PACIFIC JOURNAL OF MATHEMATICS
Vol 45, No 2, 1973

INDUCED TOPOLOGIES FOR QUASIGROUPS AND LOOPS

KENNETH BACLAWSKI AND KENNETH M. KAPP

The concept of a [semi] topological quasigroup is defined and the notions of induced groupoids and isotopes are extended to the topological case. Necessary and sufficient conditions are found in order for a continuously induced isotope of a semitopological quasigroup to be a semitopological quasigroup.

Given an injection i of a topological space $(A, \mathscr{A})$ into a set S acted on by a group, G, a topology $\mathscr{T}_A$ on S is introduced in a natural fashion under which i is continuous. When $S=Q$ is itself a semitopological quasigroup and G is generated by the left or right translations of Q the continuity or openness of i can be checked by comparing the topology $\mathscr{T}_A$ with that of Q. In particular this method is applied in § 3 to the study of topologically invariant subloops.

1. Preliminaries; continuously induced isotopes. We recall (cf. [2]) that a *quasigroup* $(Q, \cdot)$ is a closed binary system i.e., a *groupoid*, where for any $q \in Q$ the left and right translations λ_q, ρ_q, both define bijections on Q. Thus given a quasigroup $(Q, \cdot)$ we can define (cf. [2], [7]) the so called *conjugate operations* $/$, $\backslash$ on the set Q by $c/b = a$ iff $ab = c$ iff $a\backslash c = b$ for $a, b, c \in Q$. In the conjugate quasigroups $(Q, /)$ and $(Q, \backslash)$ we will denote the left and right translations determined by $q \in Q$ by $\lambda_q^/$, $\rho_q^/$, $\lambda_q^\backslash$, and $\rho_q^\backslash$ respectively. (In general, ρ_a^* will denote the right translation determined by a relative to the binary operation*.) A *loop* is a quasigroup with a two sided identity. A conjugate operation need not be associative; viz. if $G = (Z, +)$, where Z is the set of integers then $c/b = c - b$.

We now make the following definition (cf. [4], [5]):

DEFINITION 1.1. 1. A *semitopological quasigroup* [*loop*] is a triple $(Q, \cdot, \mathscr{T})$ where $(Q, \cdot)$ is a quasigroup [loop] and $(Q, \mathscr{T})$ a Hausdorff space in which the left and right (multiplications) translations, λ_a, ρ_b, $a, b \in Q$ are homeomorphisms of Q onto itself. We will write $Q = (Q, \cdot; \mathscr{T})$ is a s.t.q. [s.t.*l*.] in such cases.

2. A triple $(Q, \cdot; \mathscr{T})$ is a *topological quasigroup* [*loop*] if Q is a quasigroup [loop] and multiplication in Q and each of its conjugates is continuous in both variables with respect to the same topology $\mathscr{T}$. We will write Q is a t.q. [t.*l*.] in such cases.

One makes the obvious genralizations of a [*semi*] *topological group* (s.t.g.) ([5], p. 28) to [*semi*] *topological semigroup* (s.t.s.) and *groupoid* (s.t.gd.). We observe that a s.t.q. is homogeneous and that a s.t.g.

is also a s.t.q.

It is easily verified that a topological quasigroup, Q, is a semitopological quasigroup; indeed each conjugate of Q is a topological quasigroup and hence also a semitopological quasigroup. Moreover a topological group is a topological quasigroup and conversely any topological quasigroup which is algebraically a group is a topological group. (For proof, one need only consider the appropriate translations in a conjugate quasigroup, e.g., $\lambda_a \lambda_a^\backslash = 1_Q$ while $gh^{-1} = g/h$ in a group G.)

We now extend the definitions of topism, etc., in [6] to involve the topological structure of given s.t.gd.'s:

DEFINITION 1.2. 1. Let $(A, \cdot; \mathscr{A})$ and $(B, *; \mathscr{B})$ be two s.t.gd.'s. A triple of maps $(\gamma; \alpha, \beta)$ from A into B will be called a *continuous topism* [*isotopy*] if γ, α, and β are continuous maps [and bijections] from A into B and $(x \cdot y)^\gamma = x^\alpha * y^\beta$ for all $x, y \in A$. We will say that $(\gamma; \alpha, \beta)$ is a *topological isotopy* if in addition γ, α, and β are homeomorphisms.

2. Let $(A, \cdot; \mathscr{A})$ be a s.t.gd. and μ, ν continuous maps of A into itself. The *continuously induced groupoid* $(A, \circ; \mathscr{A}) = (A, \cdot; \mu, \nu; \mathscr{A})$ of A is the groupoid in which $x \circ y = x^\mu y^\nu$ for each $x, y \in A$. If μ and ν are in addition bijections [homeomorphisms] we will say that $(A, \circ; \mathscr{A})$ is a *continuously* [*topologically*] *induced* (*principal*) *isotope* of $(A, \cdot; \mathscr{A})$.

It is easy to see that a continuously induced groupoid of a [semi] topological groupoid is itself a [semi] topological groupoid. Since it is well known that any isotope of a quasigroup is a quasigroup one might readily expect that a similar result is true for continuous isotopes of semitopological quasigroups. The following example shows that this need not be the case even when one starts out with a topological group.

EXAMPLE 1.3. Let $Q = (Q, +; \mathscr{T})$ where Q is the set of rational numbers and $\mathscr{T}$ the topology inherited from the usual (open interval) topology of the real line. Let $\beta = 1$, the identity map, and define α as follows:

(a) α is the identity on $Q \cap (\sqrt{2}, \infty)$

(b) α maps $Q \cap (-\infty, -\sqrt{2})$ continuously and strictly order preserving onto $Q \cap (-\infty, 0)$

(c) $(0)\alpha = 0$

(d) on $[Q \cap (-\sqrt{2}, \sqrt{2})] \backslash \{0\} = [\bigcup_{n=1}^\infty Q \cap (\sqrt{2}/(n+1), \sqrt{2}/n)] \cup [\bigcup_{n=1}^\infty Q \cap (-\sqrt{2}/n, -\sqrt{2}/(n+1))]$ by mapping

(i) $Q \cap (\sqrt{2}/(n+1), \sqrt{2}/n)$ continuously and strictly order preserving onto $Q \cap (\sqrt{2}/2n, \sqrt{2}/(2n-1))$

(ii) $Q \cap (-\sqrt{2}/n, -\sqrt{2}/(n+1))$ continuously and strictly order preserving onto $Q \cap \sqrt{2}/(2n+1), \sqrt{2}/2n)$.

(We observe that such continuous strictly order preserving maps do exist.)

Thus α maps alternatively the disjoint covering of $[Q \cap (-\sqrt{2}, \sqrt{2})] \backslash \{0\}$ by $\{L_n = Q \cap (-\sqrt{2}/n, -\sqrt{2}/(n+1)), R_m = Q \cap (\sqrt{2}/(m+1), \sqrt{2}/m)\}_{n,m=1}^{\infty}$ onto $Q \cap (0, \sqrt{2})$. One readily checks that α is a continuous bijection.

Now the continuously induced groupoid $(Q, \circ; \mathscr{T}) = (Q, +; \alpha, \beta; \mathscr{T})$, which is an algebraic quasigroup, is not a semitopological quasigroup since the right translation ρ_0° is not an open map in $(Q, \circ; \mathscr{T})$. We have for $a \in Q$ $a \circ O = a^\alpha + O^\beta = a^\alpha$ so that $[Q \cap (-\sqrt{2}, \sqrt{2})]\rho_0^\circ = Q \cap [0, \sqrt{2})$ which is not open in $(Q, \mathscr{T})$.

THEOREM 1.4. *Let $(Q, \circ, \mathscr{T}) = (Q, \cdot; \alpha, \beta; \mathscr{T})$ be a continuously induced isotope of a* s.t.q. *$(Q, \cdot; \mathscr{T})$. If there exist $b, c \in Q$ such that $\rho_b^\circ, \rho_c^\circ$ are homeomorphisms then $(Q, \circ; \mathscr{T})$ is a topologically induced isotope of $(Q, \cdot; \mathscr{T})$ and $(Q, \circ; \mathscr{T})$ is a* s.t.q.

Proof. We will show that α^{-1} is continuous and hence a homeomorphism. Let $b \in Q$ for which ρ_b° is a homeomorphism. Then for $x \in Q$: $(x)\rho_b^\circ = x \circ b = x^\alpha b^\beta = (x)\alpha\rho_{b^\beta}^{\bullet}$, hence $\alpha\rho_{b^\beta}^{\bullet} = \rho_b^\circ$. Thus $\alpha^{-1} = \rho_{b^\beta}^{\bullet}(\rho_b^\circ)^{-1}$ and the first claim is verified. Dually $\beta^{-1} = \lambda_{c^\alpha}^{\bullet}(\lambda_c^\circ)^{-1}$ where λ_c° is the given homeomorphism.

In a similar manner one calculates that $\rho_a^\circ = \alpha\rho_{a^\beta}^{\bullet}$ and $\lambda_a^\circ = \beta\lambda_{a^\alpha}^{\bullet}$ for any $a \in Q$. Since an isotope of a quasigroup is a quasigroup it now follows that $(Q, \circ; \mathscr{T})$ is s.t.q.

COROLLARY 1.5. *Let $(Q, \cdot; \mathscr{T})$ be a* s.t.q. *and $(Q, \circ; \mathscr{T}) = (Q, \cdot; \alpha, \beta; \mathscr{T})$ be a continuously induced isotope of Q. Then the following are equivalent*:

1. *$(Q, \circ; \mathscr{T})$ is a* s.t.q.
2. *there exists $b, c \in Q$ such that λ_c° and ρ_b° are homeomorphisms.*
3. *$(Q, \circ; \mathscr{T})$ is a topologically induced isotope of Q.*

Example (1.3) shows that if $(Q, \circ)$ is not a loop then $(Q, \circ; \mathscr{T})$ need not be s.t.q. However the following corollary is valid since the loop identity e determines the homeomorphisms $\lambda_e^\circ = \rho_e^\circ$.

COROLLARY 1.6. *If a continuously induced isotope of a* s.t.q. *is a loop, then it is a* s.t.*l*.

We comment in conclusion that many of the results of [6], §2 have topological analogues.

2. Translation topology; embeddings of semitopological quasigroups. The algebraic problem of when a binary system of one type can be embedded in a binary system of another type has in many cases been solved. Difficulties arise, however, when the algebraic systems have in addition a topological structure. Then there is the added problem of finding an embedding which is also a topological homeomorphism. Indeed, it is also often desirable to have the image open in the second space. Solutions are then dependent on the relationship between the topologies of the two structures. This problem, it will be seen, includes the more limited one of comparing with the given topology of the image space a topology induced by the given injection from the topology of the domain space.

DEFINITION 2.1. 1. An ordered 5-tuple $(i; A, \mathscr{A}; S, G)$ is called a *system for an induced topology* of S if

1. $(A, \mathscr{A})$ is a topological space,
2. G is a group acting on the nonempty set S and
3. i is an injection of A into S.

In such a case we will say that $(i; A, \mathscr{A}; S, G)$ is an s.i.t. for brevity.

2. Let $\mathscr{T}_A = \{B \subseteqq S \mid (Bg)i^{-1} \in \mathscr{A} \text{ for each } g \in G\}$. Here $(Bg)i^{-1} = \{a \in A \mid (a)i \in Bg\}$. $\mathscr{T}_A$ is called the *induced action topology* with respect to the s.i.t. $(i; A, \mathscr{A}; S, G)$.

One readily checks that $\mathscr{T}_A$ is a topology on S; it is not necessarily Hausdorff even when A is Hausdorff. If $A \subseteqq S$ and i is the inclusion map, we see that $\mathscr{T}_A = \{B \subseteqq S \mid Bg \cap A \in \mathscr{A} \text{ for each } g \in G\}$.

EXAMPLES 2.2. 1. Let $(A, \mathscr{A})$ be a discrete space. Then $\mathscr{T}_A$ for any s.i.t. is also discrete.

2. Let $(i; A, \mathscr{A}; S, \{e\})$ be a s.i.t. Then $\mathscr{T}_A = \{B \subseteqq S \mid Bi^{-1} \in \mathscr{A}\}$. Thus $\mathscr{T}_A$ contains the set of all subsets of $S \backslash (A)i$ as well as $\{(U)i \mid U \in \mathscr{A}\}$. Whence $(S, \mathscr{T}_A)$ is Hausdorff precisely when $(A, \mathscr{A})$ is Hausdorff.

3. Let $(i; R, \mathscr{R}; C, G)$ be a s.i.t. where $(R, \mathscr{R})$ is the real line with the usual topology, C the set of complex numbers, i the natural inclusion map of R into C and G the group of all bijections on C. One checks that $\mathscr{T}_R = \{B \subseteqq C \mid C \backslash B \text{ is finite or } B = \varnothing\}$. Clearly $\mathscr{T}_R$ is not a Hausdorff topology for C.

One sees immediately that when $(i; A, \mathscr{A}; S, G)$ is a s.i.t. then $i\colon (A, \mathscr{A}) \to (S, \mathscr{T}_A)$ is continuous and each $g \in G$ determines a homeomorphism of $(S, \mathscr{T}_A)$ onto itself. Thus if G acts transitively on S the space $(S, \mathscr{T}_A)$ is homogeneous. Moreover $\mathscr{T}_A$ is the strongest topology on S under which the map $i \circ g: A \to S$ is continuous for each $g \in G$.

Following Bruck ([2], p. 54) we make the following definition:

DEFINITION 2.3. Let $(Q, \cdot)$ be a quasigroup. We set $\mathscr{M}_\rho = \mathscr{M}_\rho(Q)$, $\mathscr{M}_\lambda = \mathscr{M}_\lambda(Q)$ and $\mathscr{M} = \mathscr{M}(Q)$, the groups, respectively, generated by the right, left, and both right and left translations of Q. Thus, e.g., $\mathscr{M}_\rho$ is generated by $\{\rho_g \mid g \in Q\}$.

It is clear that for a quasigroup $(Q, \cdot)$ the three groups of translations are transitive groups acting on Q. Thus if $(i; A, \mathscr{A}; Q, G)$ is a s.i.t. where $G = \mathscr{M}(Q)$ [$\mathscr{M}_\rho(Q)$ or $\mathscr{M}_\lambda(Q)$] we will speak of the [*right, left*] *translation topology,* $\mathscr{T}_A$, of Q.

THEOREM 2.4. *Let* $(Q, \cdot\,; \mathscr{T})$ *be a* s.t.q. *and* $(i; A, \mathscr{A}; Q, G)$ *a* s.i.t. *where* $G = \mathscr{M}(Q)$ [$\mathscr{M}_\rho(Q)$, $\mathscr{M}_\lambda(Q)$]. *Then*

1. *i is continuous iff* $\mathscr{T} \subseteqq \mathscr{T}_A$
2. *if i is open then* $\mathscr{T} \supseteqq \mathscr{T}_A$ *and*
3. *if i is an open embedding then* $\mathscr{T} = \mathscr{T}_A$.

Proof. If $\mathscr{T} \subseteqq \mathscr{T}_A$ then i is clearly continuous. Conversely, suppose i is continuous. Let $B \in \mathscr{T}$. Since for each $g \in Q$, ρ_g, λ_g are homeomorphisms, G is in each case a group of homeomorphisms on Q. Thus if $t \in G$, $(B)t = Bt \in \mathscr{T}$. Whence by the continuity of i, $(Bt)i^{-1} \in \mathscr{A}$ for each $t \in G$ and thus $B \in \mathscr{T}_A$, $\mathscr{T} \subseteqq \mathscr{T}_A$ and the proof of 1 is complete.

Let us now suppose that i is a $\mathscr{T}$-open map. Let $B \in \mathscr{T}_A$, $B \neq \varnothing$. Let $b \in B$. We will find a $\mathscr{T}$-open neighborhood of b contained in B. Suppose $G = \mathscr{M}_\rho(Q)$; the proofs for the other cases are analogous. Let $a \in (A)i$. There is a unique $q \in Q$ such that $bq = a$. Now $(a)i^{-1} \in (Bq)i^{-1} \in \mathscr{A}$ by definition of $\mathscr{T}_A$. Since i is open $a \in ((Bq)i^{-1})i = U = Bq \cap (A)i \in \mathscr{T}$. Since ρ_q is a homeomorphism in $(Q, \cdot\,; \mathscr{T})$ it follows that $b = (a)(\rho_q)^{-1} \in (U)(\rho_q)^{-1} = (Bq \cap (A)i)(\rho_q)^{-1} \subseteqq B$, and $(U)(\rho_q)^{-1}$ is the desired $\mathscr{T}$-open neighborhood. Whence $B \in \mathscr{T}$ and $\mathscr{T}_A \subseteqq \mathscr{T}$.

The third result is an immediate consequence of the first two and the proof is now complete.

From the results following (2.2) we now see that when $(i; A, \mathscr{A}; Q, G)$ is a s.i.t. where Q is a quasigroup and $G = \mathscr{M}(Q)$, if $(Q, \mathscr{T}_A)$ is a Hausdorff space then $(Q, \cdot\,; \mathscr{T}_A)$ is a s.t.q.

3. Semitopological loops.

We will now apply the concepts of action and translation topologies to the study of invariant subloops of a loop.

The first two parts of the following definitions can be found in Bruck ([2], p. 61).

DEFINITION 3.1. 1. Let $(Q, \cdot)$be a loop with identity e. The subgroup of $\mathscr{M}$ defined by $\mathscr{I} = \mathscr{I}(Q) = \{t \in \mathscr{M} \mid (e)t = e\}$ is called the *inner mapping group* of $(Q, \cdot)$ and we note that $\mathscr{I}$ is the stabilizer of e in $\mathscr{M}$.

2. Let $(Q, \cdot)$ be a loop and $(H, \cdot)$ a subloop of Q. We say that H is a *normal* (*invariant*) *subloop* of Q if $(H)\mathscr{I} \subseteqq H$ and we write $H \trianglelefteq Q$ in such cases.

3. Let $(H, \cdot; \mathscr{H})$ be a topological space and a subloop of $(Q, \cdot)$. We will call $H = (H, \cdot; \mathscr{H})$ a *topologically invariant subloop* of Q if for each $K \in \mathscr{H}$ and $j \in \mathscr{I}$, $Kj \in \mathscr{H}$. We write $H \trianglelefteq_t Q$ in such cases and note that if $H \trianglelefteq_t Q$ then $H \trianglelefteq Q$.

THEOREM 3.2. *Let $(H, \cdot; \mathscr{H})$ be a* s.t.*l. and suppose $H \trianglelefteq Q$ where $(Q, \cdot)$ is loop. The followings are then equivalent*:

1. $H \trianglelefteq_t Q$
2. $\mathscr{H} \subseteqq \mathscr{T}_H$ *where $\mathscr{T}_H$ is the action topology generated from the* s.i.t. $(i; H, \mathscr{H}; Q, \mathscr{M}(Q))$.
3. *There is a topology $\mathscr{T}$ for Q under which $(Q, \cdot; \mathscr{T})$ is a* s.t.*l. with $(H, \mathscr{H})$ as a subspace.*
4. *There is a topology $\mathscr{T}$ for Q under which $(Q, \cdot; \mathscr{T})$ is a* s.t.*l. with $(H, \mathscr{H})$ as an open subspace.*

Proof. Assume 1. $H \trianglelefteq_t Q$. ($\mathscr{M} = \mathscr{M}(Q)$) is defined in (2.3)). Let $K \in \mathscr{H}$ and $t \in \mathscr{M}$. We will show that $Kt \cap H \in \mathscr{H}$. We assume for the sake of argument that $Kt \cap H \neq \varnothing$. If $h \in Kt \cap H$ then $h = (k)t$ for some $k \in K$. It is easily checked that $\rho_k \circ t \circ (\rho_h)^{-1} \in \mathscr{I}$. Whence $(H)\rho_k \circ t \circ (\rho_h)^{-1} \subseteqq H$ and $(H\rho_k)t \subseteqq H\rho_h$. It follows, since H is a closed subloop that $(H)t \subseteqq H$. In particular $Kt \subseteqq H$ and $Kt \cap H = Kt$.

Now let $m = (e)t$. Then $t \circ \rho_m^{-1} \in \mathscr{I}$ so that $(K)t \circ \rho_m^{-1} = (Kt)\rho_m^{-1} \in \mathscr{H}$. But $Ht \subseteqq H$, a s.t.*l*., thus $m \in H$ and ρ_m is a homeomorphism on H. It now follows that $Kt \in \mathscr{H}$. Whence $K \in \mathscr{T}_H$ and $1 \rightarrow 2$. (We also note that if G is any subgroup of $\mathscr{M}$ then $\mathscr{H} \subseteqq \mathscr{T}'_H$ where $\mathscr{T}'_H$ is generated from the s.i.t. $(i; H, \mathscr{H}; Q, G)$.)

Assume 2 now holds, i.e., $\mathscr{H} \subseteqq \mathscr{T}_H$. Then $i: (H, \mathscr{H}) \rightarrow (Q, \mathscr{T}_H)$, which is continuous by Theorem (2.4), is an open embedding. Thus we need only show that $\mathscr{T}_H$ is a Hausdorff topology for Q and 4 will follow. Thus let $a, b \in Q$ with $a \neq b$. If $Ha \cap Hb = \varnothing$ we are done since $Ha, Hb \in \mathscr{T}_H$. But if $Ha \cap Hb \neq \varnothing$ then $Ha = Hb$ since $H \trianglelefteq Q$. Thus $a = hb$ for some $h \in H \backslash \{e\}$. Now $(H, \mathscr{H})$ is Hausdorff so there are disjoint sets A and B such that $e \in B \in \mathscr{H}$ and $h \in A \in \mathscr{H}$. It follows that $b \in Bb$ and $a \in Ab$. But $Ab \cap Bb = \varnothing$ and $Ab, Bb \in \mathscr{T}_H$ since ρ_b is a homeomorphism. Whence $(Q, \mathscr{T}_H)$ is Hausdorff and the proof of $2 \rightarrow 4$ is complete.

Clearly $4 \rightarrow 3$. We will now prove the implication $3 \rightarrow 1$.

Assume 3 holds and let $j \in \mathscr{J}(Q)$ and $K \in \mathscr{H}$. Since $H \trianglelefteq Q$ we have $Hj = H$. Now $K = H \cap U$ for some $U \in \mathscr{T}$. Thus $Kj = (H \cap U)j = Hj \cap Uj = H \cap Uj$. Since j is a homeomorphism of Q, $Uj \in \mathscr{T}$. Hence $Kj \in \mathscr{H}$ and the proof of this implication and the theorem is complete.

COROLLARY 3.3. *Let $(Q, \cdot\,; \mathscr{T})$ be a* s.t.*l. The component E, of the identity, e, with the restricted product and inherited topology is a topologically invariant subloop of Q.*

Proof. We need only show that $E \trianglelefteq Q$. Connectivity is preserved under continuous maps so that for each $t \in \mathscr{M}(Q)$, Et is connected. Thus either $Et \cap E = \varnothing$ or $Et \subseteqq E$. In the latter case $E \subseteqq Et^{-1}$ and since $t^{-1} \in \mathscr{M}(Q)$ it also follows that $Et^{-1} \subseteqq E$. Whence $E = Et = Et^{-1}$. Now for any $g, h \in E$ we have $g \in E \cap E\rho_g$ and $h \in E \cap E\lambda_h$. Thus $E = E\rho_g = E\rho_g^{-1} = E\lambda_h = E\lambda_h^{-1}$ and it follows that E is closed. In particular for any $j \in \mathscr{J}(Q)$, $e = (e)j$, $E \cap E \cap E_j \neq \varnothing$ so that $Ej = E$ and hence $E \leqq Q$. The result now follows from Theorem (3.2).

DEFINITION 3.4. Let $(H, \cdot\,; \mathscr{H})$ be a s.t.*l*. and a subloop of a loop $(Q, \cdot)$. Then $R_H = \{Bt \mid B \in \mathscr{H}, t \in \mathscr{M}_\rho(Q)\}$ and $L_H = \{Cs \mid C \in \mathscr{H}, s \in \mathscr{M}_\lambda(Q)\}$ are called, respectively, the *right* and *lest coset coverings of* Q with respect to $(H, \cdot\,; \mathscr{H})$.

THEOREM 3.5. *Let $H \trianglelefteq_t Q$ where $(H, \cdot\,; \mathscr{H})$ is a* s.t.*l. then $R_H = L_H$ is a subbase for a topology $\mathscr{R}_H = \mathscr{L}_H$ under which $(Q, \cdot)$ is a* s.t.*l. Moreover if i is the natural inclusion map of H into Q and $G = \mathscr{M}(Q)[\mathscr{M}_\rho(Q), \mathscr{M}_\lambda(Q)]$ then $\mathscr{T}_H = \mathscr{R}_H = \mathscr{L}_H$, where $\mathscr{T}_H$ is generated from the* s.i.t. *$(i; H, \mathscr{H}; Q, G)$, and i embeds $(H, \mathscr{H})$ as an open subspace of $(Q, \mathscr{T}_H)$.*

Proof. Let $B \in \mathscr{H}$ and $t \in \mathscr{M}_\rho(Q)$. Set $m = (e)t$. Then $t \circ \lambda_m^{-1} \in \mathscr{J}$ so that $(B)t \circ \lambda_m^{-1} \in \mathscr{H}$ since $H \trianglelefteq_t Q$. Thus $Bt = ((B)t \circ \lambda_m^{-1})\lambda_m \in L_H$ and $R_H \subseteqq L_H$. Similarly $L_H \subseteqq R_H$ and equality now follows. Clearly $\cup R_H = Q$ so that R_H is a subbase for a topology on Q.

We will show that $(Q, \mathscr{R}_H)$ is Hausdorff. Let $q_1 \neq q_2 \in Q$. Since $H \trianglelefteq Q$ (cf. [2], pp. 60, 61) either $Hq_1 = Hq_2$ or $Hq_1 \cap Hq_2 = \varnothing$. In the latter case, since $Hq_1, Hq_2 \in R_H$ and $e \in H$ we have two disjoint open neighborhoods in $\mathscr{R}_H$ of q_1 and q_2. Suppose now that $Hq_1 = Hq_2$. Then $q_1 = hq_2$ for some $h \in H \backslash \{e\}$, where e is the loop identity. Since $(H, \mathscr{H})$ is Hausdorff there are open disjoint neighborhoods, $A, B \in$

$\mathscr{H}$ of e and h respectively. Then $Aq_2 \cap Bq_2 = (A \cap B)q_2 = \varnothing$, $q_2 \in Aq_2$ and $q_1 \in Bq_2$. But $Aq_2, Bq_2 \in R_H$ and it follows that $(Q, \mathscr{R}_H)$ is Hausdorff.

Now let $t \in \mathscr{M}_\rho(Q)$. Then if $\bigcap^n B_i t_i$ is a basic open set of $\mathscr{R}_H$ we have $(\cap B_i t_i)t = \cap (B_i t_i)t = \cap B_i(t_i t)$ another basic open set of $\mathscr{R}_H$. Thus t is an open map. Since $t^{-1} \in \mathscr{M}_\rho$ it is also true that t is continuous. Whence t is a homeomorphism. Since all right translations determined by $q \in Q$ are in $\mathscr{M}_\rho$ it follows that each such ρ_q is a homeomorphism. In a similar fashion since $R_H = L_H$, each λ_q is a homeomorphism. Thus $(Q, \cdot\,; \mathscr{R}_H) = (Q, \cdot\,; \mathscr{L}_H)$ is a s.t.q.

We have already seen in the proof of Theorem (3.2) that $\mathscr{H} \subseteqq \mathscr{T}_H$ where $\mathscr{T}_H$ is generated from the s.i.t. $(i; H, \mathscr{H}; Q, G)$ and $G = \mathscr{M}[\mathscr{M}_\rho, \mathscr{M}_\lambda]$. Let $Bt \in R_H$ where $B \in \mathscr{H}$, $t \in \mathscr{M}_\rho$. Since t is a homeomorphism and $\mathscr{H} \subseteqq \mathscr{T}_H$ it follows that $Bt \in \mathscr{T}_H$ and thus $\mathscr{R}_H \subseteqq \mathscr{T}_H$. Conversely since the inclusion map $i: H \to Q$ is open for the s.t.l. $(Q, \cdot\,; \mathscr{R}_H)$ by Theorem (2.4.2) we have $\mathscr{R}_H \supseteqq \mathscr{T}_H$ and equality follows. The other assertions are now clear and the proof is complete.

The following corollary is now an immediate consequence of the above and Theorem (2.4).

COROLLARY 3.6. *Let $(H, \cdot\,; \mathscr{H})$ and $(Q, \cdot\,; \mathscr{T})$ be two* s.t.*l.*'s. *with $H \trianglelefteq_t Q$. If i is the natural inclusion map of H into Q then*

1. *i is continuous iff $\mathscr{T} \subseteqq \mathscr{R}_H$*
2. *i is open iff $\mathscr{T} \supseteqq \mathscr{R}_H$*
3. *i is an opening embedding iff $\mathscr{T} = \mathscr{R}_H$.*

We conclude with an example which illustrates both the existence of a normal subloop of a s.t.*l*. which is not topologically invariant. The example will show that necessity of the topological invariance in the hypothesis of several results viz. Theorem (3.2), and Theorem (3.5).

EXAMPLE 3.7. Let C be the set of complex numbers. For $m \in (Z_2, +)$ we define an action of Z_2 on C by $z^{\underline{m}} = \begin{cases} z \text{ if } m = 0 \\ \bar{z} \text{ if } m = 1 \end{cases}$ where $z \in C$ and $\bar{z}$ denotes the conjugate $x - yi = \overline{x + yi} = \bar{z}$ ($x, y \in R$, the reals). It is easy to check that this is an action and that $(z_1 + z_2)^{\underline{m}} = z_1^{\underline{m}} + z_2^{\underline{m}}$. We will consider R, the set of real numbers, with two different topologies: $\mathscr{R}$, the usual topology of the real line, and $\mathscr{S}$, the Sorgenfrey topology (generated by half open intervals). We can now define the following two topologies on C where we make the usual identification $x + yi \to (x, y)$. The usual topology generated from $\mathscr{R} \times \mathscr{R}$ will be denoted by $\mathscr{C}$ while we will denote by $\mathscr{C}'$ the cartesian product topology generated from $\mathscr{R} \times \mathscr{S}$. It is readily checked that

$(C, +; \mathscr{C})$ and $(C, +; \mathscr{C}')$ are both semitopological groups.

Let $P = C \times Z_2$. We construct a group on P by defining a binary operation as follows:

$$(w, m) + (z, n) = (w + z^{\underline{m}}, m + n) \quad \text{for} \quad (w, m), (z, n) \in P .$$

A straightforward check shows that $(P, +)$ is a group which contains the subgroup $C \times \{0\}$. One readily checks, for example, that $(z, m) + (-z^{\underline{m}}, m) = (0, 0)$ for any $(z, m) \in P$ and that $(0, 0)$ is the identity of P. We identify now C and $C \times \{0\}$ since they are isomorphic under the obvious natural map. It is easily seen that $[P: C] = 2$ and thus $C \trianglelefteq P$. Furthermore $(z, 0) + (0, 1) = (z, 1) \neq (\bar{z}, 1) = (0, 1) + (z, 0)$ so that P is not abelian. However $(\bar{z}, 0) = (0, 1) + (z, 0) + 0, 1) = (z, 0)i_{(0,1)}$. Thus, the inner automorphism determined by $(0, 1)$ when restricted to C is just conjugation on C. Now if we consider the semitopological loop $(C, +; \mathscr{C}')$ as a subloop of $(P, +)$ we see that $(C, +; \mathscr{C}')$ is a normal but not a topologically invariant subloop, since conjugation in $(C, \mathscr{C}')$ is not a continuous map.

Let $(i; C, \mathscr{C}'; P, \mathscr{M})$ be a s.i.t. We shall show that the topology inherited by $(C)i$ from $\mathscr{T}_C$ is exactly $\mathscr{C}$. It will then follow that i is not open or even an embedding so that the converses of (2.4.2) and (2.4.3) are false when we take $(P, +; \mathscr{T}_C)$, i.e., $\mathscr{T} = \mathscr{T}_C$, in the hypothesis. Now let $K \subseteqq P$. Since $P = C \cup [C + (0, 1)]$ we can decompose K into two subsets $K = K_1 \cup [K_2 + (0, 1)]$ where $K_i \subseteqq C$. Since $(P, +)$ is a group $\mathscr{M}(P) = \{\lambda_{q_1}\rho_{q_2} \mid q_i \in P\}$ and $K \in \mathscr{T}_C$ iff $[(w, m) + K + (z, n)] \cap C \in \mathscr{C}'$. But $(w, m) + K + (z, n) = (w + K_1^{\frac{m}{}} + z^{\underline{m}}, m + n) \cup (w + K_2^{\frac{m}{}} + z^{\underline{1 \pm m}}, 1 + m + n)$. Thus if we intersect with $C = C \times \{0\}$ we get either $w + z^{\underline{m}} + K_1^{\frac{m}{}}$ when $m = n$ i.e., if $m + n = 0$ or $w + z^{\underline{1 \pm m}} + K_2^{\frac{m}{}}$ when $m \neq n$ i.e., if $m + n = 1$. Hence $K \in \mathscr{T}_C$ iff $K_1, \bar{K}_1, K_2, \bar{K}_2 \in \mathscr{C}'$ (note: $(C, +; \mathscr{C}')$ is a s.t.g.!) It then follows that K_1 and $K_2 \in \mathscr{C}$. Whence a subset $K \subseteqq P$ is open iff $K = K_1 \cup [K_2 + (0, 1)]$ where $K_i \in \mathscr{C}$. Clearly $(P, \mathscr{T}_C)$ is Hausdorff and it follows that $(P, +; \mathscr{T}_C)$ is a s.t.g.

The inclusion map $i: (C, +; \mathscr{C}') \to (P, +; \mathscr{T}_C)$ is continuous but not open since the inherited topology of $(C)i$ is $\mathscr{C}$ and $\mathscr{C} \subsetneqq \mathscr{C}'$. Indeed, i is not even an embedding (cf. (2.4)) as we were to show. Theorem (3.2) is thus false if topologically is omitted from the hypothesis since here $\mathscr{C}' \nsubseteqq \mathscr{T}_C$.

We now remark that Theorem (3.5) does not hold in this example. If K is the upper half plane including the real axis then $K \in \mathscr{C}'$. However, if $K + (0, 1) = (w, m) + L$ for some $L \in \mathscr{C}'$ then $(-w^{\underline{m}}, m) + K + (0, 1) \in \mathscr{C}'$. Hence $m + 1 = 0$ i.e., $m = 1$ and we have, since $K = K \times \{0\}$, $(-\bar{w}, 1) + K + (0, 1) = -\bar{w} + \bar{K}$ which is not in $\mathscr{C}'$. Theorem (3.5) fails for in this example neither $\mathscr{R}_C$ nor $\mathscr{L}_C$ are topologies under which $(P, +)$ is a s.t.g.: the inner automorphism $i_{(0,1)}$ is not continuous.

REFERENCES

1. J. Aczel, *Quasigroups, Nets, and Nomograms*, Advances in Mathematics, **1** (1965), Academic Press, 383-450.

2. R. H. Bruck, *A Survey of Binary Systems*, Ergebnisse der Math. Neue Folge, Vol. **20** Springer, 1958.

3. K. H. Hofmann, *Topologische loops*, Math. Z., **70** (1958), 13-37.

4. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups*, Merrill Books, Inc., Columbus, Ohio, 1966.

5. T. Husain, *Introduction to Topological Groups*, W. B. Saunders Co., Philadelphia, Pa., 1966.

6. K. Kapp and K. Baclawski, *Topisms and induced non-associative systems*, Pacific J. Math., **36** (1971), 45-54.

7. S. K. Stein, *On the foundations of quasigroups*, Trans. Amer. Math. Soc., **85** (1957), 228-256.

Received August 30, 1971 and in revised form September 14, 1972.

HARVARD UNIVERSITY
AND
UNIVERSITY OF WISCONSIN-MILWAUKEE