PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 1, 1973

WHEN ARE PROPER CYCLICS INJECTIVE?

CARL FAITH

A right R-module C is *proper cyclic* in case $R \to C \to 0$ is exact but $0 \to R \to C \to 0$ is not. A ring R is a *ring PCI* ring if every proper cyclic right R-module is injective ("$PC \Rightarrow I$ ring"). The paper is devoted to the proof of the

THEOREM. *A right PCI ring is either semisimple (artinian) or a right semihereditary simple right Ore domain.* **When R is assumed to be right noetherian, this is a theorem of A. Boyle (Rutgers Ph. D. Thesis 1971). When R is assumed to be right selfinjective, this is a theorem of B. Osofsky, (Rutgers Ph. D. thesis, 1964) and the proof uses this. Aside from the theorems of Boyle and Osofsky, interest in right PCI rings stems from Cozzens's examples (Rutgers Ph. D. thesis, 1969) of right and left principal ideal PCI domains. Boyle's theorems show that noetherian right and left PCI rings are hereditary. The above theorem shows that in any case PCI rings are semihereditary. How close they come to being hereditary (resp. principal ideal) rings is an open question. A counter-example R would have to have an injective cyclic module R/I with infinite socle! However, if we assume that R is a free right ideal ring (fir) then R must be a principal right ideal ring. In any case, we show for any right PCI ring that any finitely generated right ideal is generated by two elements. Other resemblances to Dedekind domains are noted.**

Introduction. A "good" theory of ring theory is one which generalizes the Wedderburn-Artin theorems for a ring R with radical N, and a "good" theory of modules is one which generalizes the basis theorem for abelian groups. In this paper we are concerned with the latter, in the setting of Dedekind domains, particularly the aspect which states that every finitely generated module M is a direct sum of finitely many ideals and cyclic modules. A "conceptual" way to effect such a decomposition of M is to notice that M modulo its torsion submodule $t(M)$ is a torsionfree module F embeddable in a free module P. Then, the canonical projections of $P \to R$ induce projections of F into R which furnish the requisite ideal summands. Moreover, $M/t(M)$ being projective, the torsion module $t(M)$ splits in M, and the cyclic decomposition for $t(M)$ is a consequence of the fact that R modulo any ideal $A \neq 0$ is an Artinian principal ideal ring, hence a uniserial ring over which every module is a direct sum of cyclics (as G. Köthe [22] showed).

Another way to obtain the cyclic decomposition is to observe that every proper cyclic module R/A is an injective R/A-module. Thus, every cyclic submodule "of highest order" splits off!

We adopt this point of view in this paper. However, instead of requiring that every proper right cyclic module R/A be injective modulo its annihilator ideal, we make the stronger hypothesis:

(*right PCI*) *Every proper cyclic right R-module C is injective.*

Proper cyclic means that C is cyclic but $C \not\approx R$.

Our main theorem states that any right PCI ring is either semisimple, or a semihereditary simple right Ore domain. The first main simplification in the proof is Proposition 2 which states that R must be either a von Neumann regular ring or a simple ring. The reductions to the case R is a domain are long, and not entirely satisfactory inasmuch as they are quite intricate. (In effect, we prove that if R is not a domain, then R is semisimple.)

Next, in order to prove that a right PCI domain is right Ore, we are forced to determine the structure of finitely generated modules over a (nonexistent) right PCI non Ore domain R with maximal quotient ring $\hat{R}$. (See Proposition 5.) Naturally, the properties of $\hat{R}$ are quite bizarre, to wit: $\hat{R}$ is a cyclic right R-module. Then, we show that $\hat{R} \approx R/I$ is such that I is finitely generated (and projective), that is, $\hat{R}$ is finitely presented! Together with left flatness of $\hat{R}$ over R (a consequence of R being semihereditary [the proof of which is the easiest part of the paper!]), this is enough to prove that $\hat{R} \otimes_R \hat{R}$ has zero singular submodule, consequently that $R \subsetneq \hat{R}$ is a ring epic. From this we get that every right $\hat{R}$-module M is injective iff injective as a right R-module. This is what we need, since then every cyclic R-module is injective, whence $\hat{R}$ is semisimple, in fact, a field so R must be right Ore.

Assuming R is left Ore, we show that any finitely generated right R-module M is isomorphic to a direct sum of right ideals, and cyclic modules, completely analogous to Dedekind domains. Assuming that R is right Noetherian, then R is left Ore iff R is left Noetherian. (In this case, Boyle has shown that R is also left PCI!)

Background. A ring A is *right neat* provided that the right singular ideal of A is zero. In this case, a theorem, essentially discovered by Johnson in [7], and explicitly given in Johnson and Wong [8], states that the injective hull $\hat{A}$ of A in mod-A is a right selfinjective regular ring containing A as a subring. Moreover, for any right ideal I of A, a unique injective hull $\hat{I}$ of I in mod-R is contained in $\hat{A}$. Furthermore, $\hat{I}$ is a right ideal of $\hat{A}$, generated by an idempotent.

If I is right ideal of a ring R, then a right ideal K which is maximal in the set of all right ideals Q such that $K \cap Q = 0$ is called a *complement* right ideal, and K is said to be a *relative complement* of I. In this case, $I + K$ is an essential right ideal of R. Moreover, then, I is a relative complement of K if and only if I is a complement right ideal. In this case, I and K are said to be relative complement right ideals.

For any ring A, and nonempty subset X, $X^{\perp}$ denotes the annihilator right ideal, and ${}^{\perp}X$ denotes the left annihilator. An annihilator right (left) ideal is called a right (left) annulet.

An element $x \in R$ is *right regular* if $xa = 0 \Rightarrow a = 0 \ \forall\, a \in R$. In this case $R \approx xR$ under $r \mapsto xr$.

1. LEMMA. *If I is a right ideal of R such that R/I is injective, then no ideal of R contained in I contains a right regular element.*

Proof. An injective module E is divisible by any right regular element x. If $E = R/I$, then E is annihilated by any ideal of R contained in I.

A ring R is a *right V-ring* if every simple right R-module is injective, or equivalently, provided that every right ideal is an intersection of maximal right ideals. Trivially, any right *PCI* ring is right V (see Boyle [2] for a partial converse for hereditary rings).

Let $\hat{M}$ denote the injective hull of any right A-module M, for any ring A.

PROPOSITION 2. *Let A be a right PCI ring. Then, either A is a regular ring, or else A is a simple ring. Moreover, A is a right V-ring, and every indecomposable injective is either simple or isomorphic to $\hat{A}$.*

Proof. By Brown and McCoy [3], A has a maximal regular ideal M, and A/M has no regular ideals $\neq 0$ when $A \neq M$. If I is any ideal $\neq 0$, then every cyclic module C of A/I is a proper cyclic module over A, hence C is an injective module over A, and therefore injective over A/I. Then, a theorem of Osofsky [10] implies that A/I is semisimple artinian. Hence, if A is not regular, then $M = 0$. Since every simple right module is injective, A is a right V-ring, and so by 1, a nonzero ideal B contains no right ideal isomorphic to A. Then, for any $y \neq 0$ in B, yB is injective, hence a summand of B generated by an idempotent. This implies that B is a regular ideal, contrary to the assumption that $M = 0$. Thus, A must be simple when A is not regular. If E is an indecomposable injective right A-module, then $E = \hat{C}$ for any nonzero cyclic submodule C. Thus, if $C \approx A$, then $E \approx \hat{A}$, and if $C \not\approx A$, then C is injective, whence $C = E$ is simple.

THEOREM 3. *A right PCI ring A is right neat. Moreover, A*

has a nonsimple indecomposable injective right module E if and only if A is a right Ore domain, not a field. In this case, A is a simple ring, and right semihereditary.

Proof. Simple rings and regular rings are right neat, so the first sentence follows from 2. Since, by the remarks preceding 1, $\hat{A}$ is a regular ring, then $\hat{A}$ is indecomposable as a right A-module if and only if $\hat{A}$ is a field. In this case, A is an integral domain, and $\hat{A}$ is the right quotient field of A. If A_A is not simple, then $A \neq \hat{A}$. This proves one part of the second sentence, and the converse part is trivial. Then A is either regular or simple by 2, and the former can hold only if A is a field. Thus, A is simple in either case. Moreover, A is right semihereditary by the next proposition (5).

We require a lemma.

LEMMA 4. *If R is a right neat ring, and if I is a right ideal of R, then I is a complement right ideal of R if and only if R/I embeds in $\hat{R}$.*

Proof. The complement right ideals of $\hat{R}$ are those of the form $e\hat{R}$, for some idempotent $e \in R$, and those of R are of the form $e\hat{R} \cap R$. Thus, if $U = e\hat{R} \cap R$, then the isomorphism

$$R/e\hat{R} \cap R = R/U \approx eR$$

embeds R/U into $eR \subseteqq \hat{R}$. Conversely, if $f\colon R/U \rightarrow \hat{R}$, is an embedding, and if $f(1 + U) = y$, then $U = y^{\perp} \cap R$. Write $Ry = R(1 - e)$, for some idempotent e. Then $y^{\perp} = e\hat{R}$, and then $U = e\hat{R} \cap R = eR \cap R$.

We say M is *σ-cyclic* if it is a direct sum of finitely many cyclic submodules. A module M is *torsion* provided that every element of M is annihilated by an essential right ideal. M is *torsion-free* if 0 is the largest torsion submodule.

5. PROPOSITION. *A right PCI domain R is a right semihereditary, simple domain. Moreover, if R is not right Ore, then:*

5.1. *$\hat{R}$ is a finitely presented cyclic module.*

5.2. *If I is any right ideal, then $R/I \approx \hat{R}$ iff I is a complement right ideal $\neq 0, R$. Furthermore, any complement right ideal I is finitely generated* (*by* 3 *elements*).

5.3. *Any finitely generated torsion right module M is injective, and σ-cyclic.*

5.4. *If I is a right ideal, then there is a unique least complement $\bar{I}$ containing I and $\bar{I}$ is the unique complement containing I as an* *essential submodule. Moreover any finitely generated submodule of* $\bar{I}/I$ *is cyclic and injective.*

5.5. *If* $x \in \hat{R}$, *and* $x \notin R$, *then* $xR = \hat{R}$ *iff* $xa = 1$ *for some* $a \in R$. *Moreover, in this case,* $J_a = x^{\perp} + aR$ *is a finitely generated essential right ideal, and* $R/J_a \approx \hat{R}/R$, *where* $x^{\perp} = ann_R x$.

Proof. Let P be any nonzero projective right ideal, and let $a \in R$. If $C = aR + P$ is such that $C/P \approx R$, then there exists $x \in R$ such that $x^{\perp} = \{r \in R \mid ar \in P\}$. Since R is a domain, this would imply that $aR \cap P = 0$, and then $C = aR \oplus P$ is projective. If $C/P \not\approx R$, then C/P is injective, hence a summand of R/P. Let D be a right ideal of R containing P such that D/P is a summand of R/P complementary to C/P. Then there is an exact sequence

$$0 \longrightarrow P \longrightarrow C \oplus D \longrightarrow R \longrightarrow 0 ,$$

since $R = C + D$, and $P = C \cap D$. Thus, $C \oplus D \approx R \oplus P$ is projective, and hence C (also D) is projective. This implies that any finitely generated right ideal of R is projective. An obvious induction on the number of elements required to generate a right ideal shows that any finitely generated right ideal of R is projective, so R is right semihereditary. Moreover, by Proposition 2, R is simple.

Proof of 5.1. If I is any complement right ideal $\neq 0$, and if K is the relative complement of I then K embeds canonically as an essential submodule of R/I. Since R/I is injective, then $R/I \approx \hat{K}$, so we must show that $\hat{K} \approx \hat{R}$. Now K contains a right ideal $aR \approx R$, and hence we have

$$\hat{R} \supseteqq \hat{K} \supseteqq a\hat{R} \approx \hat{R} ,$$

so by a theorem of Bumby and Osofsky (see Bumby [65]) we have $\hat{K} \approx \hat{R}$ as desired.[1] This will prove 5.1, once we complete the proof of 5.2 showing that I is finitely generated.

Proof of 5.2. Let aR be a non-zero cyclic right ideal which is not essential. Then R/aR is an injective module containing a submodule $X \approx \hat{R}$. Write $R/aR = Y/aR \oplus X$, for some module $Y \supseteqq aR$. Then, $R/Y \approx X \approx \hat{R}$, so that Y is a complement right ideal by 4. Furthermore, Y/aR is cyclic, so that $Y = aR + bR$, for some $b \in R$, is finitely generated, hence projective. Therefore, $\hat{R}$ is finitely presented. If I is any complement right ideal $\neq 0$, $\neq R$, then $R/I \approx \hat{R}$,

[1] This short argument suggested by B. L. Osofsky replaces an earlier laborious one. The same argument shows, for any domain R, that any nonzero finitely generated right ideal of $\hat{R}$ is a free right $\hat{R}$-module $\approx \hat{R}$. In particular, either R is right Ore, or else $\hat{R} \approx \hat{R}^n$, for every integer $n > 0$.

by 5.1. Then Schanuel's lemma (see Bass [68]) implies that $R \oplus Y \approx R \oplus I$, so I is generated by 3 elements. Conversely, if I is a right ideal such that $R/I \approx \hat{R}$, then I is a complement by Lemma 4. This proves 5.2.

Proof of 5.3. If $M = a_1R + \cdots + a_nR$ is finitely generated and torsion, then a_1R is injective, so $M = a_1R \oplus M_1$ for some submodule M_1. Now M_1 is generated by $a_2', \cdots, a_n'$, where b' is the image of any $b \in M$ under the canonical projection $M \to M_1$. By induction, M_1, hence M, is a direct sum of finitely many injective cyclic modules. Therefore, M is injective.

Proof of 5.4. A theorem of R. E. Johnson (see [17]) shows that in any right neat ring R any right ideal has just one injective hull $\hat{I}$ (= maximal essential extension) contained in $\hat{R}$. Then $\bar{I} = \hat{I} \cap R$ is the unique complement containing I as an essential submodule. Moreover, since any least complement containing I is essential over I, then $\bar{I}$ is the unique least complement containing I. Any finitely generated submodule of $\bar{I}/I$ is injective by 5.3, hence a summand of R/I, and therefore cyclic.

Proof of 5.5. If $xR = \hat{R}$, then $\exists a \in R$ with $xa = 1$. Conversely, if $xa = 1$ with $x \in \hat{R}$, $a \in R$, $x \notin R$, then $x^{\perp} \neq 0$ since $x(1 - ax) = 0$. Thus, $xR \approx R/x^{\perp}$ is a proper cyclic, therefore injective. But $xR \supseteqq R$ and therefore $xR = \hat{R}$.

Let $J = \{r \in R \mid xr \in R\}$. Clearly, $xJ = R$, and there is a (canonical) isomorphism $xR/xJ \approx R/J$ (sending $[xr + xJ] \mapsto r + J \forall r \in R$). Since $xR = \hat{R}$, we have the desired isomorphism $\hat{R}/R \approx R/J_a$ provided we show that J coincides with $J_a = x^{\perp} + aR$. Clearly $J \supseteqq x^{\perp} + aR$. Moreover, projectivity of $xJ = R$ yields a splitting of the canonical exact sequence

$$0 \longrightarrow x^{\perp} \longrightarrow J \longrightarrow xJ \longrightarrow 0$$

so $x^{\perp}$ is a summand of J. Moreover, if $y \in J$, then

$$y = (y - axy) + axy$$

and $y - axy \in x^{\perp}$, $axy \in aR$ which shows that $J = x^{\perp} + aR = J_a$.

Concerning 5.6, we have a theorem reminiscent of the situation for hereditary noetherian prime rings and Dedekind rings. (cf. Kaplansky [21], Levy [23]).

COROLLARY 5.6. *If R is a right PCI, right Ore domain, then every finitely generated right ideal is projective and generated by two* *elements. Furthermore, assuming that R is left Ore, then any finitely generated module M is a direct sum of a finite number of right ideals of R, and cyclic injective modules.*

Proof. For any right Ore domain, and right module M, the set $t(M)$ of torsion elements is the submodule consisting of all elements annihilated by a nonzero scalar. Moreover, over a right Ore domain, any finitely generated torsion free module F is embeddable in a free module of finite rank if and only if R is left Ore (Gentile [60]). Thus, since R is semihereditary by 5, $F = M/t(M)$ is projective, so the canonical map $M \to F$ splits. Thus, $M \approx F \oplus t(M)$. Moreover, F is isomorphic to a direct sum of right ideals ([26], Chapter 1.) Finally, the proof of 5.3 suffices to establish that $t(M)$ is σ-cyclic, and injective. In particular, if I is any finitely generated right ideal, and if a is any nonzero element of I, then I/aR, being finitely generated torsion, is injective, and, being a summand of R/aR, therefore is cyclic. Thus, $I = bR + aR$ for some $b \in R$.

The proof of 5.2 has the corollary:

COROLLARY 6. *If R is a right neat ring, and if I is a right ideal such that $\hat{R}$ embeds in R/I, then there exists an element $b \in R$ such that $Y = bR + I$ is a complement right ideal of R.*

The next several results are preparation for the main results stated in Propositions 12, 13, and 14.

LEMMA 7. *A right ideal A of R is either semisimple, or else A contains an essential right ideal $A' \neq A$. Moreover, A' is not a complement right ideal of R.*

Proof. A module is semisimple iff it has no essential proper submodules. Thus, if A is not semisimple, there is an essential submodule $A' \neq A$. Let B be a relative complement of A' in R. If A' were a relative complement right ideal, then A' would be a relative complement of B, in which case $A \cap B \neq 0$, and $(A \cap B) \cap A' = 0$ contradicting the essentiality of A' in A.

PROPOSITION 8. *Let R be a ring PCI ring. Then, either R is a right Ore domain, or else there is an exact sequence $R^2 \to \hat{R} \to 0$, or else R has nonzero right socle.*

Proof. By 3, R is a right neat ring, with injective hull $\hat{R}$ which is a regular ring containing R as a subring. If every nonzero right ideal of R is essential, then $\hat{R}$ is a field, in which case R is a right Ore domain. Otherwise, there are right ideals I and K, neither of which are 0 or R, such that $I + K$ is essential, and $I \cap K = 0$. If I' and K' are essential submodules of I and K, respectively, then $I' + K'$ is an essential right ideal of R, and by the lemma below, the injective hull of $R/I' \oplus R/K'$ contains $\hat{R}$ as a summand. Now $R/I' \approx R$ would imply that I' is a right complement right ideal of R by 4. However, by 7, I' is a complement right ideal essential in I only if $I' = I$. Thus, by 7, either I or K is semisimple, or else $R/I' \oplus R/K'$ is an injective module containing $\hat{R}$ as a summand. In the latter case, there is an exact sequence $R^2 \rightarrow \hat{R} \rightarrow 0$, as asserted.

LEMMA 9. *Let I and K be two right ideals of a ring R such that $I \cap K = 0$, and $I + K$ is essential. Then, the injective hull of $R/I \oplus R/K$ contains $\hat{R}$ as a summand.*

Proof. Clearly, $R/I \oplus R/K$ contains the direct sum $I \oplus K$, which is essential in R, hence the injective hull contains the summand $\hat{R}$.

A right R-module E is *completely injective* provided that every factor module E/K is injective.

10. LEMMA. *If R is a right PCI regular ring, and if e is an idempotent such that eR is completely injective,* (e.g., *if e is contained in a nonzero ideal $S \neq R$*), *then eRe is a semisimple ring, and eR contains a minimal right ideal of R.*

Proof. First, the canonical map $eR \otimes_R Re \rightarrow eRe$ is an isomorphism. To see this, let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be finitely many elements of R such that $\sum_{i=1}^n ea_ib_ie = 0$. Then:

$$\sum_{i=1}^{n} ea_i \otimes b_ie = \left(\sum_{i=1}^{n} ea_ib_ie\right) \otimes e = 0 .$$

Henceforth, let $Q = eRe$, and let $M = Q/K$ be a cyclic right Q-module. Since Q is a regular ring along with R, then every Q-module is flat, and therefore the canonical exact sequence $0 \rightarrow K \rightarrow Q \rightarrow M \rightarrow 0$ implies exactitude of

$$0 \longrightarrow K \underset{Q}{\otimes} eR \longrightarrow Q \underset{Q}{\otimes} eR \longrightarrow M \underset{Q}{\otimes} eR \longrightarrow 0 .$$

Since $Q \otimes_Q eR \approx eR$ as a (Q, R) bimodule, this implies that $C = M \otimes_Q eR$ is an epimorphic image of eR in mod-R, and hence is injective. (If

e is contained in a nonzero ideal $S \neq R$, then $eR/L \approx R$, for some right ideal L, would imply via projectivity of R that eR contains a submodule $\approx R$, contrary to 1 which states that S contains no right ideals $\approx R$. Thus, eR is completely injective in this case.)

In order to show that M is injective in mod-Q, it suffices to show that M is isomorphic to a summand of any over-module. Since $C = M \otimes_Q eR$ is injective in mod-R, if Y is any right Q-module containing M, then the induced inclusion $C \to Y \otimes_Q eR$ splits. Write $Y \otimes_Q eR = C \otimes X$, for some right R-module S. Then, flatness of eR in mod-R, yields an isomorphism

$$\left(Y \underset{Q}{\otimes} eR\right) \underset{R}{\otimes} Re \approx \left(C \underset{R}{\otimes} Re\right) \oplus \left(X \underset{R}{\otimes} Re\right).$$

Since $eR \otimes_R Re \approx Q$ canonically, it follows that

$$C \underset{R}{\otimes} Re \approx \left(M \underset{Q}{\otimes} eR\right) \underset{R}{\otimes} Re \approx M \underset{Q}{\otimes} eR \underset{R}{\otimes} Re \approx M \underset{Q}{\otimes} Q \approx M .$$

Thus, M is isomorphic in mod-Q to a summand of $Y \approx (Y \otimes_Q eR) \otimes_R Re$ and so M, hence every cyclic right Q-module, is injective. Then, Q is semisimple. Let I be a minimal right ideal of Q. Since S is semisimple, $I = fQ$ for an idempotent $f \in Q$, and then $fQf = fRf$ is a field $\approx$ End fQ_Q. Since R is semiprime, this implies that fR is a minimal right ideal of R contained in eR (Jacobson [16, p. 65]).

Lemma 11. *If R is right PCI, and if S is any ideal $\neq 0, R$, then S contains any right ideal I for which there is an isomorphism $f\colon R/I \to R$. Moreover, $I = x^{\perp}$ for $x = f(1 + I)$, and $xy = 1$, for some $y \in R$.*

Proof. Let $f\colon R/I \to R$ be an isomorphism. Then, $I = x^{\perp}$, where $x = f([1 + I])$, and $xR = R$. Thus, $xy = 1$ for some $y \in R$. Since every cyclic right R/S-module is injective, then R is semisimple artinian by Osofsky's theorem. Therefore, every one-sided inverse in R/S is two-sided, and consequently, $yx = 1 - s$, for some $s \in S$. Thus, if $xa = 0$, then $a = sa \in S$. This proves that $I = x^{\perp}$ is contained in S.

Proposition 12. *If R is a right PCI ring, and if R is regular, then R is semisimple.*

Proof. Let S denote the intersection of all nonzero ideals of R. The proof falls into three cases:

(1) $S = 0$ but R is not simple.

By Lemma 11, if R/I is a cyclic right R-module isomorphic to R, then I is contained in every nonzero ideal of R. Thus, (1) implies that R/I is injective for every right ideal $I \neq 0$. If R has idempotents $e \neq 0, e \neq 1$, then $R/eR \approx (1-e)R$ is injective, and, similarly, eR is injective, in which case, $R = eR \oplus (1-e)R$ is injective. Thus, every cyclic right R-module is injective, so R is semisimple.

(2) $S \neq 0$ but R is not simple.

First, R/S is semisimple, since every cyclic R/S module is injective. Second, by Lemma 10, R has nonzero socle. The fact that $S = S^2$ is the least ideal of R implies that R is prime, and hence that S is the right socle of R. If V is any minimal right ideal of R, then V is injective, and hence $V\hat{R} = V$ is a right ideal of $\hat{R}$, and in fact, V is a simple right ideal of $\hat{R}$. Thus, right socle $\hat{R} \supseteq S$. Furthermore, if W is a minimal right ideal of $\hat{R}$, then $W \cap R = U$ is a minimal complement right ideal of R such that $W = U\hat{R}$. But every nonzero right ideal of R contains a minimal right ideal, and the latter is a minimal complement. Thus, U is a minimal right ideal of R, and, as indicated above, $U = U\hat{R}$. Hence $W = U$ is a minimal right ideal of R. This proves that $S =$ socle $\hat{R}$. Therefore, S is an ideal of $\hat{R}$. If the socle S is finite, then $S = R$ is semisimple, and the proof is complete. Otherwise, $S \approx V^{(a)}$ for some infinite cardinal a, and hence, there is a right ideal K contained in S such that $K \approx V^{(a)} \approx S \approx K^2$. Then, $R/K \approx R$ would imply that $K = eR$ for an idempotent $e \in R$, in which case K, whence S, is a finite sum of injective simple modules. But this case has just been discarded. Thus, R/K is an injective module containing a submodule $K_1 \approx S$, and hence containing the injective hull $\hat{R}$ of S. Since $\hat{R}$ is then a summand of cyclic module, then $\hat{R}$ is cyclic, and hence there exists $x \in \hat{R}$ such that $\hat{R} = xR$. Let $a \in R$ be such that $xa = 1$. Since S is an ideal of $\hat{R}$, if $c \in R$, and if $ac \in S$, then $c = xac \in S$. This proves that $[a + S]$ is not a left zero divisor in R/S. Since R/S is semisimple, this means that $[a + S]$ is a unit of R/S, with inverse, say $[a' + S]$. Then, $aa' = 1 - s$, for some $s \in S$, and

$$xaa' = x(1 - s) = a' .$$

Thus, $x = xs + a'$. Since S is an ideal of $\hat{R}$, then $x \in R$, and therefore, $\hat{R} = xR = R$. Since every cyclic module is therefore injective, R is semisimple in this case too.

(3) R is simple.

This case follows from the next proposition.

PROPOSITION 13. *A simple right PCI ring R is either semisimple, or else a semihereditary domain.*

Proof. *Case* (1). R contains a nonzero injective right ideal A.

Since R is simple, then A is a generator of mod-R, which then implies that $R = \hat{R}$. Then R is semisimple by Osofsky's theorem.

Case (2). If (1) does not hold, then, since every simple module is injective, then R has zero right socle. Then, by 8, either R is a right Ore domain, or else $R^2 \to \hat{R} \to 0$ is exact. In the former case, R is right semihereditary by 5. Hence, assume:

Case (3) $R^2 \to \hat{R} \to 0$ is exact. Since R has no injective right ideals $\neq 0$, every principal right ideal $aR \neq 0$ is isomorphic to R. Let $f: aR \to R$ be an isomorphism. Then, $f(a) = x$ is an element of R such that $xR = R$, and $x^{\perp} = a^{\perp}$. Let $y \in R$ be such that $xy = 1$. Then, $e = yx$ is idempotent. If $e \neq 1$, then $eR \approx R$ and $(1 - e)R \approx R$, hence $R \approx R^2$. This implies that $R \to \hat{R} \to 0$ is exact, and $R \to R \oplus \hat{R} \to 0$ is exact. Since $C = R \oplus \hat{R}$ is thereby cyclic, then C is either injective, whence R is injective, or else $C \approx R$, whence R contains an injective right ideal $A \approx \hat{R}$. But, in either case, R is semisimple. This proves that $e = yx = 1$, whence that $x^{\perp} = a^{\perp} = 0$, for every nonzero $a \in R$. Then R is an integral domain, which must be simple and right semihereditary by 5.

THEOREM 14. *A right PCI ring R is either semisimple, or a right semihereditary simple domain.*

Proof. By 2, R is either regular, whence semisimple by 12, or else R is simple, in which case the last proposition applies.

Propositions 15 and 16 which follow explore the possibility that a right *PCI* ring is right Noetherian, and are not needed in the proof of the main Theorem (17).

PROPOSITION 15A. *Let R be any ring, and let V be a simple injective right R-module which is not Σ-injective. Then, some cyclic right R-module R/J has essential socle $\approx V^{(\omega)}$, a direct sum of countably infinitely many V's.*

Proof. An injective module M is Σ-injective if and only if $M^{(\omega)}$ is injective (Faith [5]). Assuming that V is not Σ-injective, then $V^{(\omega)}$ has injective hull $E \neq V^{(\omega)}$. Since $V^{(\omega)}$ is essential in E, this implies that E is not semisimple, and hence there is a cyclic submodule xR which is not semisimple. Since E has essential socle, then xR has essential socle H. Since every finite direct sum of injective modules is injective, essentiality of H in xR implies that H does not have finite length. Since H is contained in $V^{(\omega)}$, this implies that $H \approx V^{(\omega)}$.

The proof has the corollary.

COROLLARY 15B. *If R is right PCI, and if R is not right noetherian, then there is a cyclic injective right R-module E with infinite essential socle.*

Proof. By 12, R must be a domain, not a field. By Kurshan's lemma [9], not every semisimple module is injective. Hence, there is a semisimple module M with $\hat{M}$ not semisimple, hence containing a cyclic submodule xR which is not semisimple. Then, xR has infinite socle, since any finite direct sum of simple modules is injective. Moreover, since R is a domain, socle $R = 0$, so $xR \not\approx R$, that is xR is injective.

PROPOSITION 16A. *Let R be a right PCI ring, let I be a finitely generated essential right ideal, and let $E = R/I$. Then, $B = \operatorname{End} E_R$ is a right selfinjective regular ring.*

Proof. It is clear that E is completely injective. If $b \in B$, then $C = bE$ is injective, hence a summand of E. Write $C = X/I$, for a right ideal X, and let Y/I be the complementary summand of E. Then Y/I is cyclic, so Y is finitely generated. This implies that $C \approx R/Y$ is finitely presented. If $\ker b = K/I$, then there is an exact sequence $0 \to K \to R \to C \to 0$, hence by Schanuel's lemma, K is finitely generated. But, by induction, every finitely generated submodule of E is injective. Thus, $\ker b$ is essential only if $b = 0$. By Utumi's theorem, this proves the lemma.

COROLLARY 16B. *Let R be a right PCI domain, with a unique simple right R-module V. Assume that R is not right noetherian, and let I be the right ideal given by* 15B *such that $E = R/I$ is injective with infinite essential socle. Then I is not finitely generated. Moreover, $E \approx E^2$.*

Proof. For, $E = R/I$ is not semisimple, hence there exists a maximal right ideal $M \supseteqq I$ such that M/I contains the socle of R/I. The socle S of $E \approx$ direct sum of copies of V, so there exists a nonzero map $R/M \to E$. Thus, M/I is an essential submodule of R/I, and is the kernel of an endomorphism f of E. Then, f is contained in the radical of $\operatorname{End} E_R$, so the proposition shows that I cannot be finitely generated. Since $E \approx \hat{S}$, and $S \approx S^2$, then $E \approx E^2$.

Then $B = \operatorname{End} E_R$ has nonzero radical J, and $B/J \approx \operatorname{End} S_R$ is a full right linear ring.

17. THEOREM. *A right PCI domain is right Ore.*

Proof. If R is not right Ore, then we may assume that $\hat{R} \approx R/I$ is a finitely presented cyclic module, by 5.1, that is, that I is finitely generated. Write

$$(1) \qquad 0 \longrightarrow I \longrightarrow R \longrightarrow \hat{R} \longrightarrow 0$$

exact. Now, any finitely generated left R-submodule M of $\hat{R}$ can be embedded in R (via a right multiplication by a nonzero element of R), and so $\hat{R}$ is a direct limit of projective modules whence $\hat{R}$ is a flat R-module (all of which is well-known; e.g. Sandomierski [24], or Cateforis [13]). Then, we have exactness of

$$(2) \qquad 0 \longrightarrow I \underset{R}{\otimes} \hat{R} \longrightarrow R \underset{R}{\otimes} \hat{R} \longrightarrow \hat{R} \underset{R}{\otimes} \hat{R} \longrightarrow 0 \,.$$

Since $\hat{R}$ is left flat, and I is finitely generated, then $I \otimes_R \hat{R} \approx I\hat{R}$ is a finitely generated right ideal of $\hat{R}$, which by the remark in Footnote 1, is isomorphic to $\hat{R}$, whence a summand of $\hat{R}$. This proves that $\hat{R} \otimes_R \hat{R}$ is isomorphic to a summand of $\hat{R}$, whence has zero singular submodule on the right. Now the kernel of the canonical map $\hat{R} \otimes_R \hat{R} \rightarrow \hat{R}$ is contained in the singular submodule, hence must $=0$. This proves that $R \subsetneq \hat{R}$ is a ring epic as asserted (see Silver [25]). This, together with left flatness of R, suffices to show that a right $\hat{R}$-module M is injective as a right $\hat{R}$-module if it is injective as a right R-module. In particular, if M is a cyclic right R-module, we conclude that M is injective, since M is a cyclic right R-module. Then, by Osofsky's theorem, $\hat{R}$ is semisimple, which can happen iff R is a field. This proves that R is right Ore.[2]

A ring R is a *right fir* provided that every right ideal of R is a free module, and every free right module has invariant basis number (*IBN*). Thus, an isomorphism $R^{(I)} \approx R^{(J)}$, for sets I and J, implies $|I| = |J|$.

18. PROPOSITION. *A right PCI domain R is a principal right ideal ring iff R is a right fir.*

Proof. The sufficiency is trivial. Conversely, assume that R is

[2] My original proof of this theorem was pointed out to be defective by Osofsky, and the present proof was made only 18 months later after numerous attempts by many people. I have to thank J. H. Cozzens for several stimulating conversations, and for pointing out the sufficiency of a fact I already knew, namely that $R \subsetneq \hat{R}$ is sepic. The proof of 17, and certain other material (e.g. 5.4 and the new proof of 5.1), have been added in proof.

a right fir. In order to prove that R is a principal right ideal ring, it is necessary and sufficient to prove that R is right Ore, but this is Theorem 17.

19. PROPOSITION. *If R is a right noetherian right PCI domain with right quotient field Q, then R is right hereditary and:*

19.1. *Q/R is semisimple*

19.2. *R embeds canonically in* End Q/R *under a map: $r \mapsto r'$, where r' sends $[x+R] \mapsto [rx+R]$ when $Q = R$.*

19.3. *For any right ideal I, the factor module R/I is a semisimple module of finite length.*

19.4. *Any finitely generated torsion module is semisimple of finite length.*

Proof. A right *PCI* domain R is right semihereditary by 5, and hence noetherian implies right hereditary.

Proof of 19.1. The module Q/R is a torsion module, and by 5.6, every finitely generated torsion module is injective (Cf. 5.3). Since R is right noetherian, this implies that for any finitely generated submodule M of Q/R, every submodule is a summand, and hence that M is semisimple. It readily follows that Q/R is semisimple.

Proof of 19.2. Obvious. (Note: since R is simple, and the map nonzero, then the map is an embedding.)

Proof of 19.3-4 are similar to that of 19.1 (cf. 5.3-4)

The propositition was first observed under conditions for the left and right by Cozzens [4] for rings of differential polynomials over universal fields (and certain twisted polynomial rings) and by Boyle [2] for right and left noetherian hereditary right *PCI* (or right *V*) rings. Assuming that R is also left Ore in 19 (as we do in the following Theorem 22), the increased generality is only apparent, since R is then left noetherian.

20. COROLLARY. *If R is a right PCI right noetherian left Ore domain, then every finitely generated right module M is isomorphic to a finite direct sum of right ideals and a semisimple module.*

Proof. Apply 19.4 to 5.6

Any right noetherian hereditary ring R is left semihereditary, and, moreover, R satisfies the restricted minimum condition on the left, namely for any finitely generated essential left ideal A, the module R/A is artinian. (Cf. Chatters [14].) A converse to this is the following. (For convenience, we switch sides.)

21. PROPOSITION (Cf. [18]). *If R is a left semihereditary left Ore domain, and if R satisfies the right restricted minimum condition, then R is left noetherian.*

We apply this in the following.

22. THEOREM. *A left Ore right noetherian right PCI ring is left noetherian.*

Proof. By 19.3, R satisfies the restricted right minimum condition, so R is left noetherian by 21.

We cite a theorem of Boyle [2].

Boyle's Theorem. *A right and left noetherian ring R is right PCI iff R is right hereditary right V-ring. Furthermore, in this case, R is left PCI.*

Thus, the ring in Theorem 22 is left *PCI*. The question remains, however, whether a right noetherian right *PCI* ring is left Ore. (Conjecture: affirmative.)

Cozzens has conjectured that an ultraproduct P of infinitely many copies of a noetherian V-domain (not a field) (Cozzens [4]) is a right *PCI* ring which is not noetherian. (Oral communication.) He has proved, however, that in any case P is a semihereditary simple domain. (The property of being a simple ring in a V-ring can be shown to be a first-order property; semihereditary is known to be first-order.)

Added in Proof. I also have the pleasure of thanking Carleton University for the opportunity to read this paper on "Algebra Day" October 14, 1972.

REFERENCES

1. H. Bass, *Algebraic K-theory*, Benjamin, New York and Amsterdam, 1968.
2. A. Boyle, Ph. D. thesis, Rutgers, The State University, New Brunswick, New Jersey, 1971.
3. B. Brown and N. McCoy, The maximal regular ideal of a ring, Proc. Amer. Math. Soc., **1** (1950), 165-171.
4. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc., **76** (1970), 75-79.
5. C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179-191.

6. N. Jacobson, *Structure of rings*, Revised Colloquium Publication, vol. 37, Amer. Math. Soc., Providence, 1964.
7. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc., **2** (1951), 891-895.
8. R. E. Johnson and E. T. Wong, *Self-injective rings*, Canad. Math. Bull., **2** (1959), 167-173.
9. R. P. Kurshan, *Rings whose cyclic modules have finitely generated socles*, J. Algebra, **15** (1970), 376-386.
10. B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math., **14** (1964), 646-650.
11. Y. Utumi, *On quotient rings*, Osaka Math. J. **8** (1956), 1-18.
12. R. Bumby, *Modules which are isomorphic to submodules of each other*, Arch. Math., **16** (1965), 184-5.
13. V. C. Cateforis, *Flat regular quotient rings*, Trans. Amer. Math. Soc., **138** (1969), 241-250.
14. A. W. Chatters, *The restricted minimum condition in Noetherian hereditary rings*, J. London Math. Soc., **4** (1971), 83-87.
15. P. M. Cohn, *Torsion modules over free ideal rings*, Prod. London Math. Soc., **17** (1967), 577-599.
16. ———, *Free rings and their relations*, Academic Press, New York and London, 1971.
17. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Mathematics, Springer-Verlag, New York Berlin, 1967.
18. ———, *Lecture Notes on a Theorem of Chatters*, preprint 1972.
19. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math Soc., **VIII** (1958), 589-608.
20. ———, *Semi-prime rings with maximum conditions*, Proc. London Math. Soc., **X** (1960), 201-220.
21. I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc., **72** (1952), 327-340.
22. G. Köthe, *Verallgemeinerte Abelsche Gruppen mit Hyperkomplexen operatorenring*, Math. Z., **39** (1935) 31-44.
23. L. Levy, *Torsion free and divisible modules over non-integral domains*, Can. J. Math., **15** (1963), 132-157.
24. F. L. Sandomierski, *Semisimple maximal quotient rings*, Trans. Amer. Math. Soc., **128** (1967), 112-120.
25. L. Silver, *Noncommutative localizations and applications*, J. Algebra, **7** (1967), 44-76.
26. H. Cartan and S. Eilenberg, *Homological Algecra*, Princeton University Press, Princeton, 1956.

Received July 20, 1971. Research on this paper was supported in part by a grant from the National Science Foundation. This paper was researched and written while the author was at the Institute for Advanced Study, Summer 1971. I take this opportunity to thank the faculty of the Institute for Advanced Study for visiting privileges granted Summer 1971, and the staff for much help during my tenure. The abstract of this paper indicates the obvious debt this work owes to the inspirations provided by my three named students.

RUTGERS, THE STATE UNIVERSITY