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UNIMBEDDABLE NETS OF SMALL DEFICIENCY

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We construct some new geometrical examples of unimbeddable nets N of order p^2 with p an odd prime. The deficiency of N is $p-j$ where either $j=0$ or $j=1$. In particular, the examples show that a bound of Bruck is best possible for nets of order 9,25. Our proof also shows that deriving a translation plane of order p^2 is equivalent to reversing a regulus in the corresponding spread.

2. **Background, summary.** Let N be a net of order n, degree k so that N has deficiency $d = n + 1 - k$. Let the polynomial $f(x)$ be given by $f(x) = x/2[x^3 + 3 + 2x(x + 1)]$. The following result is shown in [1].

THEOREM 1 (Bruck). *Suppose N is a finite net of order n, deficiency d. Then N is embeddable in an affine plane of order n provided $n > f(d-1)$.*

Thus a net of small deficiency is embeddable. However, as is pointed out in [1], little is known concerning the bound above. It is our purpose here to remedy this. In Theorem 2 we describe a construction used in [2] to obtain maximal partial spreads W of $PG(3, q)$. W yields a net N of order q^2 and deficiency $q - j$ where either $j = 0$ or $j = 1$. Our main result is that N is not embeddable if $q = p$ is an odd prime. This will show that Bruck's bound is best possible for nets of order 9,25 and is fairly good, if not best possible, for other nets of order p^2.

3. **The construction.** For definitions and proofs of Theorems 2, 3 we refer to [2].

THEOREM 2. *Let S be a spread of $\Sigma = PG(3, q)$ with $q \geqq 3$, such that S is not regular. Let u be a line of Σ with u not in S, such that the $q+1$ lines A of S passing through the $q+1$ points of u do not form a regulus. Let W_1 be the partial spread of Σ which is got by removing A from S and adjoining u: in symbols $W_1 = H \cup \{u\}$ where $H = S - A$. Then there exists a maximal partial spread W of Σ which contains W_1. Furthermore, either*

(i) *$W = W_1$ so that $|W| = q^2 - q + 1$ or*

(ii) *$W = W_1 \cup \{v\}$ where v is a line of Σ which is skew to each line of W_1. In this case $|W| = q^2 - q + 2$.*

THEOREM 3. *For any (prime power) $q \geqq 3$ there exist examples of* *case* (i). *For any odd q with $q \geqq 5$ there exist examples of case* (ii).

We can think of Σ in terms of a 4-dimensional vector space $V = V_4(q)$ over $GF(q)$. The points and lines of Σ are precisely the 1-dimensional and the 2-dimensional subspaces of V respectively. The lines or *components* of W in Σ correspond to the components of a maximal partial spread W of V, that is, a maximal collection W of 2-dimensional subspaces of V such that any 2 distinct members (components) of W have only the origin of V in common. For a proof of the next result see [7, p. 8], [4, p. 219].

THEOREM 4. *Let U be a partial spread of $V = V_4(q)$ having exactly k components. Then there is defined a net $N = N(U)$ of order q^2 and degree k. The points of N are the q^4 vectors in V. The lines of N are the components of U and their translates (cosets) in V. Furthermore, if U is a spread of V, then $N(U)$ is a translation plane.*

Our main result is that if W is the maximal partial spread of Theorem 2 and q is an odd prime, then $N(W)$ is not embeddable.

4. **The main result.** In what follows, if J is a set of vectors, then $\{J\}$ will denote the subspace spanned by the vectors in J.

LEMMA 5. *Let $\Sigma = PG(3, q)$ and let $(V, +) = V_4(q)$ be the corresponding vector space. Let a, b, c be 3 distinct and pairwise skew lines of Σ. Then we may choose a basis e_1, e_2, e_3, e_4 of V in such a manner that a corresponds to $\{e_1, e_2\}$, b corresponds to $\{e_3, e_4\}$ and c corresponds to $\{e_1 + e_3, e_2 + e_4\}$.*

The following is crucial in our argument.

THEOREM 6. *Let n be a square and let N be a net of order n and deficiency $\sqrt{n} + 1$, which is embedded in an affine translation plane π. Suppose further that N is embedded in another affine plane π_1. Then π_1 is also an affine translation plane.*

Proof. π_1 is related to π by Ostrom's technique of derivation (see [2, p. 383] and [6, p. 1382]). From this the result will follow, for it is easy to show that a plane π_1 obtained by deriving a translation plane π is itself a translation plane [4, p. 224].

We revert to the notation of Theorem 2. Recall that W is a maximal partial spread of $\Sigma = PG(3, q)$ with $q \geqq 3$. $W = H \cup \{u, v\}$ where (sometimes) $u = v$. H is a partial spread contained in the nonregular spread S of Σ. H contains exactly $q^2 - q$ lines. Since

$q \geqq 3$ we have $|H| = q^2 - q > 3$. Thus H contains 3 pairwise skew lines a, b, c which we will refer to as the *fundamental components*. Corresponding to Σ we have $V = V_4(q)$. As in Lemma 5 we have a basis e_1, e_2, e_3, e_4 of V with $a = \{e_1, e_2\}$, $b = \{e_3, e_4\}$, $c = \{e_1 + e_3, e_2 + e_4\}$. Let $L = \{e_1, e_2\}$ and $M = \{e_3, e_4\}$. We can write $V = L \oplus M$ the direct sum of L and M. Each vector in V is uniquely expressible as an ordered pair (x, y) with x in L, y in M. The fundamental components are then sets $y = 0$, $x = 0$, $y = x$ respectively. In the sequel it will be convenient to identify M with L and write $V = L \oplus L$. We also let 0 denote the null vector in L, so that $(0, 0)$ is the null vector of V.

THEOREM 7 (Main Result). *Let W be the maximal partial spread of $PG(3, q)$ constructed in Theorem 2. Assume that $q = p \geqq 3$ is a prime. Then the net $N = N(W)$ obtained from W as in Theorem 4 has order p^2 and deficiency $p - j$ where either $j = 0$ or $j = 1$. Moreover, N is not embeddable in a plane.*

Proof. By way of contradiction assume that N is embeddable in an affine plane π_1. Choose the origin of π_1 to be the origin of V. In the construction of W recall that $H \subset S$. Denote the translation plane obtained from S by π. Thus $N(H) \subset \pi$. Also $N(H) \subset N(W) \subset \pi_1$. Therefore, by Theorem 6, π_1 is a translation plane. We may use the fundamental components a, b, c to set up Hall coordinates for π_1 using the set L (see [5]). Actually it is easy to see that a vector λ has in π_1 Hall coordinates (s, t) if and only if λ has vector space coordinates (s, t) in $V = L \oplus L$. Also the Hall addition is precisely the vector space addition $+$ on L (see [7, p. 4]). Thus the translation plane π_1 is then coordinatized by a quasifield $Q = (L, +, \cdot)$. Those lines of π_1 through the origin which are also lines of $N = N(W)$ correspond to the components of W. Let l be a line of π_1 through the origin of π_1 such that l is not a line of N. Then l consists of all points with coordinates of the form $(x, x.m)$ for some m in L. Since Q is a quasifield we have $(x + y).m = x.m + y.m$. Therefore l is a set of p^2 vectors in V which is closed under addition. Since p is a prime, l is a 2-dimensional subspace of V. And l has only the origin of V in common with any component of W. Thus l yields a line of $PG(3, q)$ which is skew to each line of W. But this is a contradiction, since W is maximal.

Comments. 1. Our argument in Theorem 7 above can be modified to show the following. Let π_1 be obtained from the translation plane π of order p^2 by deriving with respect to a derivation set D of $p + 1$ points on the line at infinity. Then the $p + 1$ lines of π joining the origin to D yield a regulus in the spread corresponding to π. Thus, in this case, derivation implies reversing a regulus. It can be shown (see [2]) that reversing a regulus implies derivation for translation planes of order q^2, whether or not q is a prime. Thus the procedures of derivation and reversing a regulus are equivalent for the case of translation planes of order p^2. However, as is proved in [3], they are not in general equivalent if q is not a prime. The reason is that l above is not always a subspace in this general case. So it is not clear whether or not N is embeddable if q is not a prime.

2. For $q = p$ we have shown that $N = N(W)$ is unimbeddable. However except for $p = 3, 5$ we do not know whether $N(W)$ is contained in a larger net or even whether there exists a transversal T of N (that is, a set of p^2 points of N no two of which are joined by a line of N). However, it follows from the work in [2], [6] that T would have to be an affine subplane of π having order p.

3. For $p = 3$, $N(W)$ has deficiency 3 or 2. By Theorem 3.3 in [2], $N(W)$ must have deficiency 3. We have shown that $N(W)$ is not embeddable. It follows that $N(W)$ is not contained in any larger net, and that the bound in Theorem 1 is best possible for nets of order 9.

4. For $p = 5$ we can obtain an unimbeddable net $N = N(W)$ of deficiency 4 using Theorem 3. By Theorem 1, N is not contained in a larger net and so Bruck's bound is also the best possible for nets of order 25. Another way of putting it is to say that *we have produced a maximal set of* 20 *mutually orthogonal latin squares of order* 25.

References

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