

AN OBSTRUCTION TO FINDING A FIXED POINT FREE MAP ON A MANIFOLD

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The problem of whether a manifold M admits a fixed point free map is an old one. One well known result is that if the Euler characteristic $\chi(M) = 0$, then M has such a map. In the case where M is a closed differentiable manifold this follows easily from the fact that $\chi(M) = 0$ if and only if the tangent bundle of M admits a nonzero cross-section (see Hopf [4]). But $\chi(S^{2n}) = 2$, and S^{2n} certainly admits a fixed point free map, namely, the antipodal map. Therefore, the vanishing of the Euler characteristic of a manifold is only a sufficient, though hardly a necessary, condition for the manifold to have a fixed point free map. In the search for other invariants it is natural to generalize somewhat and state the problem in terms finding coincidence free maps.

The object of this paper is to give an elementary proof of the fact that, given a continuous map $f: (W^n, \partial W^n) \rightarrow (M^n, \partial M^n)$ between oriented C^∞ -manifolds, there is a well defined obstruction $o(f)$ to finding a special sort of map $F: M \rightarrow M$ with the property that $F(x) \neq f(x)$ for all $x \in W$. This is the content of Theorem 1 in §2. F will not necessarily be homotopic to f , but then this is something that should not be required in view of the fact that the antipodal map on S^{2n} is not homotopic to the identity map either. In Theorem 2 we prove that $o(\text{identity})$ behaves naturally with respect to tangential maps.

The author would like to thank the referee for bringing the work of F. B. Fuller ([2] and [3]) and E. Fadell ([1]) to his attention. In §3 we discuss the relationship between this paper and their work and how Theorem 1 might be generalized to the case $f: K^n \rightarrow M^n$, where K is an n -complex and M is a 1-connected closed manifold. One of the main differences between our approach and that of Fuller, Fadell, and others who have worked on the question of coincidences of maps is that they have allowed themselves to make changes only by a homotopy. They obtain fairly complete results with that restriction in terms of Lefschetz numbers (see [1]). On the other hand, we partially free ourselves from this requirement (in the sense that our maps will be homotopic only on the $(n - 1)$ -skeleton in general), so that there are more possibilities for F .

Finally, we have restricted ourselves to differentiable manifolds because all the constructions and proofs, which are quite simple from

a geometric point of view, follow from a few elementary facts about C^∞ -manifolds. In particular, it is because we have an exponential map from the tangent bundle of these manifolds to the manifold that allows us to use a nonzero vector field over the $(n - 1)$ -skeleton (which always exists) to homotope any given map to another which is coincidence free with the first on the $(n - 1)$ -skeleton. Thus, the whole problem reduces to being able to make maps on a disk coincidence free in a certain way, which also is not hard. Most probably, the results themselves are true for topological manifolds, but the proofs would be more technical.

1. **The definition of $o(f, g, \sigma)$.** Throughout this paper all manifolds will be compact, connected, oriented and C^∞ . If M is a manifold, then TM_x will denote the tangent space of M at x and $\exp_x: TM_x \rightarrow M$ will be the standard exponential map. (NOTE. \exp_x will not be defined everywhere if M has a boundary.) We assume that a base point p has been chosen in the interior of M , $\text{int}(M)$. Set $M_0 = M - p$. Furthermore, the metric on M will be induced from a suitable Riemannian metric on its tangent bundle. A good reference for most of the facts we need to know about vector fields is Milnor [8].

Consider a continuous map $f: (W^n, \partial W^n) \rightarrow (M^n, \partial M^n)$, $n \geq 1$.

Let D_1 be the closed n -disk in $\text{int}(M)$ centered at p and of radius $\delta > 0$. Choose a differentiable map $g: (W, \partial W) \rightarrow (M, \partial M)$ which is an ε -approximation to f for some $\varepsilon > 0$, i.e., $\text{dist}(f(x), g(x)) < \varepsilon$ for all $x \in W$. Just how small δ and ε have to be will be determined as we go along. That we are able to make the desired choices will be justified by the fact that we can always change the Riemannian metric on M . First of all, δ and ε must satisfy

(I) $\varepsilon < \text{dist}(x, \exp_x(v)) < \delta/3$ for each $x \in M$ and each $v \in TM_x$ in the domain of \exp_x with $\|v\| = 1$.

By choosing a small enough δ , we may assume that g is transverse regular on ∂D_1 , and that if B_1, \dots, B_k are the components of $g^{-1}(D_1)$ then B_i is an n -disk in $\text{int}(W)$ and $g|_{B_i}$ is a diffeomorphism of B_i onto D_1 . Let D be the closed n -disk centered at p and of radius $\varepsilon/2$. Set $C_i = g^{-1}(D) \cap B_i$ and $N = W - \text{int}(C_1 \cup \dots \cup C_k)$. Figure 1 below should help in keeping track of the various definitions.

Next, let σ be a nondegenerate vector field on M which points inward at all boundary points of M and such that the isolated zeros of σ all lie in $\text{int}(D)$. Fix distinct $v_i \in TM_p$, $1 \leq i \leq 2(k - 1)$, with $\|v_i\| = 1$ and let $S_i = \{\exp_p(tv_i) | \varepsilon/2 \leq t \leq \delta\}$. We require further that $\sigma(x)$ is transverse to the tangent vector of S_i at every $x \in S_i$ and "pointing away from p ." Define $h: N \rightarrow M$ by

$$h(x) = \exp_{g(x)}(\sigma(g(x))/\|\sigma(g(x))\|).$$

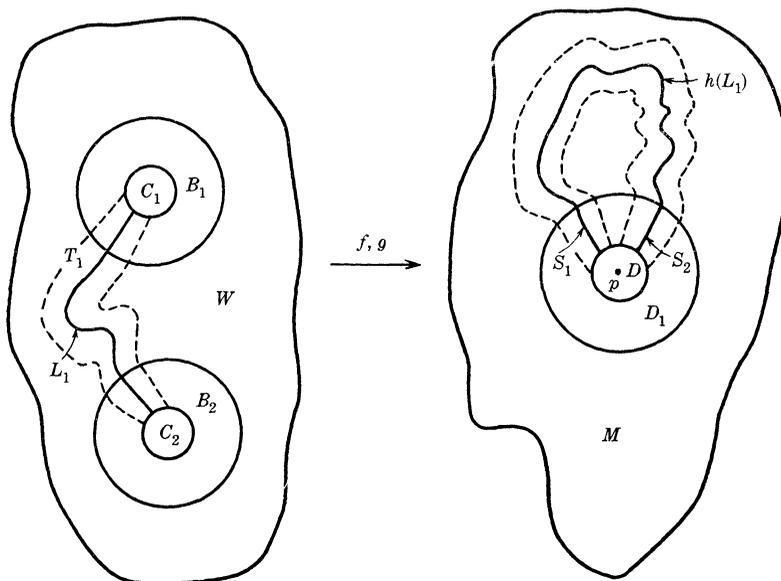


FIGURE 1

With the right Riemannian metric on M , h will be well defined since σ points inward on ∂M . Clearly, h is homotopic to $f|N$. It also follows from the definition of σ on the S_i and (I) that

$$(II) \quad h(x) \neq f(x), \text{ for } x \in N, \text{ and } h((\partial N - \partial W) \cup g^{-1}(S_1) \cup \dots \cup g^{-1}(S_{2(k-1)})) \subseteq M - \text{int}(D); \text{ in fact, } h(\partial N - \partial W) \subseteq D_1 - \text{int}(D).$$

Case 1. Assume that $n \geq 3$. Then we can connect the C_i with small tubes. Let $\varphi_i: [0, 1] \rightarrow N$ be disjoint imbeddings which meet ∂N transversally and such that $\varphi_i(0) \in \partial C_i, \varphi_i(1) \in \partial C_{i+1}, g(\varphi_i([0, 1]) \cap B_i) = S_{2i-1}$, and $g(\varphi_i([0, 1]) \cap B_{i+1}) = S_{2i}$. Set $L_i = \varphi_i([0, 1])$ and let T_i be a tubular neighborhood of L_i in N . We may suppose that $T_i \cap T_j = \emptyset = g(T_i) \cap g(T_j)$, for $i \neq j$. Define $B = g^{-1}(D) \cup T_1 \cup \dots \cup T_{k-1}$ and $Y = \partial B \cup T_1 \cup \dots \cup T_{k-1}$. B is a closed n -disk. (I) and (II) imply that if we choose the T_i small enough, then

$$(III) \quad h(Y) \subseteq M - \text{int}(D).$$

Assume that this is done. Therefore, the problem of extending h to a map $F: W \rightarrow M$ with $F(x) \neq f(x)$ for all $x \in W$ reduces to finding an extension $h_0: B \rightarrow M - \text{int}(D)$ of $h|_{\partial B}$ with $h_0(x) \neq f(x)$ for $x \in B$.

Let $q \in \partial D$ and let $G = \pi_{n-1}(M_0, q)$. The orientation of W induces an orientation on ∂C_i ; and therefore, in the notation of Hu [5] we get well defined obstructions $\gamma_i = \gamma^n(h|_{\partial C_i}) \in H^n(C_i, \partial C_i; G)$ with the property that $\gamma_i = 0$ if and only if $h|_{\partial C_i}$ extends to a map $h_i: C_i \rightarrow M - \text{int}(D)$, since $M - \text{int}(D)$ is a deformation retract of M_0 .

Next, let $W_0 = W - z$, where $g(z) = p$, and let $Q_i = B - \text{int} C_i$. Consider

$$\begin{array}{ccccc}
 H^n(W, \partial W; G) & & & & \\
 \uparrow j_5^* \approx & & & & \\
 H^n(W, W_0; G) & & & & H^n(B, Q_i; G) \\
 \downarrow j_2^* \approx & \swarrow j_3^* \approx & & & \downarrow j_4^* \approx \\
 H^n(B, \partial B; G) & \longleftarrow H^n(B, Y; G) & \longrightarrow & H^n(C_i, \partial C_i; G) & \\
 & \downarrow j_0^* \approx & & & \\
 & H^n(C_1, \partial C_1; G) \oplus \cdots \oplus H^n(C_k, \partial C_k; G) & & &
 \end{array}$$

where $j_i^*, i = 0, 1, \dots, 5$, are induced from the natural inclusion maps. Define $\lambda_i \in H^n(W, \partial W; G)$ by the equation

$$j_4^* j_3^{*-1} j_2^* j_5^{*-1}(\lambda_i) = \gamma_i .$$

DEFINITION. $o(f, g, \sigma) = \lambda_1 + \dots + \lambda_k \in H^n(W, \partial W; G)$.

Let $a_i \in \pi_1(M_0, q)$ be the elements determined by $g(L_i)$. Recall that $\pi_1(M_0, q)$ acts on G . This action induces an action of $\pi_1(M_0, q)$ on $H^n(B, \partial B; G)$. Then it is easy to check from the definitions that the obstruction $\gamma^n(h|\partial B) \in H^n(B, \partial B; G)$ satisfies $\gamma^n(h|\partial B) = j_1^*(j_5^{*-1}(\gamma_k)) + a_{k-1} \cdot j_1^*(j_5^{*-1}(\gamma_{k-1})) + \dots + a_1 \cdot j_1^*(j_5^{*-1}(\gamma_1))$. It follows that $j_2^* j_5^{*-1}(o(f, g, \sigma)) = \gamma^n(h|\partial B)$ if the a_i act trivially on $H^n(B, \partial B; G)$.

DEFINITION. f is said to satisfy (*) if the above construction can be performed in such a way that $a_i \cdot j_1^*(j_5^{*-1}(\gamma_i)) = j_1^*(j_5^{*-1}(\gamma_i))$.

LEMMA 1. Let $n \geq 3$. f will satisfy (*) in any one of the following situations:

- (a) M_0 is $(n - 1)$ -simple; in particular, if M is 1-connected.
- (b) $f_\sharp: \pi_1(W, z) \rightarrow \pi_1(M, p)$ is onto, or the a_i above are zero.
- (c) For some $g, k = 1$ above. For example, this is so if $f =$ identity.

Proof. If M is 1-connected, then M_0 is 1-connected because $n \geq 3$. Therefore, (a) is obvious. (c) and the second part of (b) are also obvious. The first part of (b) follows from the fact that in that case we can always choose our paths L_i so that $a_i = 0$. We need $n \geq 3$ here in order to be able to realize homotopy classes of paths by imbeddings.

Let us summarize our discussion in a lemma.

LEMMA 2. *Let $n \geq 3$. Suppose f satisfies (*) and we have performed the constructions above with respect to a vector field σ and approximation g . Then there is a well defined obstruction $o(f, g, \sigma) \in H^n(W, \partial W; G)$ with the property that $o(f, g, \sigma) = 0$ if and only if there exists a map $h_1: W \rightarrow M$ such that*

- (a) $h_1|W - \text{int}(B) = h$, and
- (b) $h_1(B) \cong M - \text{int}(D)$.

By construction h_1 will also have the property that $h_1(x) \neq f(x)$ for $x \notin T_1 \cup \dots \cup T_{k-1}$. We would like to have h_1 differ from f on the T_i also, but we can arrange this if we are a little more careful about how we define h and h_1 . (NOTE. The h_1 we get may not agree with h on $T_i - (\partial B \cap T_i)$.)

Since $n \geq 3$, we may assume that $g|L_i$ are disjoint imbeddings. Furthermore, we can also require that $\sigma(x)$ is transverse to the tangent vector of $g(L_i)$ for every $x \in g(L_i)$ and "pointing away from p ." Then by changing the Riemannian metric on M and making the T_i still smaller, if necessary, we can assume that $h(T_i) \cap f(T_j) = \emptyset$ for all i and j . Therefore, $h_1(\partial T_i) \cap f(T_j) = \emptyset$. Now homotope $h_1(T_i)$ into a neighborhood of $h_1(\partial B)$ using $h_1(B)$ and keeping $h_1|_{\partial B}$ fixed. This can be done simultaneously for all i . In effect, we push $h_1(T_{k-1} \cap \partial C_k)$ through $h_1(B - C_k)$ towards $h_1(\partial B - \partial C_k)$. We can do this in such a way so that now $h_1(x) \neq f(x)$ for $x \in T_1 \cup \dots \cup T_{k-1}$. But then $h_1(x) \neq f(x)$ for $x \in W$, since we have not disturbed anything on $M - \text{int}(B)$.

We have therefore proved the following:

LEMMA 3. *Let $n \geq 3$ and suppose f satisfies (*). Let σ be a vector field on M and g a differentiable approximation to f as described above. Then there is a well defined obstruction $o(f, g, \sigma) \in H^n(W, \partial W; \pi_{n-1}(M_0, q))$ with the property that $o(f, g, \sigma) = 0$ if and only if there exists a map $F: W \rightarrow M$ such that*

- (a) $F(x) \neq f(x)$ for all $x \in W$, and
- (b) $F(x) = h(x)$ for x outside some open n -disk in W , where h is obtained from σ and g as above.

NOTE. $o(f, g, \sigma)$ actually depends also on p, q , and the orientations, but the vanishing of $o(f, g, \sigma)$ is independent of this data. Also, since $H^n(W, \partial W; \pi_{n-1}(M_0, q)) \approx \pi_{n-1}(M_0, q)$, we can think of $o(f, g, \sigma)$ as being an element of $\pi_{n-1}(M_0, q)$.

DEFINITION. Any map F which satisfies the conclusion of Lemma 3 with respect to some σ will be called a σ -map for f to indicate its

intimate relation to σ , namely (b).

Case 2. Assume that $n \leq 2$. In certain cases we can still define an obstruction $o(f, g, \sigma) \in H^n(W, \partial W; \pi_{n-1}(M_0, q))$ having the properties described in Lemma 3. $\pi_{n-1}(M_0, q)$ may not be abelian when $n = 2$. In that case we identify that cohomology group with $\pi_{n-1}(M_0, q)$. Nevertheless one can manipulate the group formally just like other cohomology groups.

(a) Suppose M^2 is arbitrary and f is the identity map. If we work directly with $\pi_1(M_0, q)$ rather than cohomology groups, then $o(\text{identity}, \text{identity}, \sigma)$ can be defined just like in Case 1, because the only difficulties that were encountered there were due to the fact that k could be larger than 1. (k was the number of components of $g^{-1}(D)$ for some n -disk D .)

(b) Suppose f is arbitrary and $M = S^n$ or D^n . If $M = S^n$, then M_0 is contractible and it is easy to show that there is no obstruction at all to getting a σ -map $F: W \rightarrow S^n$ with $F(x) \neq f(x)$ for all $x \in W$ and all vector fields σ . If $M = D^n$, then there is again no difficulty in using the procedure of Case 1 to define an obstruction $o(f, g, \sigma) \in \pi_{n-1}(D^n - p) \approx Z$ or Z_2 with the desired properties.

DEFINITION. f is said to satisfy (**) if one of the following holds:

- (a) $n \geq 3$ and f satisfies (*).
- (b) $n = 2$ and f is the identity.
- (c) $M^n = S^n$ or D^n .

2. The obstruction $o(f)$. Suppose we are given a continuous map $f: (W^n, \partial W^n) \rightarrow (M^n, \partial M^n)$ so that f satisfies (**). Then we defined in §1 an obstruction $o(f, g, \sigma) \in H^n(W, \partial W; \pi_{n-1}(M_0, q))$ to finding a σ -map $F: W \rightarrow M$ with $F(x) \neq f(x)$ for all $x \in W$. Now we would like to show that in fact $o(f, g, \sigma)$ does not depend on σ or g . We retain the notation developed in §1.

Assume that $n \geq 2$. Observe that $H^n(W, \partial W; \pi_{n-1}(D_1 - p, q)) \approx \pi_{n-1}(D_1 - p, q) = Z$. Let $\eta = \text{degree}(h|\partial C_1) + \dots + \text{degree}(h|\partial C_k) \in H^n(W, \partial W; \pi_{n-1}(D_1 - p, q))$. (NOTE. $\text{degree}(h|\partial C_i) = \gamma^n(h|\partial C_i)$, which is the obstruction to extending $h|\partial C_i$ to a map from $C_i \rightarrow D_1 - \text{int}(D)$.) It follows from the definitions that if

$$\begin{aligned} \alpha: H^n(W, \partial W; Z) &= H^n(W, \partial W; \pi_{n-1}(D_1 - p, q)) \\ &\rightarrow H^n(W, \partial W; \pi_{n-1}(M_0, q)) \end{aligned}$$

is induced by the coefficient homomorphism $\pi_{n-1}(D_1 - p, q) \rightarrow \pi_{n-1}(M_0, q)$, then $\alpha(\eta) = o(f, g, \sigma)$. Let $\chi(M)$ denote the Euler characteristic of M and set $d = \text{degree}(f) = \text{degree}(f|B: (B, \partial B) \rightarrow (M, M_0))$. Then

it is known that $\eta = d \cdot (\text{index of } \sigma) = d \cdot \chi(M)$, where the index of σ is the sum of the indices at the zeros of σ . (See Milnor [8]; in particular, the Poincaré-Hopf theorem.) This shows that η , and hence $o(f, g, \sigma)$, is independent of σ and g .

If $n = 1$, then $M = S^1$ or D^1 and it is trivial to see that $o(f, g, \sigma)$ is again independent of σ and g .

DEFINITION. $o(f) \equiv o(f, g, \sigma)$ for some σ and g . Let $o(M) \equiv o(\text{identity}: M \rightarrow M)$.

We can now state the first theorem of this paper.

THEOREM 1. *Suppose that $f: (W^n, \partial W^n) \rightarrow (M^n, \partial M^n)$ is a continuous map and that f satisfies (**). Then there is a well defined obstruction $o(f) \in H^n(W, \partial W; \pi_{n-1}(M_0, q))$ with the following properties:*

(a) $o(f) = 0$ if and only if there is a vector field σ on M and a σ -map $F: W \rightarrow M$ with $F(x) \neq f(x)$ for all $x \in W$.

(b) $\alpha(\text{degree}(f) \cdot \chi(M)) = o(f)$.

Proof. Theorem 1 is a consequence of Lemma 3 and the remarks made at the beginning of this section.

Observe that although $F|W_0$ is homotopic to $f|W_0$, F need not be homotopic to f . For example, if $W = M = S^n$ and $f = \text{identity}$, then F is homotopic to the antipodal map and therefore has degree $(-1)^{n+1}$.

COROLLARY 1. *Let $f: (W, \partial W) \rightarrow (M, \partial M)$. Then there is a map $F: W \rightarrow M$ such that $F(x) \neq f(x)$ for $x \in W$ if any one of the following conditions is satisfied:*

(a) $\chi(M) = 0$.

(b) $\pi_{n-1}(M_0, q) = 0$.

(c) $\text{degree}(f) = 0$.

In particular, every homotopy sphere admits a fixed point free map.

Proof. If f satisfies (**), then the corollary follows from Theorem 1. If f does not satisfy (**), then one has to go back and look at the various constructions in the proof of Theorem 1 to see that the corollary is still true.

NOTE. (a) and (c) in the corollary follow also from the work of [1], and [3] in the case where M^n is closed, 1-connected, and $n \geq 3$.

The condition $\pi_{n-1}(M_0, q) = 0$ is of course very strong. Unfortunately, the map α above is often one-to-one so that $o(M)$ gives us no more information than $\chi(M)$ in that case. However, the next corollary

is useful for certain spherical manifolds. (See [6] for a definition of "spherical.")

COROLLARY 2. *Let M^n be a closed 1-connected manifold and $f: M \rightarrow M$. Suppose that all the nonzero elements of $\pi_{n-1}(M_0, q)$ have order 2. Then there is a map $F: M \rightarrow M$ such that $F(x) \neq f(x)$ for $x \in M$. In particular, M admits a fixed point free map.*

Proof. The conditions on M imply that M is spherical and that $\chi(M) = 2$ or 0. The case $\chi(M) = 0$ is clear. If $\chi(M) = 2$, then $o(f) = \alpha(\text{degree}(f) \cdot 2) = 0$, and we can apply Theorem 1.

Next, let $f: (M^n, \partial M^n) \rightarrow (W^n, \partial W^n)$ be a continuous map which preserves the tangent bundles. It is well known that the Euler characteristic behaves naturally with respect to such maps, i.e., $\chi(M) = \text{degree}(f) \cdot \chi(W)$. We want to show now that the obstructions defined in this paper satisfy an analogous formula.

Approximate f by a differentiable map $g: (M, \partial M) \rightarrow (W, \partial W)$. If σ is a vector field on W , then g will induce a vector field $g^*\sigma$ on M . We can also find a closed n -disk $D \subseteq \text{int} W$ so that $g^{-1}(D) = C_1 \cup \dots \cup C_k$, where C_i is a closed n -disk in the interior of M and $g|C_i: C_i \rightarrow D$ is a diffeomorphism. Assume that the zeros of σ are contained in $\text{int}(D)$. Then all the zeros of $g^*\sigma$ will lie in $g^{-1}(D)$. In fact, for each zero of σ , there will be k zeros of $g^*\sigma$ with the same index.

Let p and z be the base points of M and W , respectively, and let $n \geq 2$. By connecting the C_i with little tubes we can find a closed n -disk B such that $C_1 \cup \dots \cup C_k \subseteq B \subseteq \text{int} M$ and $p \in B$. If we assume that M_0 and W_0 are $(n-1)$ -simple, then we can ignore base points. Therefore, g , and hence f , induces a well defined homomorphism $\beta_f: \pi_{n-1}M_0 \rightarrow \pi_{n-1}W_0$ because $M - \text{int}(B)$ is deformation retract of M_0 . If we now recall our definitions of $o(M)$ and $o(W)$, then the next theorem follows easily:

THEOREM 2. *Let $f: (M^n, \partial M^n) \rightarrow (W^n, \partial W^n)$, $n \geq 2$, be a continuous tangential map. Assume that M_0 and W_0 are $(n-1)$ -simple. Then $\beta_f(o(M)) = \text{degree}(f) \cdot o(W)$, where $\beta_f: \pi_{n-1}M_0 \rightarrow \pi_{n-1}W_0$ is the map induced by f as above.*

Theorem 2 enables us to compute $o(W)$ once we know $o(M)$ and we have a tangential map of degree 1.

COROLLARY 3. *If M_0 is $(n-1)$ -simple, then $o(M)$ is an invariant of the tangential homotopy type of $(M, \partial M)$. (Actually, all we need is a tangential homotopy equivalence $f: M \rightarrow W$ such that $f(\partial M) \subseteq \partial W$*

and $\deg f = 1$.)

3. Relationships to other obstructions. In this section we would like to discuss the relationship between the obstruction $o(f)$ and the ones defined in [1] and [3]. We shall indicate how one might extend Theorem 1 and then make a concluding remark.

Suppose that M^n , $n \geq 3$, is a 1-connected closed manifold and $f: M \rightarrow M$. It follows from [1] and [3] that $\eta = \text{degree}(f) \cdot \chi(M) \in H^n(M; \pi_{n-1}(D_1 - p, q)) = H^n(M; Z) = Z$ is the obstruction to finding a map $F: M \rightarrow M$ such that F is homotopic to f and $F(x) \neq f(x)$ for all $x \in M$. This and Theorem 1(b) shows that $o(f)$ is a weaker obstruction than that in [1] and [3], since the latter maps onto the former under α . Note that the F in Theorem 1(a) will be homotopic to f if and only if $\text{degree}(f) \cdot \chi(M) = 0$.

Next, if we drop our hypothesis of differentiability and use the techniques of [1], [2], and [3] instead, it is most probable that the following extension of Theorem 1 is true:

CONJECTURE. Let M^n , $n \geq 3$, be a 1-connected closed topological manifold. Suppose that K is a finite n -complex and that $f: K \rightarrow M^n$ is a continuous map. Then there is a well defined obstruction $o(f) \in H^n(K; \pi_{n-1}(M_0, q))$ such that $o(f) = 0$ if and only if there exists a map $F: K \rightarrow M$ with $F(x) \neq f(x)$ for all $x \in K$ and F homotopic to f on the $(n-1)$ -skeleton of K .

Finally, this paper suggests that the next step in the program for determining when a manifold admits a fixed point free map is to allow maps to have even more freedom and require that they be homotopic (via a vector field) only over the $(n-2)$ -skeleton, then only over the $(n-3)$ -skeleton, and so on. This process may lead to a nice well defined sequence of obstructions which might be computable in certain cases.

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