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A GENERALIZATION OF KNESER'S CONJECTURE[1]

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Let M be a closed connected 3-manifold such that $\pi_2(M) = 0$. Suppose that $\pi_1(M)$ is a nontrivial free product with amalgamation across the group of a closed connected surface S other than the projective plane or 2-sphere. Then it is shown that there is an embedded surface S in M "realizing" the group structure above.

Our theorem also considers the case when M has boundary and gives an answer to a problem of Neuwirth.

1. Introduction. In 1929 H. Kneser stated the following result[1] in [8].

THEOREM K. *Let M be a closed, connected 3-manifold. Suppose $\pi_1(M)$ is the nontrivial free product of two groups A_1 and A_2. Then there exists an embedding of the 2-sphere in M which separate M into 3-submanifolds M_1 and M_2 such that $\pi_1(M_j) = A_j$ for $j = 1, 2$.*

Theorem K was confirmed by J. Stallings in his thesis. One would like to generalize Theorem K so that one could realize geometrically more complicated algebraic splittings of $\pi_1(M)$. In [3] we made a generalization of this form which required that M be orientable and closed and that the splitting surface be a closed orientable surface not the 2-sphere.

In the "Splitting Theorem" below we eliminate most of these restraints.

2. Preliminaries. All spaces discussed in this paper are simplicial complexes and all maps are piecewise linear. As usual we will write $G = A \underset{C}{*} B$ when the group G is the free product of A and B with amalgamation over C. We shall restrict our attention to the case when A and B are proper subgroups of G or equivalently C is a proper subgroup of A and B. As usual, we shall denote the boundary, closure, and interior of a subspace X of a space Y by bd(X), cl(X), and int(X) respectively.

Let X be a connected subspace of the space Y. Then we shall denote the natural inclusion map from X into Y by ρ and the induced homomorphism from $\pi_1(X)$ into $\pi_1(Y)$ by ρ_*. Let a closed connected surface S not the 2-sphere or projective plane be embedded in a space X. If $\rho_*: \pi_1(S) \to \pi_1(X)$ is one-to-one, we shall say that S *is*

[1] Papakyriakopoulos gives an interesting discussion of this conjecture in [9].

incompressible in X. Similarly if S is a system of closed surfaces embedded in a space X, we shall say that *S is incompressible in X* if each component of S is incompressible in X. We will denote $[0,1]$ by I throughout this paper.

Let S be a closed, connected surface not the 2-sphere or projective plane. Let M be a connected 3-manifold and

$$\pi_1(M) = A_1 \underset{\pi_1(S)}{*} A_2 \,.$$

Then we shall way that *$\pi_1(M)$ splits across $\pi_1(S)$*. We shall say that the splitting of $\pi_1(M)$ above *respects the peripheral structure of M* if for each component F of the boundary of M, a conjugate of $\rho_*\pi_1(F)$ is contained either in A_1 or in A_2.

Let S be a closed, connected, incompressible surface embedded in M. Suppose that S separates M into two 3-submanifolds M_1 and M_2. Let $\pi_1(S) \cong A_0$ and suppose $\pi_1(M) = A_1 \underset{A_0}{*} A_2$. Consider the group diagrams given in Figures 1 and 2. The group diagram in Figure 1 is obtained by applying Van Kampen's theorem to the decomposition of M into M_1 and M_2. The group diagram in Figure 2 is obtained from the splitting of $\pi_1(M)$ by A_0.

$$\begin{tikzcd} & \pi_1(S) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ \pi_1(M_1) \arrow[dr, "i_3"'] & & \pi_1(M_2) \arrow[dl, "i_4"] \\ & \pi_1(M) & \end{tikzcd}$$

FIGURE 1

$$\begin{tikzcd} & A_0 \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ A_1 \arrow[dr, "h_3"'] & & A_2 \arrow[dl, "h_4"] \\ & \pi_1(M) & \end{tikzcd}$$

FIGURE 2

Then we shall say that the surface S *geometrically realizes* the algebraic splitting above if there is an isomorphism

$$\varphi\colon\ \pi_1(M) \longrightarrow \pi_1(M)$$

such that

(1) $\varphi(\pi_1(S)) = A_0$

(2) $\varphi(\pi_1(M_j)) = A_j$ for $j = 1, 2$

(3) $h_k = \varphi i_k \varphi^{-1}$ for $k = 1, 2, 3, 4$.

3. The splitting theorem.

THEOREM 1. *Let M be a compact, connected 3-manifold such that $\pi_2(M) = 0$. Let S be a closed, connected surface not the 2-sphere or projective plane. Suppose*

$$\pi_1(M) = A_1 \underset{\pi_1(S)}{*} A_2$$

and that this splitting preserves the peripheral structure of M. Then there is a geometric splitting realizing the algebraic splitting above.

The proof of Theorem 1 in this paper is similar to the proof of Theorem 1 in [3]. We shall need three lemmas in the proof of Theorem 1 and we shall consider these at this point.

Lemma 1 is the result of a number of well known techniques and is similar in content to Lemma 1.1 in [12]. We shall omit most of the details of the proof.

LEMMA 1. *Let M be a compact, connected 3-manifold such that $\pi_2(M) = 0$. Let X be a connected complex and S a closed incompressible surface embedded in X and having a neighborhood homeomorphic to $S \times I$. We suppose that no component of S is a 2-sphere or projective plane. Let X_k, $k = 1, \cdots, n$ be the components of $X - S$. We suppose that $\pi_i(X) = \pi_i(X_k) = 0$ for $i \geqq 2$ and $k = 1, \cdots, n$. Let f: $M \rightarrow X$ be a map such that f_*: $\pi_1(M) \rightarrow \pi_1(X)$ is one-to-one and f $\mathrm{bd}(M)$ does not meet S. Then there is a homotopy, constant on $\mathrm{bd}(M)$, of f to a map g such that $g^{-1}(S)$ is an incompressible surface in M.*

Proof. One first uses the simplicial approximation theorem to find a map g_1 homotopic to f such that $g_1^{-1}(S)$ is a surface in M. Next one uses techniques developed by J. Stallings in his thesis to fiind a map g homotopic to g_1 such that $g^{-1}(S)$ is an incompressible surface in M. We note that the homotopies used could be held constant on $\mathrm{bd}(M)$ since $f(\mathrm{bd}(M)) \cap S$ was empty. The lemma follows.

A 3-manifold M will be called *p^2-irreducible* if there are no embedded projective planes in M and every 2-sphere in M bounds a 3-ball embedded in M.

LEMMA 2. *Let S_1 and S_2 be disjoint, incompressible, connected surfaces which are embedded in a P^2-irreducible 3-manifold M. Then if S_1 is homotopic to S_2 in M, $S_1 \cup S_2$ bounds an $S_1 \times I$ embedded in M.*

Proof. It is a consequence of 1.1.5 in [13] that $\pi_j(M) = 0$ for $j \geqq 2$ and that the higher homotopy groups of each component of $M - (S_1 \cup S_2)$ are trivial. Let H: $S \times I \rightarrow M$ be a homotopy of S_1 to S_2. It is a consequence of Lemma 1 that we may assume $H^{-1}(S_1 \cup S_2)$ is incompressible in $S \times I$. Let S^* be a component of $H^{-1}(S_1 \cup S_2)$. Then we claim S^* separates $S \times \{0\}$ from $S \times \{1\}$.

Assume that S^* does not separate $S \times I$. Then $[S^*]$ is not homologous to $[S \times \{0\}]$ in $C_2(S \times I;\ Z_2)$. Since $H_2(S \times I;\ Z_2) = Z_2$, $[S^*]$ bounds a 3-chain in $C_3(S \times I;\ Z_2)$. Thus S^* bounds a 3-submanifold $N \subset S \times I$. Let $N_1 = \mathrm{cl}\,(S \times I - N)$. Now by Van Kampen's theorem we have the commutative diagram shown in Figure 3. All homomorphisms in Figure 3 are induced by inclusion.

$$\begin{tikzcd}
& \pi_1(S^*) \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\
\pi_1(N) \arrow[dr, "\rho_3"'] & & \pi_1(N_1) \arrow[dl, "\rho_4"] \\
& \pi_1(S \times I) &
\end{tikzcd}$$

FIGURE 3

Since $N_1 \supset S \times \{0\}$, ρ_4 is onto. Since ρ_1 and ρ_2 are one-to-one it is a consequence of 2.5 in [1] that ρ_1 is onto. It is a consequence of the corollary to Theorem A in [5] that N is a product line bundle.

Thus S^* separates $S \times I$. By an argument similar to the one above we can show that the closure of either component of $S \times I - S^*$ is a product line bundle.

It is now easy to show that $H^{-1}(S_1 \cup S_2)$ can be assumed to be $\mathrm{bd}\,(S \times I)$; one simply considers the restriction of H to a submanifold of $S \times I$. Lemma 2 is now a consequence of the corollary to Theorem A in [5].

A result similar to Lemma 3 was suggested to the author by an unknown referee. This suggestion enabled us to greatly simplify the proof to Theorem 1. Lemma 3 is proved using standard arguments in obstruction theory and could be stated in terms of cell complexes and relative homotopy groups. However, it will be immediately obvious in the proof of Theorem 1 that the hypotheses of our Lemma 3 are met.

LEMMA 3. *Let M_1 be a compact, connected, 3-manifold, X a connected complex, and F and S incompressible connected surfaces in M_1 and X respectively. We suppose that S is neither a 2-sphere or projective plane and $\pi_i(X) = 0$ for $i \geqq 2$.*

Let f: $(M_1,\ F) \to (X, S)$ be a map of pairs such that for some $x \in F$

$$f_*\pi_1(M_1,\ x) \subset \pi_1(S,\ f(x))\ .$$

Then f is homotopic under a deformation constant on F to a map into S.

Proof. We wish to define a map $H: (N \times I, F \times I) \to (X, S)$ such that

(1) $H(n, 0) = f(n)$ for $n \in N$

(2) $H(n, t) = f(n)$ for $n \in F, t \in I$

(3) $H(N \times \{1\}) \subset S$.

Of course such an H will be the desired homotopy.

Let N^i be the i-skeleton of some subdivision of the pair (N, F) for $i = 1, 2, 3$. We define H to satisfy (1) and (2) above on

$$F \times I \cup N \times \{0\} .$$

If α is any arc embedded in N^1 which meets F in its endpoints, the arc $f(\alpha)$ can be deformed modulo its boundary to lie in S. Thus H can be extended to $\alpha \times I$. It follows that H can be extended to $N^1 \times I$. If D is a disk embedded in N^2 and meeting N^1 in $\mathrm{bd}(D)$, we have defined H on $\mathrm{bd}(D) \times I \cup D \times \{0\}$. Since $H(\mathrm{bd}(D) \times \{1\})$ is nullhomotopic in X, it is nullhomotopic in S since S is incompressible. It follows that if D is not contained in F, we may extend H to $D \times \{1\}$. Since $\pi_2(X) = 0$, H can be extended to $D \times I$. It follows that H can be extended to $N^2 \times I$. Similarly we can extend H to $N^3 \times I$ since $\pi_2(S) = 0$ and $\pi_3(X) = 0$. This completes the proof of the lemma.

Proof of Theorem 1. It follows from generalization 1 in [9] that we can replace finitely many prime homotopy 3-cells in M and obtain an irreducible 3-manifold. Since an incompressible surface can be made to miss a finite collection of disjoint 2-spheres, we may assume that M is irreducible. It is a consequence of Theorem 1 in [2] that M does not admit any embeddings of the projective plane since $\pi_2(M) = 0$ and $\pi_1(M)$ is not finite.

Let (M_{A_j}, q_j) be the covering space of M associated with $A_j \subset \pi_1(M, x)$ for $j = 1, 2$. Let $g: (S, y) \to (M, x)$ be a map such that

$$g_* \pi_1(S, y) = A_1 \cap A_2 .$$

Let $g_j: S \to M_{A_j}$ be a map covering g, i.e., $q_j g_j = g: S \to M$. Let X_j be the mapping cylinder of g_j over M_{A_j}, i.e., X_j is the union of M_{A_j} with $S \times I$ with identification $g_j(n) = (n, 0)$ for n in S and $j = 1, 2$. Let $X = X_1 \cup X_2$ identifying $(n, 1)$ in X_1 with $(n, 1)$ in X_2 for n in S. As was shown in [2] $\pi_i(X) = \pi_i(X_j) = 0$ for $j = 1, 2$ and $i \geqq 2$. Also $\pi_1(X) \cong \pi_1(M)$.

Let $G: X \to M$ be defined by

(1) $G \mid M_{A_j} = q_j$ for $j = 1, 2$

(2) $G(n, t) = g(n)$ for (n, t)
in $S \times I \subset X_j$ $j = 1, 2$. Then $G_*\colon \pi_1(X) \to \pi_1(M)$ is an isomorphism. We denote $S \times \{1\} \subset X$ by S.

We wish to construct a map $\hat{G}\colon M \to X$ such that

(1) $\hat{G}_*\colon \pi_1(M) \to \pi_1(X)$ is G_*^{-1}

(2) $\hat{G}(\mathrm{bd}(M))$ does not meet $S \subset X$.

Let $\mathrm{bd}(M) = \bigcup_{k=1}^m F_k$ where F_k is a closed connected surface. By assumption $\rho_{k*}\colon \pi_1(F_k) \to \pi_1(M)$ is conjugate to a subgroup either of A_1 or A_2. We assume A_1. It follows that there is a map $\hat{\rho}_k\colon F_k \to M_{A_1}$ covering ρ_k. We define $\hat{G}(F_k) = \hat{\rho}_k \rho_k^{-1}$.

Let $\{\alpha_k\colon k = 1, \cdots, n\}$ be a collection of simple arcs in X such that

(1) $G(\alpha(\alpha_{k_0}) \cap G(\alpha_{k_1}) = x$ for $k_0 \neq k_1$

(2) α_k runs from y in S to a point in $\hat{\rho}(F_k)$

(3) $G(\alpha_k)$ is a simple arc.

We extend $\hat{G}$ to $G(\alpha_k)$ to be any homeomorphism onto α_k. Note that $\hat{G}$ has been defined on $Y = \mathrm{bd}(M) \cup \bigcup_{k=1}^n G(\alpha_k)$ such that for each loop $l \subset Y$ based at x $(G\hat{G})_*[l] = [l]$ in $\pi_1(M, x)$. Thus we can extend G by using well known techniques so that $\hat{G}_* = G_*^{-1}$ since G_*^{-1} is an isomorphism and $\pi_i(X) = 0$ for $i \geqq 2$.

We have now established the existence of the desired map $\hat{G}$. It is a consequence of Lemma 1 that we may assume $\hat{G}^{-1}(S)$ is an incompressible surface in M. Denote the components of $\hat{G}^{-1}(S)$ by S_i for $i = 1, \cdots, m$. Since S_i and S are incompressible in M and X respectively,

$$(\hat{G} \mid S_i)_*\colon \pi_1(S_i) \longrightarrow \pi_1(S) \subset \pi_1(X)$$

is one-to-one. It is a consequence of Theorem 1 in [5] that $\hat{G} \mid S_i$ is homotopic to a covering map. Thus one can assume that $\hat{G} \mid S_i$ is a local homeomorphism. (One can change $\hat{G}$ on a small neighborhood of S_i to achieve this result.)

We may choose any point z in M and have that

$$\hat{G}_*\colon \pi_1(M, z) \longrightarrow \pi_1(X, \hat{G}(z))$$

is an isomorphism. Suppose $\hat{G} \mid S_i$ is not a homeomorphism and z in S_i. Let Φ be the isomorphism of $\pi_1(M, z)$ onto $\pi_1(X, \hat{G}(z))$ induced by $\hat{G}$. Then $\Phi(\rho_*\pi_1(S_i, z)) \subsetneqq \rho_*\pi_1(S, \hat{G}(z))$. Since $\Phi^{-1}(\rho_*\pi_1(S, \hat{G}(z))$ is a subgroup of $\pi_1(M, z)$ properly containing $\rho_*\pi_1(S_i, z)$ we would have by Theorem 6 in [7] that S_i bounds a twisted line bundle $N \subset M$. One can easily show using the techniques in [7], as has been done in [4], that $p_*\pi_1(N, z)$ may be taken to be $\Phi^{-1}\, \rho_*\pi_1(S, \hat{G}(z))$. It is now a consequence of Lemma 3 that we may assume that $\hat{G} \mid S_i$ is a homeomorphism for $i = 1, \cdots, m$.

As was shown in the proof of Theorem 1 in [3] every pair of components S_1 and S_2 in $\hat{G}^{-1}(S)$ are homotopic. Thus by Lemma 2, $S_1 \cup S_2$ bounds an $S \times I$ embedded in M. As was shown in the proof of Theorem 1 in [3] we may assume that there is a single component S_1 in $\hat{G}^{-1}(S)$ and S_1 is homeomorphic to S. The desired algebraic result now follows as in the proof of Theorem 1 in [3] and Theorem 1 is now established.

4. An application. In [10] Neuwirth asks, "Every knot group contains the group $(a, b; [a, b])$. This subgroup may be obtained from the natural inclusion of the fundamental group of a nonsingular torus in the knot group. Suppose a knot contains the group of a closed surface of genus g. Does there exist a nonsingular closed surface of genus g whose fundamental group is injected monomorphically into the knot group by the natural inclusion?".

W. Heil has shown in [6] that, if the subgroup in question is normal, such a surface does not exist.

THEOREM 2. *A knot complement admits an incompressible embedding of a closed surface of genus $g > 1$ if and only if its fundamental group splits across the group of the surface in question and said splitting preserves the peripheral structure of the fundamental group of the knot complement.*

Proof. Since any closed surface embedded in S^3 separates S^3, one half of the theorem follows from Van Kampen's theorem. The other half follows from Theorem 1.

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