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# A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

JULIEN HENNEFELD

**For a Banach space $X$, let $B(X)$ be the space of all bounded linear operators on $X$, and $\mathscr{C}$ the space of all compact linear operators on $X$. In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for $X = c_0$ or $l^p$ with $1 < p < \infty$, each bounded linear functional on $\mathscr{C}$ has a unique norm-preserving to $B(X)$. This is proved by using a decomposition theorem for $B(X)^*$, which takes on a special form for $X = c_0$ or $l^p$ with $1 < p < \infty$.**

1. DEFINITION 1.1. A basis $\{e_i\}$ for a Banach space $X$ having coefficient functionals $e_i^*$ in $X^*$ is called unconditional if, for each $x$, $\sum_{i=1}^{\infty} e_i^*(x)e_i$ converges unconditionally. The basis is called monotone if $||U_m x|| < ||x||$ for all $x \in X$ and positive integers $m$, where $U_m x = \sum_{i=1}^{m} e_i^*(x)e_i$.

PROPOSITION 1.2. *If $X$ has a monotone, unconditional basis $\{e_i\}$, then $B(X)^* = \mathscr{C}^* + \mathscr{C}^\perp$, where $\mathscr{C}^*$ is a subspace of $B(X)^*$ isomorphically isometric to the space of bounded linear functionals on $\mathscr{C}$, and $\mathscr{C}^\perp$ annihilates $\mathscr{C}$. Furthermore, the associated projection from $B(X)^*$ onto $\mathscr{C}^*$ has unit norm.*

*Proof*. If $T \in B(X)$, then $T(x) = \sum_{i=1}^{\infty} f_i^T(x)e_i$ for each $x \in X$, where $f_i^T \in X^*$. For each $T$ and $i$, let $T_i$ be defined by $T_i(x) = f_i^T(x)e_i$ for all $x$. Also, for each $F \in B(X)^*$, define $G \in B(X)^*$ by $G(T) = \sum_{i=1}^{\infty} F(T_i)$. Note that this sum converges. Otherwise, we have $\sum_{i=1}^{\infty} |F(T_i)| = \lim_{n\to\infty} F[\sum_{i=1}^{\infty} SgF(T_i)\cdot T_i] = +\infty$, and then

$$\lim_{n\to\infty} ||\sum_{i=1}^{n} SgF(T_i)\cdot T_i|| = \infty .$$

Then by using an absolutely convergent series, it is easy to construct an element $y \in X$: $\lim_{n\to\infty} ||\sum_{i=1}^{n} SgF(T_i)\cdot T_i(y)|| = \infty$. Therefore, $\sum_{i=1}^{\infty} f_i^T(y)e_i$ converges while $\sum_{i=1}^{\infty} SgF(T_i)\cdot f_i^T(y)e_i$ does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of $G$ restricted to $\mathscr{C}$ is equal to the norm of $G$ on $B(X)$, since by monotonicity $||\sum_{i=1}^{n} T_i|| \leqq ||T||$ for each $n$ and $T \in B(X)$. Also, $F$ and $G$ agree on $\mathscr{C}$, because $\mathscr{C}$ is the closure of the set of all $T$ for which only a finite number of the $f_i^T$ are nonzero. Hence the projection defined by $PF = G$ has unit norm, since

$\|F\|_{B(X)} \geqq \|F\|_{\mathscr{C}} = \|G\|_{\mathscr{C}} = \|G\|_{B(X)}$.

COROLLARY 1.3. *If $X$ has an unconditional basis $\{e_i\}$, then there is a bounded projection from $B(X)^*$ onto a subspace isomorphic to $\mathscr{C}^*$.*

*Proof.* Renorm $X$ so that the basis $\{e\}_i$ is monotone. See [1], p. 73.

2. THEOREM 2.1. *Let $X$ have an unconditional, shrinking basis $\{e_i\}$, for which there is a function $N$ of two real variables such that:*
(i) $N(a, b) \leqq N(\alpha, \beta)$ *if* $0 \leqq a \leqq \alpha$ *and* $0 \leqq b \leqq \beta$;
(ii) $N(\|x\|, \|y\|) = \|x + y\|$ *for which* $x = \sum_{i=1}^{n} a_i e_i$ *and* $y = \sum_{n+1}^{\infty} a_i e_i$. *Then for each $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, where $F = G + H$ with $G \in \mathscr{C}^*$ and $H \in \mathscr{C}^{\perp}$.*

*Proof.* Note that the existence of $N$ implies that the basis is monotone, and so we have a decomposition for $B(X)^*$. The operators whose matrices have a finite number of nonzero entries form a dense subset of $\mathscr{C}$. Hence, for $\varepsilon > 0$, there exists an operator $D$ of unit norm whose image lies in the subspace $[e_1, e_2, \cdots, e_m]$, and whose kernel contains $[e_{m+1}, e_{m+2}, \cdots]$: $G(D) > \|G\| - \varepsilon/3$. Also, there exists an operator $T \in B(X)$ of unit norm: $H(T) > \|H\| - \varepsilon/3$. Let $Q_r$ be the projection onto $[e_{r+1}, e_{r+2}, \cdots]$. Define $T^{(r)}x = \sum_{i=r+1}^{\infty} f_i^T(Qx)e_i$. Note that the matrix for $T^{(r)}$ is simply the matrix for $T$, with the first $r$-rows and $r$-columns replaced by zeros.

Then $\lim_{r\to\infty} G(T^{(r)}) = 0$. To see this, first note that the existence of $N$ and the basis being shrinking imply that the functionals in $\mathscr{C}^*$ with a finite number of nonzero entries form a dense subset of $\mathscr{C}^*$. See [2], Propositions 3.1 and 3.3. Thus, for any $\delta > 0$, $\exists J \in B(X)^*$, for which $\|J - G\| < \delta$ and: $\lim_{r\to\infty} J(T^{(r)}) = 0$. Hence $\lim_{r\to\infty} G(T^{(r)}) = 0$.

Then pick $r > m$: $|G(T^{(r)})| < \varepsilon/3$. Observe that $\|D + T^{(r)}\| = 1$, since and $z \in X$ can be written as $z = x + y$ where $x \in [e_1, \cdots, e_r]$ and $y \in [e_{r+1}, \cdots]$. Then

$$\begin{aligned} \|(D + T^{(r)})(x + y)\| &= \|Dx + T^{(r)}y\| = N(\|Dx\|, \|T^{(r)}y\|) \\ &\leqq N(\|x\|, \|y\|) = \|x, y\| . \end{aligned}$$

Using the fact that $H$ annihilates $\mathscr{C}$, we have

$$\begin{aligned} F(D + T^{(r)}) &= G(D) + G(T^{(r)}) + H(T^{(r)}) > \|G\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} + \|H\| - \frac{\varepsilon}{3} \\ &= \|G\| + \|H\| - \varepsilon . \end{aligned}$$

Hence $\|F\| = \|G\| + \|H\|$.

COROLLARY 2.2. *If $X$ is $(c_0)$ or $l^p$ with $1 < p < \infty$, then, for each $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, where $F = G + H$ with $G \in \mathscr{C}^*$ and $H \in \mathscr{C}^\perp$.*

*Proof.* Let $\{e_i\}$ be the standard basis. Let $N(a, b) = [|a^p| + |b^p|]^{1/p}$ for $l^p$. Let $N(a, b) = \mathrm{Max}(|a|, |b|)$ for $c_0$.

THEOREM 2.3. *Each bounded linear functional on $\mathscr{C}$ has a unique normpreserving extension to $B(X)$ for $X = c_0$ or $l^p$ with $1 < p < \infty$.*

## REFERENCES


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2. J. Hennefeld, *The Arens products and an imbedding theorem*, Pacific J. Math., **29** (1969), 551-563.
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BOSTON COLLEGE