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DIFFERENTIAL SIMPLICITY AND EXTENSIONS OF A DERIVATION

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Let R be an integral domain containing the rational numbers, K its quotient field and Ω an algebraic closure of K; let D be a derivation on R such that R is D-simple. The valuation rings V such that $R \subseteqq V \subseteqq \Omega$ on which D is regular are determined.

Introduction. Let R' be the complete integral closure of R in K. Seidenberg has shown that D is regular on R' [3]. We want here to continue his work and determine all the valuation rings V such that $R \subseteqq V \subseteqq \Omega$ on which D is regular.

First we determine in paragraph 2 the valuation rings of K that have property, and we show that they are in 1-1 correspondence with the proper prime ideals of R.

Then, in paragraph 4 we show that if V is a valuation ring such that $R \subseteqq V \subseteqq \Omega$, then D is regular on V if and only if V is unramified over K and D is regular on $V \cap K$. To do that, we have to show first in paragraph 3 that if B is a valuation ring of Ω such that $B \cap K$ is rank-1 discrete and contains the rational numbers, then its inertia field over K can be obtained as the intersection of a formal power series field with Ω.

1. Preliminaries. Let R be a commutative ring with identity. A derivation D of R is a map from R into R such that $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$. An ideal I of R is a D-ideal if $D(I) \subseteqq I$; R is D-simple if it has no D-ideal other than (0) and (1). If R is a D-simple ring of characteristic $p \neq 0$, R is a primary ring [2, Theorem 1.4], hence is equal to its total quotient ring; this case will not be of interest in our considerations.

Thus, let R be a D-simple ring of characteristic 0, which is then a domain containing the rational numbers [2, Corollary 1.5]; let K be its quotient field and Ω an algebraic closure of K. The derivation D can be uniquely extended to a derivation of Ω, which we also call D, and if N is any field between K and Ω, we have $D(N) \subseteqq N$ [6, Corollary 2', p. 125]. If S is a ring with quotient field N such that $D(S) \subseteqq S$, we shall say that D is regular on S, or that (N, S) is D-regular, or that D can be extended to S.

We note that if D is regular on a ring S and if M is a multiplicative system of S, then D is regular on S_M. We note also that if R is D-simple, and if S is a ring such that $R \subseteqq S \subseteqq \Omega$, then to say that

D is regular on S is equivalent to saying that S is D-simple, indeed:

PROPOSITION 1.1. *Let R be a D-simple ring with quotient field K; let Ω be an algebraic closure of K, and S a ring such that $R \subseteqq S \subseteqq \Omega$. If D is regular on S, then S is D-simple.*

Proof. It will be enough to show that if I is a nonzero ideal of S, then $I \cap R$ is a nonzero ideal of R. Let $0 \neq x \in I$, and let $X^n + k_1X^{n-1} + \cdots + k_n \in K[X]$ be its minimal polynomial over K where we note that $k_n \neq 0$; then, from the equality $x^n + k_1x^{n-1} + \cdots + k_n = 0$, we can get $r_0x^n + r_1x^{n-1} + \cdots + r_n = 0$ with $r_i \in R \subseteqq S$ for $i = 0, 1, \cdots, n$, and $r_n \neq 0$, so that we have $0 \neq -r_n = r_0x^n + r_1x^{n-1} + \cdots + r_{n-1}x \in I \cap R$.

Let L be a field, N an algebraic extension of L, and V a valuation ring of N. We shall denote the inertia degree of V over L by $f(V|L)$, and the ramification index of V over L by $e(V|L)$. If A is a valuation ring of L, following Endler's terminology in [1], we shall say that A is indecomposed in N if there is only one valuation ring of N lying over A, and, when N is a finite extension of L, we shall say that A is defectless in N if $[N:L] = \sum_{i=1}^{m} e(V_i|L)\, f(V_i|L)$ where $\{V_1, \cdots, V_m\}$ is the set of valuation rings of N lying over A.

An ideal I of a ring S will be said to be proper if it is different from S. We shall use $D^{(0)}(x)$ to denote x, and for $n \geqq 1$, $D^{(n)}(x)$ to denote $D(D^{(n-1)}(x))$, i.e., the nth derivative of x.

2. Extensions of the derivation in the quotient field.

LEMMA 2.1. *Let R be a ring, D a derivation on R, P a prime ideal of R containing no D-ideal other than* (0). *Define $v\colon R\backslash\{0\} \rightarrow$ {nonnegative integers} by $v(x) = n$ if $D^{(i)}(x) \in P$ for $i = 0, \cdots, n-1$ and $D^{(n)}(x) \notin P$. Then,*

(i) *R is a domain.*

(ii) *v is the trivial valuation if $P = (0)$, and is a rank-1 discrete valuation if $P \neq (0)$.*

(iii) *The valuation ring R_v of v contains R, and its maximal ideal $\mathfrak{M}_v$ lies over P.*

Proof. See [2, Theorem 3.1]. Note that for $x \in R\backslash\{0\}$ we indeed have $v(x) < \infty$ for otherwise the ideal generated by $\bigcup_{i=0}^{\infty} D^{(i)}(x)$ would be a nonzero D-ideal contained in P, which cannot be. Note also that the property for P to contain no D-ideal other than (0) is equivalent to R_P being D-simple.

LEMMA 2.2. *Let R, D, P, v, R_v, $\mathfrak{M}_v$ be as in* 2.1. *Let K be the* *quotient field of* R. *Let* S *be a ring between* R *and* K *such that* D *is regular on* S. *Then, the following statements are equivalent*:

(i) $S \subseteqq R_v$.

(ii) *There is a prime ideal* Q *of* S *lying over* P.

In this case, Q *is the only prime ideal of* S *lying over* P *and is equal to* $\mathfrak{M}_v \cap S$.

Proof. If $S \subseteqq R_v$, take $Q = \mathfrak{M}_v \cap S$. Conversely, suppose there exists a prime ideal Q of S such that $Q \cap R = P$. Being regular on S, D is also regular on S_Q; furthermore, $S_Q \supseteqq R_P$, and R_P is D-simple, thus by 1.1 S_Q is D-simple. Then, by 2.1, we can define a valuation w: $S\backslash\{0\} \rightarrow \{\text{nonnegative integers}\}$ by $w(y) = m$ if $D^{(j)}(y) \in Q$ for $j = 0, \cdots, m-1$ and $D^{(m)}(y) \notin Q$; calling S_w the valuation ring of w, we have $S \subseteqq S_w$. At the same time, we will have the valuation v defined with the prime ideal P of R, and for an element $x \in R\backslash\{0\}$ we have $D^{(i)}(x) \in P$ if and only if $D^{(i)}(x) \in Q$ since $P = Q \cap R$; thus, $v = w$ on R, hence also $v = w$ on K, and $S \subseteqq S_w = R_v$. Furthermore, by 2.1, we have $Q = \mathfrak{M}_w \cap S$, hence also $Q = \mathfrak{M}_v \cap S$, so that $\mathfrak{M}_v \cap S$ is the unique prime ideal of S lying over P.

LEMMA 2.3. *Let* A *be a* D*-simple valuation ring. Then,* A *is a field or is a rank*-1 *discrete valuation ring.*

Proof. If A is not a field, and $\mathfrak{A} \neq (1)$ is any ideal of A, then $\bigcap_{n=0}^{\infty} \mathfrak{A}^n \neq (1)$ is a D-ideal; thus, A being D-simple, we have $\bigcap_{n=0}^{\infty} \mathfrak{A}^n = (0)$ and S is a rank-1 discrete valuation ring.

THEOREM 2.4. *Let* R *be a* D*-simple ring with quotient field* K. *Let* $\mathscr{P} = \{$*proper prime ideals of* $R\}$, *and* $\mathscr{V} = \{$*valuation rings of* K *containing* R *to which* D *can be extended*$\}$. *Define* $\varphi\colon \mathscr{P} \rightarrow \mathscr{V}$ *by* $\varphi(P) = R_v$ *where* v *is the valuation associated to* P *by* 2.1. *Then,* φ *is a bijection.*

Proof. Let us show first that D is regular on R_v. Let ab^{-1} be any element of R_v with $a, b \in R$, $b \neq 0$, $v(a) \geqq v(b)$; then $D(ab^{-1}) = [bD(a) - aD(b)]b^{-2}$. If $v(a) > v(b)$, then $v(D(a)) = v(a) - 1 \geqq v(b)$ and $v(D(b)) \geqq v(b) - 1$, so that $v(bD(a) - aD(b)) \geqq \inf\{v(b) + v(D(a)), v(a) + v(D(b))\} \geqq 2v(b)$ and $D(ab^{-1}) \in R_v$. If $v(a) = v(b) = 0$, then $v(bD(a) - aD(b)) \geqq 0 = 2v(b)$ and $D(ab^{-1}) \in R_v$. If $v(a) = v(b) = n > 0$, then $v(bD(a)) = v(aD(b)) = 2n - 1$ so that $v(bD(a) - aD(b)) \geqq 2n - 1$; furthermore we have $D^{(2n-1)}(bD(a)) = \sum_{i=0}^{2n-1} C_{2n-1}^{i} D^{(i)}(b) D^{(2n-i)}(a) = \alpha_1 + C_{2n-1}^{n} D^{(n)}(b) D^{(n)}(a)$ with $\alpha_1 \in P$, and similarly $D^{(2n-1)}(aD(b)) = \alpha_2 + C_{2n-1}^{n} D^{(n)}(a) D^{(n)}(b)$ with $\alpha_2 \in P$, so that $D^{(2n-1)}(bD(a) - aD(b)) = \alpha_1 - \alpha_2 \in P$; hence $v(bD(a) -$

$aD(b)) \geqq 2n$ and $D(ab^{-1}) \in R_v$. Thus, D is regular on R_v.

If $\mathfrak{M}_v$ is the maximal ideal of R_v, we have $P = \mathfrak{M}_v \cap R$ by 2.1, thus φ is injective.

Now, let A be a valuation ring of K containing R to which D can be extended. If $A = K$, we clearly have $A = \varphi((0))$. If $A \neq K$, let Q be its maximal ideal. Let $P = Q \cap R$, let v be the valuation associated to P by 2.1, and let R_v be the valuation ring of v. Since P is different from (0), R_v is different from K; by 2.2, we have $A \subseteqq R_v$; by 1.1 A is D-simple, and hence has rank-1 by 2.3. Thus $A = R_v$, $A = \varphi(Q \cap R)$, and φ is surjective.

COROLLARY 2.5. *Let R be a D-simple ring with quotient field K. Let A be a valuation ring of K which contains R, Q its maximal ideal, P its center over R, and v the valuation associated to P by* 2.1. *Then, the following statements are equivalent:*

(i) *D can be extended to A.*

(ii) *For any $a, b \in P$ such that $v(a) \geqq v(b)$, then $ab^{-1} \in A$.*

(iii) *For any $x \in A$, there exists $a, b \in R$, such that $x = a/b$ and $v(a) \geqq v(b)$.*

Remember that for an element a of R, $v(a)$ is the number of successive applications of the derivation D necessary to get a out of the center P.

Proof. The condition (ii) is equivalent to $R_v \subseteqq A$; the condition (iii) is equivalent to $A \subseteqq R_v$. But in both cases A and R_v have the same center on R; thus, both conditions (ii) and (iii) are equivalent to $A = R_v$, i.e., equivalent to (i).

3. On the inertia field. Let N be a normal algebraic extension of K (possibly infinite), and G its Galois group. Let B be a valuation ring of N, $\mathfrak{M}_B$ its maximal ideal; let π be a place of N corresponding to B and μ its residue field; let v be a valuation of N corresponding to B and Δ its value group. Let $A = B \cap K$, Λ its residue field and Γ its value group; μ is a normal algebraic extension of Λ [1, (14.5)]. The inertia group of B over K is $G^T(B|K) = \{\sigma \in G/\sigma x - x \in \mathfrak{M}_B \forall x \in B\} = \{\sigma \in G/\pi \circ \sigma = \pi\}$; it is a closed subgroup of G [1, (19.2)]; its fixed field $K^T(B|K) = \{y \in N/\sigma y = y \forall \sigma \in G^T(B/K)$ is the inertia field of B over K.

In this section, we shall only be concerned with the case of $A = B \cap K$ being a rank-1 discrete valuation ring which contains the rational numbers. Note that B has to be of rank-1 too [1, (13.14)]. We have:

PROPOSITION 3.1. *$K^T(B/K)$ is the smallest field L between K and N such that $B \cap L$ is indecomposed in N and such that μ is purely inseparable over the residue field Λ^L of $B \cap L$.*

Proof. See [1, (19.11)].

PROPOSITION 3.2. *$K^T(B|K)$ is the unique field L between K and N such that $B \cap L$ is indecomposed in $N, f(B|L) = 1$ and $e(B \cap L|K) = 1$.*

Proof. Since A contains the rational numbers, Λ has characteristic zero, μ is a separable extension of Λ, and, by 3.1, $K^T(B|K)$ is the smallest field L between K and N such that $B \cap L$ is indecomposed in N and $f(B|L) = 1$. Now, N is also separable over K so that $\Gamma^T = \Gamma$, and $B \cap K^T(B|K)$ is a rank-1 discrete valution ring; then $B \cap K^T(B|K)$ is defectless in all the finite extensions of $K^T(B|K)$ contained in N [6, Corollary, p. 287], and $K^T(B|K)$ is maximal among the fields L that have the property $f(B|L) = 1$ and $e(B \cap L|K) = 1$.

PROPOSITION 3.3. *$K^T(B|K)$ is the biggest field L between K and N such that $e(B \cap L|K) = 1$.*

Proof. Let L be a field between K and N such that $e(B \cap L|K) = 1$. Let $L^T(B|L)$ be the inertia field of B over L; by 3.2, $B \cap L^T(B/L)$ is indecomposed in $N, f(B|L^T(B|L)) = 1$ and $e(B \cap L^T(B|L)|L) = 1$, hence also $e(B \cap L^T(B|L)|K) = 1$ since $e(B \cap L|K) = 1$. Thus, by 3.2, $L^T(B|L) = K^T(B|K)$ and $L \subseteqq K^T(B|K)$.

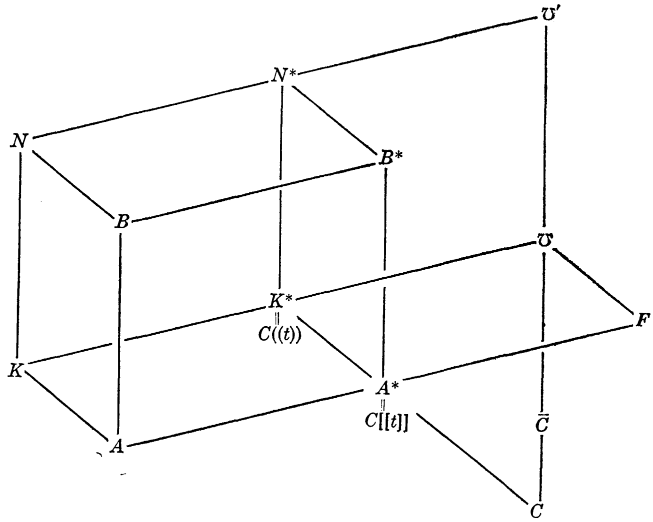
COROLLARY 3.4. *Let V be a valuation ring contained in N lying over A. Then, the following statements are equivalent:*

(i) $e(V/K) = 1$.

(ii) *There exists a valuation ring E of N lying over V such that $V \subseteqq K^T(E/K)$.*

(iii) *For every valuationring E of N lying over $V, V \subseteqq K^T(E/K)$.*

Now, let (N^*, B^*) be a completion of (N, B); by this, we mean that N^* is a B-completion of N [5, (1–7–1), p. 27], and B^* the topological closure of B in N^*; let (K^*, A^*) be the completion of (K, A) contained in (N^*, B^*). A being a rank-1 discrete valuation ring, we let t be a generator of the maximal ideal of A. Let $\mho'$ be an algebraic closure of N^*, and $\mho$ the algebraic closure of K^* contained in $\mho'$. Let C be a field of representatives of A^* and $\bar{C}$ the algebraic closure of C contained in $\mho$; by [7, Theorem 27, p. 304], we have $A^* = C[[t]]$ and $K^* = C((t))$. Let F be the unique valuation ring of $\mho$ which lies over A^* [5, (2–1–3), p. 44]. The situation can be resumed by the following diagram:



PROPOSITION 3.5. *$K^T(B/K) = \bar{C}((t)) \cap N$ and $B \cap K^T(B/K) = \bar{C}[[t]] \cap N$.*

Proof. We shall do it in several steps.

Step 1. $\bar{C}((t)) \cap \bar{\mathcal{O}}$ is the inertia field of F over K^* and $\bar{C}[[t]] \cap \bar{\mathcal{O}} = F \cap (\bar{C}((t)) \cap \bar{\mathcal{O}})$.

Proof. $\bar{C}[[t]] \cap \bar{\mathcal{O}}$ is a valuation ring of $\bar{C}((t)) \cap \bar{\mathcal{O}}$ which lies over $A^* = C[[t]]$; thus it is indecomposed in $\bar{\mathcal{O}}$ and is equal to $F \cap (\bar{C}((t)) \cap \bar{\mathcal{O}})$. Let ξ (respectively w-) be a place (respectively a valuation) of $\bar{\mathcal{O}}$ corresponding to the valuation ring F; since $\bar{C} \subsetneqq \bar{\mathcal{O}}$, we have $\xi(C) = \xi(C[[t]]) \subsetneqq \xi(\bar{C}) \subsetneqq \xi(\bar{C}[[t]] \cap \bar{\mathcal{O}}) \subsetneqq \xi(F)$; furthermore $\xi(F)$ is algebraic over $\xi(C[[t]])$ by [1, (14.5)], and $\xi(\bar{C}) \cong \bar{C}$ is algebraically closed; thus $\xi(\bar{C}[[t]] \cap \bar{\mathcal{O}}) = \xi(F)$. On the other hand we have clearly $w(C[[t]]) = w(\bar{C}[[t]] \cap \bar{\mathcal{O}})$. Thus by 3.2, $\bar{C}((t)) \cap \bar{\mathcal{O}}$ is the inertia field of F over K^*.

Step 2. Let N_α be a finite normal extension of K contained in N. Let N_α^* be the completion of N_α contained in N^*. Then $\bar{C}((t)) \cap N_\alpha^*$ is the inertia field of $B^* \cap N_\alpha^*$ over K^* and $\bar{C}[[t]] \cap N_\alpha^* = (B^* \cap N_\alpha^*) \cap (\bar{C}((t)) \cap N_\alpha^*)$.

Proof. N_α^* is a finite normal extension of K^* [4, Corollary 4, p. 41]; hence $N_\alpha^* \subseteqq \bar{\mathcal{O}}$. $B^* \cap N_\alpha^*$ is a valuation ring of N_α^* which lies over A^*; hence it has to be equal to $F \cap N_\alpha^*$. Now, the inertia field of $F \cap N_\alpha^*$ over K^* is equal to the intersection of the inertia field of F over K^* with N_α^* [1, (19.10)], i.e., is equal to $(\bar{C}((t)) \cap \bar{\mathcal{O}}) \cap N_\alpha^* =$

$\bar{C}((t)) \cap N_\alpha^*$. Finally, $\bar{C}[[t]] \cap N_\alpha^*$ is a valuation ring of $\bar{C}((t)) \cap N_\alpha^*$ which lies over A^*, thus it has to lie under $B^* \cap N_\alpha^*$, i.e., we need to have $\bar{C}[[t]] \cap N_\alpha^* = (B^* \cap N_\alpha^*) \cap \bar{C}((t)) \cap N_\alpha^*)$.

Step 3. $\bar{C}((t)) \cap N_\alpha$ is the inertia field of $B \cap N_\alpha$ over K and $\bar{C}[[t]] \cap N_\alpha = (B \cap N_\alpha) \cap (\bar{C}((t)) \cap N_\alpha)$.

Proof. $B \cap N_\alpha \cap \bar{C}((t)) \subseteqq B^* \cap N_\alpha^* \cap \bar{C}((t)) = \bar{C}[[t]] \cap N_\alpha^*$ by Step 2; then, being contained in $\bar{C}((t)) \cap N_\alpha$, $B \cap N_\alpha \cap \bar{C}((t))$ has also to be contained in $\bar{C}[[t]] \cap N_\alpha$; being a rank-1 valuation ring, $B \cap N_\alpha \cap \bar{C}((t))$ has to be equal to $\bar{C}[[t]] \cap N_\alpha$.

Now, if we still call w the valuation of $\bar{\mathfrak{O}}$ corresponding to F, we have $w(K) \subseteqq w(\bar{C}((t)) \cap N_\alpha) \subseteqq w(\bar{C}((t)) \cap N_\alpha^*)$; but $w(K^*) = w(\bar{C}((t)) \cap N_\alpha^*)$ by Step 2, and $w(K) = w(K^*)$ because, by [5, (1–7–5), p. 31], the completion is an immediate extension; hence $w(K) = w(\bar{C}((t)) \cap N_\alpha)$, and $\bar{C}((t)) \cap N_\alpha \subseteqq K^T(B \cap N_\alpha/K)$ by 3.3. Then, $\bar{C}((t)) \cap N_\alpha = K^T(B \cap N_\alpha/K)$, because if not, the completion L of $K^T(B \cap N_\alpha/K)$ contained in N_α^* would be such that $L \not\subseteqq C((t)) \cap N_\alpha^*$ and $e(B^* \cap L/K^*) = 1$, which is impossible by 3.3, since $\bar{C}((t)) \cap N_\alpha^*$ is the inertia field of $B^* \cap N_\alpha^*$ over K^* by Step 2.

Step 4. $\bar{C}((t)) \cap N$ is the inertia field of B over K and $\bar{C}[[t]] \cap N = B \cap (\bar{C}((t)) \cap N)$.

Proof. Let $\{N_\alpha; \alpha \in J\}$ be the set of all the finite normal subextensions of N over K. Let us show that $K^T(B|K) = \bigcup_{\alpha \in J} K^T(B \cap N_\alpha|K)$. For any $\alpha \in J$, the homomorphism θ_α^T: $G^T(B|K) \to G^T(B \cap N_\alpha|K)$ defined by $\theta_\alpha^T(\rho) = \rho|_{N_\alpha}$ = the restriction of ρ to N_α, is surjective [1, (19.7)]. Let $x \in K^T(B|K)$, N_α a finite normal extension of K containing x and $\sigma \in G^T(B \cap N_\alpha|K)$; since θ_α^T is surjective, there exists $\rho \in G^T(B|K)$ such that $\rho|_{N_\alpha} = \sigma$, so that $\sigma(x) = \rho(x) = x$ and $x \in K^T(B \cap N_\alpha|K)$. Conversely, let $\alpha \in J$, and $x \in K^T(B \cap N_\alpha|K)$; for any $\rho \in G^T(B|K)$ we have $\rho|_{N_\alpha} \in G^T(B \cap N_\alpha|K)$, so that $\rho(x) = \rho|_{N_\alpha}(x) = x$ and $x \in K^T(B|K)$. Hence, $K^T(B|K) = \bigcup_{\alpha \in J} K^T(B \cap N_\alpha|K) = \bigcup_{\alpha \in J} (\bar{C}((t)) \cap N_\alpha) = \bar{C}((t)) \cap (\bigcup_{\alpha \in J} N_\alpha) = \bar{C}((t)) \cap N$, and $B \cap K^T(B|K) = B \cap (\bigcup_{\alpha \in J} K^T(B \cap N_\alpha|K)) = \bigcup_{\alpha \in J} (B \cap K^T(B \cap N_\alpha|K)) = \bigcup_{\alpha \in J} (\bar{C}[[t]] \cap N_\alpha) = \bar{C}[[t]] \cap N$.

4. Extensions of the derivation in the algebraic closure of the quotient field.

LEMMA 4.1. *Let A be a ring, I a finitely generated ideal of A such that $\bigcap_{n=0}^\infty I^n = (0)$, A^* the I-adic completion of A. Let D: $A \to A^*$ be a map such that $D(x + y) = D(x) + D(y)$ and $D(xy) = xD(y) + yD(x)$. Then,*

(i) *D can be extended to a derivation D' on A^* by $D'(\lim_n x_n) = \lim_n D(x_n)$, where $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in A.*

(ii) *D' is the only derivation of A^* that extends D.*

Proof. (i) Let $\{x_n\}_{n\geq 0}$ be a Cauchy sequence in A; for any positive integer m, there exists q such that $r, s > q \Rightarrow x_r - x_s \in I^m$; $x_r - x_s \in I^m \Rightarrow x_r - x_s = \sum_i u_{i1} \cdots u_{im}$ with $u_{ij} \in I$, hence $Dx_r - Dx_s = D(x_r - x_s) = \sum_i \sum_{j=1}^m u_{ij} \cdots u_{i(j-1)} D(u_{i1}) u_{i(j+1)} \cdots u_{im} \in (IA^*)^{m-1}$; then as I is finitely generated, the topology of A^* is the $(IA^*)-$adic topology [7, Corollary 1, p. 257], and $\{Dx_n\}_{n\geq 0}$ is a Cauchy sequence in A^*; set $D'(\lim_n x_n) = \lim_n D(x_n)$. Defined that way, D' is a function of A^* for if $\{z_n\}_{n\geq 0}$ is another Cauchy sequence such that $\lim_n x_n = \lim_n z_n$, then for any positive integer m, there exists q such that $n > q \Rightarrow (x_n - z_n) \in I^m$, so that $D(x_n) - D(z_n) = D(x_n - z_n) \in (IA^*)^{m-1}$, and $\lim_n D(x_n) = \lim_n D(z_n)$. Furthermore, D' is a derivation of A^* for if $\{x_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ are two Cauchy sequences of A, then $\lim_n \mathrm{D}(x_n + z_n) = \lim_n D(x_n) + \lim_n D(z_n)$ and $\lim_n D(x_n \cdot z_n) = \lim_n x_n \cdot \lim_n D(z_n) + \lim_n D(x_n) \cdot \lim_n z_n$ since, for every n, we have $D(x_n + z_n) = D(x_n) + D(z_n)$ and $D(x_n \cdot z_n) = x_n \cdot D(z_n) + D(x_n) \cdot z_n$. Finally, for any $y \in A$, we clearly have $D'(y) = D(y)$.

(ii) Let D'' be a derivation of A^* which extends D. Let y be any element of A^*, and $\{x_n\}_{n\geq 0}$ a Cauchy sequence in A such that $y = \lim_n x_n$; then, for any positive integer m, there exists q such that $n > q \Rightarrow y - y_n \in (IA^*)^m$, so that $D''(y) - D(y_n) = D''(y) - D''(y_n) = D''(y - y_n) \in (IA^*)^{m-1}$, and $D''(y) = \lim_n D(y_n) = D'(y)$.

REMARK. In the case of D being a derivation of A, the procedure used in the preceding lemma allows to extend D to a derivation D' of A^* even if I is not finitely generated. To get the uniqueness property however, we again need I to be finitely generated.

THEOREM 4.2. *Let A be a rank-1 discrete valuation ring containing the rational numbers with quotient field K; let Ω be an algebraic closure of K and D a derivation of A. Let B be a valuation ring of Ω lying over A; let V be a valuation ring contained in Ω, lying over A and unramified over K. Then,*

(i) *$(K^T(B|K), B \cap K^T(B|K))$ is a D-regular extension of (K, A) contained in (Ω, B).*

(ii) *$(N, B \cap N)$ is D-regular for any field N between K and $K^T(B|K)$.*

(iii) *D is regular on V.*

Proof. (i) Let (Ω^*, B^*) be a completion of (Ω, B) and (K^*, A^*)

the completion of (K, A) contained in (Ω^*, B^*); let $\bar{\Omega}'$ be an algebraic closure of Ω^* and $\bar{\Omega}$ the algebraic closure of K^* contained in $\bar{\Omega}'$. Let t be a generator of the maximal ideal of A; let C be a field of representatives of A^*, and $\bar{C}$ the algebraic closure of C in $\bar{\Omega}$; of course we have $A^* = C[[t]]$ and $K^* = C((t))$ [7, Corollary, p. 307]. By 4.1, let D' be the unique derivation of A^* which is an extension of D, and, as usual, call again D' its extension to $\bar{\Omega}$. For an element y of $\bar{C}$, we have $D'(y) \in \bar{C}[[t]]$; indeed, if $X^n + c_1X^{n-1} + \cdots + c_n \in C[X]$ is the minimal polynomial of y over C, differentiating the equation $y^n + c_1y^{n-1} + \cdots + c_n = 0$, we get $(ny^{n-1} + c_1(n-1)y^{n-2} + \cdots + c_{n-1})D'(y) + (D(c_1)y^{n-1} + \cdots + D(c_n)) = 0$; the first factor of the first term is an element of $\bar{C}$, different from zero since y is separable over C; the second term is an element of $\bar{C}[[t]]$; thus $D'(y) \in \bar{C}[[t]]$. We also have $D'(t) \in \bar{C}[[t]]$, so that the restriction D'' of D' to $\bar{C}[t]$ is a function with values in $\bar{C}[[t]]$ which satisfies the properties $D''(x + z) = D''(x) + D''(z)$ and $D''(xz) = xD''(z) + zD''(x)$; furthermore, $\bar{C}[[t]]$ is the $(t)-$adic completion of $\bar{C}[t]$; thus, by 4.1, D'' can be extended to a derivation of $\bar{C}[[t]]$, which we call D'' again, by $D''(\sum_{i=0}^{\infty} d_it^i) = \sum_{i=0}^{\infty} D''(d_it^i) = \sum_{i=0}^{\infty} D'(d_it^i)$. As $C[[t]]$ is the completion of $C[t]$ for the $(t)-$adic topology, by 4.1 also, we know that for an element $\sum_{i=0}^{\infty} c_it^i$ of $C[[t]]$ we must have $D'(\sum_{i=0}^{\infty} c_it^i) = \sum_{i=0}^{\infty} D'(c_it^i)$, so that $D' = D''$ on $A^* = C[[t]]$; thus $D = D''$ on A, hence also on K. But we can even see that $D = D''$ on $\bar{C}((t)) \cap \Omega$; indeed, if $X^m + k_1X^{m-1} + \cdots + k_m \in K[X]$ is the minimal polynomial over K of an element z of $\bar{C}((t)) \cap \Omega$, we have $z^m + k_1z^{m-1} + \cdots k_m = 0$, thus $D(z) = [D(k_1)z^{m-1} + \cdots + D(k_m)] \times [mz^{m-1} + \cdots + k_{m-1}]^{-1} = [D''(k_1)z^{m-1} + \cdots + D''(k_m)][mz^{m-1} + \cdots + k_{m-1}]^{-1} = D''(z)$. Then, since D is regular on Ω, since D'' is regular on $\bar{C}[[t]]$, and since $D = D''$ on $\bar{C}((t)) \cap \Omega$, we get that $(\bar{C}((t)) \cap \Omega, \bar{C}[[t]] \cap \Omega)$ is D-regular; but by 3.5 we know that $\bar{C}((t)) \cap \Omega = K^T(B|K)$ and $\bar{C}[[t]] \cap \Omega = B \cap K^T(B|K)$; thus $(K^T(B|K), B \cap K^T(B|K))$ is D-regular.

(ii) Let N be any field between K and $K^T(B|K)$. D is regular on N and is regular on $B \cap K^T(B|K)$; thus D is regular on $(B \cap K^T(B|K)) \cap N = B \cap N$.

(iii) Let B' be a valuation ring of Ω lying over V; by 3.4 we have $V \subseteqq K^T(B'|K)$, so that D is regular on V.

THEOREM 4.3. *Let A be a D-simple valuation ring with quotient field K; let Ω be an algebraic closure of K, and B a valuation ring of Ω lying over A. Then, $(K^T(B|K), B \cap K^T(B|K))$ is the biggest D-regular extension of (K, A) contained in (Ω, B).*

Proof. Being D-simple, A contains the rational numbers; thus, by 4.2, we know that $(K^T(B|K), B \cap K^T(B|K))$ is D-regular. Now let (L, E) be a D-regular extension of (K, A) contained in (Ω, B); of course E is rank-1, and thus B lies over E; also E is D-simple by 1.1. If t is a generator of the maximal ideal of A, then t is also a generator of the maximal ideal $\mathfrak{M}_E$ of E; indeed, otherwise we would have $t \in \mathfrak{M}_E^2$, hence also $D(t) \in \mathfrak{M}_E$ which cannot be since $D(t)$ is a unit in A. Thus, the index of ramification of E over K is equal to 1, and by 3.3 $(L, E) \subsetneqq (K^T(B|K), B \cap K^T(B|K))$.

COROLLARY 4.4. *Let R be a D-simple ring with quotient field K; let Ω be an algebraic closure of K. Let V be a valuation ring which contains R and is contained in Ω; let $e(V|K)$ be its ramification index over K. Then, the following statements are equivalent:*

(i) *D is regular on V.*

(ii) *$e(V|K) = 1$ and D is regular on $V \cap K$.*

Proof. If D is regular on V, then D is regular on $V \cap K$ since D is also regular on K. Furthermore, $V \cap K$ contains R which is D-simple; thus, by 1.1, $V \cap K$ is D-simple and, as already noticed in the proof of 4.3, this implies that $e(V|K) = 1$. Conversely, if D is regular on $V \cap K$ and if $e(V|K) = 1$ we know that D is regular on V by 4.2.

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