. The hyperquasicenter $Q^*(G)$ of a group G is the terminal member of the upper quasicentral series $1 = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = Q_{n+1} = Q^*(G)$ of G, where Q_{i+1} is defined by $Q_{i+1}/Q_i = Q(G/Q_i)$. Mukherjee has shown [3, 4] that the quasicenter of a group is nilpotent and the hyperquasicenter is the largest supersolvably immersed subgroup of a group. The proofs of these structure theorems rely on the fact that the powers of QC-elements are again QC-elements.

In this paper we generalize the notion of a quasicentral element in a way which allows the results about the quasicenter and the hyperquasicenter [3, 4] to be extended. All groups mentioned are assumed to be finite.

For a given group G and a fixed prime p, the definition of QC-element might suggest that an element x of G be called a p-quasicentral element provided $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ holds for all p-elements y of G. An apparent difficulty with this definition is that the powers of p-quasicentral elements need not again be p-quasicentral elements. For example, consider the group of order 18 defined by $G = \langle a, b, x \mid a^3 = b^3 = 1 = x^2, [a, b] = 1 = [a, x], [b, x] = a\rangle$. A simple calculation shows that ax is 3-quasicentral while $x = (ax)^3$ is not 3-quasicentral—otherwise $\langle x\rangle\langle b\rangle = \langle b\rangle\langle x\rangle$ shall imply that x normalizes $\langle b\rangle$, which is not the case however. Because of this example we choose to generalize the notion of a QC-element as follows.

DEFINITION 1. Let G be a given group and p a fixed prime. Suppose x is an element of G and let the order of x be written as $|x| = p^r m$ where $(p, m) = 1$. Then x is called a p-quasicentral (p-QC) element of G provided $\langle x^m\rangle\langle y\rangle = \langle y\rangle\langle x^m\rangle$ and $\langle x^{p^r}\rangle\langle y\rangle = \langle y\rangle\langle x^{p^r}\rangle$ hold for all p-elements y of G. (It should be noted that every element of a p'-group is p-QC.)

THEOREM 1. *If x is a p-QC element of a group G and k is a fixed integer, then x^k is also a p-QC element of G.*

Proof. Suppose $|x| = p^b m$ where $(p, m) = 1$. Since $|x^m| = p^b$, $|x^{p^b}| = m$ and x^{p^b} commutes with x^m, $\langle x\rangle = \langle x^{p^b}\rangle\langle x^m\rangle = \langle x^m\rangle\langle x^{p^b}\rangle$. If $|x^k| = p^c n$ where $(p, c) = 1$, then $(x^k)^{p^c}$ is a p'-element of $\langle x\rangle$ and $(x^k)^n$ is a p-element of $\langle x\rangle$. It follows that $(x^k)^{p^c}$ is some power of x^{p^b} and $(x^k)^n$ is some power of x^m. To show that x^k is a p-QC element of G, it will suffice to show that $\langle (x^m)^i\rangle\langle y\rangle = \langle y\rangle\langle (x^m)^i\rangle$ and $\langle (x^{p^b})^i\rangle\langle y\rangle = \langle y\rangle\langle (x^{p^b})^i\rangle$ hold for all integers i and all p-elements y of G.

Let y be any p-element in G. Since x is a p-QC element of G, $\langle x^m\rangle\langle y\rangle = \langle y\rangle\langle x^m\rangle$. Therefore $\langle x^m\rangle\langle y\rangle$ is some subgroup H of G whose order divides $|x^m|\cdot|y|$. Since x^m is then a p-QC element of the p-group H, x^m is a QC-element of H. It follows [3, 4] that every power of x^m is a QC-element of H. In particular, $\langle (x^m)^i\rangle\langle y\rangle = \langle y\rangle\langle (x^m)^i\rangle$ holds for every integer i.

Now proceed by induction on the order of G to show that $\langle (x^{p^b})^i\rangle\langle y\rangle = \langle y\rangle\langle (x^{p^b})^i\rangle$ holds for every integer i and every p-element y. Let y be a fixed p-element of G of order p^r. If $\langle x^{p^b}\rangle\langle y\rangle = \langle y\rangle\langle x^{p^b}\rangle$ is a proper subgroup of G, induction completes the argument. Assume therefore that $G = \langle x^{p^b}\rangle\langle y\rangle = \langle y\rangle\langle (x^{p^b})\rangle$. Then G is a supersolvable group (Theorem 13.3.1, [5]).

Let π denote the set of prime divisors of $|x^{p^b}| = m$ which are larger than p. Since G is supersolvable with order $|G| = p^r m$, G has a normal Hall π-subgroup K. Distinguish two cases.

Case 1. π is empty. Then p is the largest prime dividing $|G|$. Since $\langle y\rangle$ is a Sylow p-subgroup of G, $\langle y\rangle$ must be normal in G. Clearly $\langle (x^{p^b})^i\rangle\langle y\rangle = \langle y\rangle\langle (x^{p^b})^i\rangle$ holds for all integers i in this case.

Case 2. π is nonempty. Let s and t denote integers such that $x_1 = (x^{p^b})^s$ is a π-element, $x_2 = (x^{p^b})^t$ is a π'-element and $x^{p^b} = x_1x_2 = x_2x_1$ (Theorem 4, [2], p. 23). Then $\langle x_1\rangle$ is a Hall π-subgroup of G. Since G is supersolvable, $\langle x_1\rangle \trianglelefteq G$. It follows that $\langle x_1^i\rangle\langle y\rangle = \langle y\rangle\langle x_1^i\rangle$ holds for every integer i. Since $\langle (x^{p^b})^i\rangle = \langle x_1^i\rangle\langle x_2^i\rangle$ for all integers i, the argument will be complete if we show $\langle x_2^i\rangle\langle y\rangle = \langle y\rangle\langle x_2^i\rangle$ holds for all i. Since $\langle x_1\rangle$ is a normal Hall π-subgroup of G, the Schur-Zassenhaus theorem shows that G possesses a π-complement R. Since y is a π'-element of G, we may choose R so that $y \in R$. Then $\langle y\rangle$ is a Sylow p-subgroup of R. Since R is supersolvable and p is the largest prime dividing $|R|$, $\langle y \rangle \trianglelefteq R$. We now use the fact that x_2 is a π'-element. Since R is a Hall π'- subgroup of the solvable group G, some conjugate x_2^g of x_2 lies in R. It now follows from $G = \langle x^{p^b} \rangle \langle y \rangle$ that $x_2 \in R$, since every element g in G can be written as $(x^{p^b})^u y^v$ for some integers u, v. Therefore $\langle x_2^i \rangle \langle y \rangle = \langle y \rangle \langle x_2^i \rangle$ holds for every integer i. This completes the proof of the theorem.

LEMMA 1. *Let θ be a homomorphism from a group G onto a group $\bar{G}$. If x is a p-QC element of G, the image x^θ of x is a p-QC element of $\bar{G}$.*

Proof. Let $|x| = p^b m$ where $(p, m) = 1$ and let $|x^\theta| = p^c n$ where $(p, n) = 1$. It follows that $\langle x \rangle = \langle x^{p^b} \rangle \langle x^m \rangle$ and $\langle x^\theta \rangle = \langle (x^\theta)^{p^c} \rangle \langle (x^\theta)^n \rangle$. Now $\langle x^\theta \rangle = \langle x \rangle^\theta$ implies $\langle x^{p^b} \rangle^\theta = \langle (x^\theta)^{p^c} \rangle$ and $\langle x^m \rangle^\theta = \langle (x^\theta)^n \rangle$.

Let $\bar{u}$ be any p-element of $\bar{G}$. Then there is a p-element y of G with $y^\theta = \bar{u}$. Since x is a p-QC element of G, $\langle x^{p^b} \rangle \langle y \rangle = \langle y \rangle \langle x^{p^b} \rangle$ and $\langle x^m \rangle \langle y \rangle = \langle y \rangle \langle x^m \rangle$. This shows $\langle x^{p^b} \rangle^\theta \langle y \rangle^\theta = \langle y \rangle^\theta \langle x^{p^b} \rangle^\theta$ and $\langle x^m \rangle^\theta \langle y \rangle^\theta = \langle y \rangle^\theta \langle x^m \rangle^\theta$. Now $\langle y \rangle^\theta = \langle y^\theta \rangle = \langle \bar{u} \rangle$ implies $\langle (x^\theta)^{p^c} \rangle \langle \bar{u} \rangle = \langle \bar{u} \rangle \langle (x^\theta)^{p^c} \rangle$ and $\langle (x^\theta)^n \rangle \langle \bar{u} \rangle = \langle \bar{u} \rangle \langle (x^\theta)^n \rangle$. The proof of the lemma is therefore complete.

DEFINITION 2. Let G be a given group and p a fixed prime. The p-quasicenter $Q_p(G)$ is the subgroup of G generated by all p-QC elements of G.

We mention a few simple consequences of the definition of the p-quasicenter. For any group G and any prime p, the quasicenter of G is contained in the p-quasicenter of G. The p-quasicenter of a group is always a characteristic subgroup of the group. It should be noted that if a prime p does not divide the order of a group G then $Q_p(G) = G$.

THEOREM 2. *For any group G and every prime p, the p-quasicenter $Q_p(G)$ is p-supersolvable.*

Proof. First we notice that $Q_p(G) = G$ is p-supersolvable if p does not divide $|G|$. Consequently we assume that p divides $|G|$. The proof is by induction on $|G|$.

It suffices to show that G contains a nontrivial normal subgroup N of order p or of order prime to p. For, by induction, $Q_p(G/N)$ is then p-supersolvable. Since Lemma 1 shows $Q_p(G)N/N \subseteq Q_p(G/N)$ it will follow that $Q_p(G)$ is p-supersolvable. (This is because of the fact that normal subgroups of p-supersolvable groups are p-supersolvable and N being of order p or prime to p, the p-supersolvability of $Q_p(G)N/N$ implies $Q_p(G)N$ is p-supersolvable.) Since $Q_p(Q_p(G)) = Q_p(G)$, induction lets us assume that $Q_p(G) = G$. Thus G is generated by p-QC elements $x_1, x_2, \cdots, x_n$. First we show that G contains a proper normal subgroup. Distinguish two cases.

Case 1: Some x_i has order divisible by p. Assume p divides the order of x_1. Then there is an integer d such that $|x_1^d| = p$. Since x_1^d is a p-QC element of G, $\langle x_1^d \rangle$ permutes with each Sylow p-subgroup of G. Therefore $\langle x_1^d \rangle$ lies in the maximum normal p-subgroup $O_p(G)$ of G. Therefore $O_p(G)$ is a proper normal subgroup of G or $O_p(G) = G$ and G is a p-group. If G is a p-group, the theorem is trivially true.

Case 2: No x_i has order divisible by p. Then $x_1, x_2, \cdots, x_n$ are p-QC elements of G with p'-orders. Since $|G|$ is divisible by p, G must contain nonidentity p-elements. Let T denote the subgroup of G generated by all the p-elements of G. Since $T \trianglelefteq G$, we can assume $T = G$. Therefore G contains nonidentity p-elements $y_1, y_2, \cdots, y_m$ with $\langle y_1, y_2, \cdots, y_m \rangle = G$. Let q be the largest prime dividing the product $|x_1| \cdot |x_2| \cdots |x_n|$. First suppose $p > q$. Since x_i is a p-QC element and y_1 is a p-element, $\langle x_i \rangle \langle y_1 \rangle = \langle y_1 \rangle \langle x_i \rangle$ holds for all $i = 1, 2, \cdots, n$. It follows (theorem 13.3.1, [5]) that $\langle x_i \rangle \langle y_1 \rangle$ is supersolvable of order $|x_i| \cdot |y_1|$ for $i = 1, 2, \cdots, n$. Since x_i is a p'-element and $p > q$, $\langle y_1 \rangle$ is a normal Sylow p-subgroup of each group $\langle x_1 \rangle \langle y_1 \rangle$. Then $x_1, x_2, \cdots, x_n$ normalize $\langle y_1 \rangle$ and $\langle y_1 \rangle$ is a normal subgroup of $G = \langle x_1, x_2, \cdots, x_n \rangle$. Now suppose $p < q$ and let $|x_1|$ be divisible by q. Let s be an integer such that $\langle x_1^s \rangle$ is a Sylow q-subgroup of $\langle x_1 \rangle$. Since $\langle x_1 \rangle \langle y_j \rangle = \langle y_j \rangle \langle x_1 \rangle$ is a supersolvable group and q is the largest prime dividing $|y_j| \cdot |x_1|$, y_j normalizes $\langle x_1^s \rangle$ for $j = 1, 2, \cdots, m$. Therefore $\langle x_1^s \rangle \trianglelefteq G = \langle y_1, y_2, \cdots, y_m \rangle$. This shows that in every case G contains a proper normal subgroup M. If M has order prime to p, we are finished. Assume now that M is a minimal normal subgroup of G and p divides $|M|$. We will show that $|M| = p$.

Since $Q_p(G) = G$, G is generated by p-QC elements $x_1, x_2, \cdots, x_n$ of G. For each i, $1 \leqq i \leqq n$, $\langle x_i \rangle = \langle v_1 \rangle \langle v_2 \rangle \cdots \langle v_{d_i} \rangle$ where $v_1, v_2, \cdots, v_{d_i}$ are powers of x_i, v_1 is a p-element, and $v_2, v_3, \cdots, v_{d_i}$ are p'-elements of prime power orders. Since powers of p-QC elements are also p-QC elements, it follows that G can be written as $G = \langle a_1, a_2, \cdots, a_h, b_1, b_2, \cdots, b_k \rangle$ where each a_i is a p-QC p-element of G and each b_j is a p-QC p'-element of G having prime power order.

Let P denote the subgroup of G generated by all p-QC p-elements of G. Clearly P is a characteristic p-subgroup of G with $\langle a_1, a_2, \cdots, a_h \rangle \subseteqq P$. Since M is a minimal normal subgroup of G, $P \cap M = 1$ or $P \cap M = M$. First suppose that $P \cap M = 1$. Then $[P, M] \subseteqq P \cap M = 1$ and P centralizes M. Let $w \in M$ with $|w| = p$. Clearly a_i normalizes $\langle w \rangle$ for $i = 1, 2, \cdots, h$. Since each b_j is a p-QC element of G, $\langle b_j \rangle \langle w \rangle = \langle w \rangle \langle b_j \rangle$ holds for $j = 1, 2, \cdots, k$. It follows that each group $\langle b_j \rangle \langle w \rangle$ is supersolvable of order $|b_j| \cdot |w|$.

Since $|b_j|$ is a power of a prime other than p, $\langle b_j\rangle$ is a Sylow subgroup of $\langle b_j\rangle\langle w\rangle$. Hence b_j normalizes $\langle w\rangle$ or w normalizes $\langle b_j\rangle$ for each $j = 1, 2, \cdots, k$. Since $\langle b_j\rangle \cap M = 1$ implies b_j normalizes $\langle w\rangle$, $\langle w\rangle \trianglelefteq G$ unless $\langle b_j\rangle \cap M \neq 1$ for some j. Assume that $\langle b_d\rangle \cap M \neq 1$ for some integer d, $1 \leqq d \leqq k$. This implies that some prime different from p divides the order of M. Since every power of b_d is a p-QC element of G, $Q_p(M) \neq 1$. From the minimality of M it follows that $Q_p(M) = M$, since $Q_p(M)$ is characteristic in M and M is normal in G. Induction applied to M then shows that M is p-supersolvable. If N is a minimal normal subgroup of M then $|N|$ is either p or is prime to p. Then $T = \langle N^g \mid g \in G\rangle$ is a normal subgroup of G contained in M and $T = N^{g_1} \cdots N^g$ where $g_1, \cdots, g_t$ are elements of G. But M being minimal normal in G it follows that $T = M$. Therefore M is either a p-group or a p'-group, since T is so. But p divides the order of M and therefore M must be a p-group. This however contradicts the assumption that $\langle b_d\rangle \cap M \neq 1$. Thus $\langle w\rangle \trianglelefteq G$. Since $\langle w\rangle \subseteqq M$, $M = \langle w\rangle$ and M has order $|w| = p$. Now suppose $P \cap M = M$. Then M is a normal subgroup of the p-group P and $M \cap Z(P) \neq 1$. Let z be a nonidentity element of $M \cap Z(P)$ with $|z| = p$. Since $z \in Z(P)$, surely $\langle a_1, a_2, \cdots, a_h\rangle$ normalizes $\langle z\rangle$. On the other hand, M being a p-group it is evident that $\langle b_j\rangle \cap M = 1$ for each $j = 1, 2, \cdots, k$. As before, $\langle z\rangle \trianglelefteq G$ unless $\langle b_j\rangle \cap M \neq 1$ for some j. Therefore $\langle z\rangle \trianglelefteq G$. Since $1 \neq \langle z\rangle \subseteqq M$, the minimality of M shows $M = \langle z\rangle$. Therefore M has order $|z| = p$ and the proof is complete.

Since the quasicenter of a group is nilpotent it is natural to ask if the p-quasicenter of a group must be p-nilpotent. We give an example to show that this need not be the case. Let S_3 denote the symmetric group of degree 3. The 3-quasicenter of S_3 is S_3 itself. Clearly $Q_3(S_3) = S_3$ is not 3-nilpotent.

DEFINITION 3. Let G be a given group and p a fixed prime. The upper p-quasicentral series $1 = H_0 \subset H_1 \subset \cdots \subset H_n = H_{n+1}$ of G is the characteristic series where H_{i+1} is defined by $H_{i+1}/H_i = Q_p(G/H_i)$. The number of distinct nontrivial terms in the upper p-quasicentral series of G is called the p-quasicentral length of G. The terminal member of the upper p-quasicentral series of G is called the p-hyperquasicenter of G. We denote this characteristic subgroup of G by $Q_p^*(G)$.

THEOREM 3. *In any group G, the p-hyperquasicenter $Q_p^*(G)$ is the intersection of all normal subgroups N with $Q_p(G/N) = N/N$.*

Proof. Let $S = \bigcap \{N \mid N \trianglelefteq G$ and $Q_p(G/N) = N/N\}$. Clearly

$S \subseteqq Q_p^*(G)$. We now show that $Q_p^*(G)$ is included in every normal subgroup N for which $Q_p(G/N) = N/N$. Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_m = Q_p^*(G)$ be the upper p-quasicentral series of G. Trivially $H_0 \subseteqq N$. Assume that $H_i \subseteqq N$ and $H_{i+1} \not\subseteqq N$. Then for some p-QC element yH_i of G/H_i, $y \notin N$. This implies that under the natural homomorphism of G/H_i to G/N, the p-QC element yH_i is mapped onto the p-QC element yN of G/N. Therefore $Q_p(G/N)$ is nontrivial, a contradiction. Hence $H_{i+1} \subseteqq N$ and $Q_p^*(G) \subseteqq N$ follows by induction.

We shall now investigate the structure of the p-hyperquasicenter $Q_p^*(G)$.

LEMMA 2. *Let G be a group and p a fixed prime. If $N \trianglelefteq G$ and $N \subseteqq Q_p^*(G)$ then $Q_p^*(G/N) = Q_p^*(G)/N$.*

Proof. Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = Q_p^*(G)$ be the upper p-quasicentral series of G and let $N/N = L_0/N \subset L_1/N \subset \cdots \subset L_k/N = Q_p^*(G/N)$ be the upper p-quasicentral series of G/N. By Lemma 1, $H_1N/N = Q_p(G)N/N \subseteqq Q_p(G/N) = L_1/N$. Thus $H_1 \subseteqq L_1 \subseteqq L_k$. Now assume $H_i \subseteqq L_k$ and deduce $H_{i+1} \subseteqq L_k$. Since $H_i \subseteqq L_k$, G/L_k is a homomorphic image of G/H_i. Let θ be the natural homomorphism described by $(xH_i)^\theta = xL_k$. Then Lemma 1 shows that $(Q_p(G/H_i))^\theta \subseteqq Q_p(G/L_k) = L_k/L_k$. Since $Q_p(G/H_i) = H_{i+1}/H_i$, $(Q_p(G/H_i))^\theta = H_{i+1}L_k/L_k \subseteqq L_k/L_k$. Therefore $H_{i+1} \subseteqq L_k$ and by induction $H_n \subseteqq L_k$. We complete, the proof by showing $L_i \subseteqq H_n = Q_p^*(G)$ for each $i = 1, 2, \cdots, k$. By hypothesis $L_0 = N \subseteqq Q_p(G)$. Now assume $L_i \subseteqq Q_p(G)$ and deduce $L_{i+1} \subseteqq Q_p^*(G)$. Since $L_i \subseteqq Q_p^*(G)$, $G/Q_p^*(G)$ is a homomorphic image of G/L_i. The argument above can be repeated to obtain $L_{i+1} \subseteqq Q_p^*(G)$.

THEOREM 4. *For any group G and any prime p, $Q_p^*(G)$ is p-solvable.*

Proof. If $Q_p^*(G) = Q_p(G)$, $Q_p^*(G)$ is p-supersolvable and the theorem is proved. Assume now that $Q_p(G) \subsetneqq Q_p^*(G)$. Let N denote any minimal normal subgroup of $Q_p(G)$. Since $Q_p(G)$ is p-supersolvable, N has p'-order or $|N| = p$. Set $S = \langle N^g \mid g \in G \rangle$. Since $N \trianglelefteq Q_p(G) \trianglelefteq G$, $N^g \trianglelefteq Q_p(G)$ for each $g \in G$. It follows that S has order prime to p or order a power of p. Since $S \trianglelefteq G$ and $S \subseteqq Q_p(G) \subseteqq Q_p^*(G)$ induction shows that $Q_p^*(G/S) = Q_p^*(G)/S$ is p-solvable. Therefore $Q_p^*(G)$ is p-solvable.

It is possible to characterize the p-hyperquasicenter in terms of the normal subgroups included in it. We begin with the following definition.

DEFINITION 4. Let G be a group and p a fixed prime. A normal subgroup N of G is called p-hyperquasicentral (p-HQ) if $N/M \cap$

$Q_p^*(G/M) \neq M/M$ holds for each normal subgroup M of G which is properly contained in N.

The lemmas proved next will be useful for the proof of Theorem 5.

LEMMA 3. *Let G be any group and p a fixed prime. If $N \trianglelefteq G$ then $Q_p^*(G)N/N \subseteqq Q_p^*(G/N)$.*

Proof. Let $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = Q_p^*(G)$ be the upper p-quasicentral series of G. By Lemma 1, $H_1N/N = Q_p(G)N/N \subseteqq Q_p(G/N) \subseteqq Q_p^*(G/N) = L/N$. Thus $H_1N \subseteqq L$. Now assume $H_iN \subseteqq L$ and deduce that $H_{i+1}N \subseteqq L$. Since $H_i \subseteqq H_iN$, G/H_iN is a homomorphic image of G/H_i. Let ϕ be the natural homomorphism of G/H_i onto G/H_iN described by $(xH_i)^\phi = xH_iN$. Then Lemma 1 shows $(Q_p(G/H_i))^\phi \subseteqq Q_p(G/H_iN)$. Since $Q_p(G/H_i) = H_{i+1}/H_i$, $H_{i+1}N/H_iN = (H_{i+1}/H_i)^\phi \subseteqq Q_p(G/H_iH)$. Next let θ be the natural homomorphism of G/H_iN onto G/L given by $(xH_iN)^\theta = xL$. By Lemma 1, $(Q_p(G/H_iN))^\theta \subseteqq Q_p(G/L) = L/L$. Since $H_{i+1}N/H_iN \subseteqq Q_p(G/H_iN)$, $(H_{i+1}N/H_iN)^\theta = H_{i+1}NL/L \subseteqq L/L$. Therefore $H_{i+1}N \subseteqq L$ and the assertion follows.

LEMMA 4. *If any two groups G_1 and G_2 are isomorphic under a map θ then $(Q_p(G_1))^\theta = Q_p(G_2)$.*

LEMMA 5. *For any group G and any prime p, the product of p-HQ subgroups of G is a p-HQ subgroup of G.*

Proof. It suffices to show that for any p-HQ subgroups A and B of G, the product AB is a p-HQ subgroup of G. Let M be any normal subgroup of G with $M \subsetneqq AB$. If $M \subsetneqq A$ or $M \subsetneqq B$ then $AB/M \cap Q_p^*(G/M) \neq M/M$. Now suppose M is not a proper subgroup of either A or B. Since $A \cap M = A$ and $B \cap M = B$ together imply $AB \subseteqq M$, we may assume $R = A \cap M \subsetneqq A$. Since A is p-HQ, $A/R \cap Q_p^*(G/R) \neq R/R$. Let yR be any nonidentity element of $A/R \cap Q_p^*(G/R)$. Then $y \in A$ and $y \notin R$ show $y \notin M$. Since $M/R \trianglelefteq G/R$, Lemma 3 shows $Q_p^*(G/R) \cdot M/R/M/R \subseteqq Q_p^*(G/R/M/R)$. It now follows from the isomorphism of $G/R/M/R$ and G/M that yM is a nonidentity element of $Q_p^*(G/M)$. Therefore $AB/M \cap Q_p^*(G/M) \neq M/M$ and the assertion is proved.

THEOREM 5. *For any group G and any prime p, $Q_p^*(G)$ is the product of all p-HQ subgroups of G.*

Proof. Let S denote the product of all p-HQ subgroups of G. From Lemma 2 and the definition of p-HQ subgroup it is easily seen that $Q_p^*(G)$ is a p-HQ subgroup of G. Therefore $Q_p^*(G) \subseteqq S$.

Assume for the sake of contradiction that $Q_p^*(G) \subsetneqq S$. Since S is a p-HQ subgroup of G (Lemma 5) $S/Q_p^*(G) \cap Q_p^*(G/Q_p^*(G)) \neq Q_p^*(G)/Q_p^*(G)$. Since $Q_p^*(G/Q_p^*(G)) = Q_p^*(G)/Q_p^*(G)$, this is the desired contradiction.

It should be remarked that for a set of primes π, π-quasicentrality can be defined in a manner analogous to p-quasicentrality. The p-quasicenter and p-hyperquasicenter can be extended in the natural way to obtain the notions of π-quasicenter and π-hyperquasicenter. It is easily checked that the results about the p-quasicenter and the p-hyperquasicenter of a group remain valid when p is replaced by π.

REFERENCES

1. B. Huppert, *Endliche Gruppen I*, Springer-Verlag (1967).
2. W. Ledermann, *Introduction to the Theory of Finite Groups*, Interscience Publishers, Inc. (1964).
3. N. P. Mukherjee, *The hyperquasicenter of a finite group I*, Proc. Amer. Math. Soc., **26**, No. 2 (1970), 239-243.
4. ———, *The hyperquasicenter of a finite group II*, Proc. Amer. Math. Soc., **32**, No. 1 (1972), 24-28.
5. W. R. Scott, *Group Theory*, Prentice Hall, Inc. (1964).

Received April 12, 1971 and in revised form January 13, 1972.

WEST VIRGINIA UNIVERSITY