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RADIAL QUASIHARMONIC FUNCTIONS

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A function s on a Riemannian manifold is called quasiharmonic if it satisfies $\Delta s = 1$, where Δ is the Laplace-Beltrami operator $d\delta + \delta d$. Existence of quasiharmonic functions with various boundedness properties has thus far been investigated by means of useful implicit tests. We now ask: Can such functions be formed by direct construction, in a manner accessible to computation if need be?

1. We shall present our approach to the problem in the setup of a Riemannian N-ball

$$B_\alpha = \{r < 1 \,|\, ds\} \tag{1}$$

endowed with the generalized Poincaré metric

$$ds = \lambda(r)\,|dx|,\ \lambda(r) = (1 - r^2)^\alpha,\ \alpha \in \boldsymbol{R}\ , \tag{2}$$

where $r = |x|$, $x = (x^1, \cdots, x^N)$. In [16] we proved that there exist bounded quasiharmonic functions on B_α if and only if $\alpha \in (-1, 1/(N-2))$. We shall now show that this in turn is necessary and sufficient for the boundedness of an explicitly constructed function $s(r)$, given in No. 3 below. *Thus the boundedness of this single function characterizes the existence of bounded quasiharmonic functions on B_α.*

We shall call, for brevity, a function radial if it depends on r only. A simple consequence of our result is that there exist bounded radial quasiharmonic functions if and only if there exist bounded quasiharmonic functions.

We expect that our approach is extendable to other classes of quasiharmonic and biharmonic functions as well, and to other Riemannian manifolds which are invariant under rotation. In particular, *there exist negative radial quasiharmonic functions on every B_α.*

2. The proof of our main result will be divided into Lemmas 1–6. We start by formulating the equation:

LEMMA 1. *A function $s(r)$ satisfies*

$$\Delta s = 1 \tag{3}$$

on B_α if and only if

$$s'' + \left(\frac{N-1}{r} - \frac{2(N-2)\alpha r}{1-r^2}\right)s' + (1-r^2)^{2\alpha} = 0\ . \tag{4}$$

Proof. The metric tensor (g_{ij}) is diagonal, with elements

$$\lambda^2, \lambda^2 r^2, \lambda^2 r^2 \varphi_1, \cdots, \lambda^2 r^2 \varphi_{N-2} ,$$

where $\varphi_1, \cdots, \varphi_{N-2}$ are functions of the coordinate angles $\theta^1, \cdots, \theta^{N-1}$. We set $\varphi = (\varphi_1 \cdots \varphi_{N-2})^{1/2}$, and have $\sqrt{g} = \lambda^N r^{N-1}\varphi$, $g^{rr} = \lambda^{-2}$, and

$$\begin{aligned} \Delta s &= -\frac{1}{\sqrt{g}}\frac{\partial}{\partial r}(\sqrt{g}\, g^{rr} s') \\ &= -\lambda^{-2}\left[s'' + \left(\frac{N-1}{r} + \frac{(N-2)\lambda'}{\lambda}\right)s'\right], \end{aligned}$$

hence the lemma.

For convenience in later calculation, we rewrite (4) in the form:

$$\begin{aligned} r^2(1-r^2)s'' + r[(N-1)(1-r^2) \\ - 2(N-2)\alpha r^2]s' + r^2(1-r^2)^{2\alpha+1} = 0 . \end{aligned} \tag{4'}$$

3. We are ready to give the function $s(r)$ referred to in the introduction. Here and later $\sum_m^n$ with $n < m$ will mean 0.

LEMMA 2. *Equation* (3) *is satisfied by the function*

$$s(r) = -\sum_{i=0}^{\infty} b_i r^{2i+2} , \tag{5}$$

where

$$b_0 = \frac{1}{2N} \tag{6}$$

and for $i > 0$,

$$b_i = \frac{1}{2N}\prod_{j=1}^{i} p_j + \sum_{j=1}^{i-1} q_j \prod_{k=j+1}^{i} p_k + q_i , \tag{7}$$

with

$$p_i = \frac{2i[2i + (N-2)(2\alpha+1)]}{(2i+2)(2i+N)} , \tag{8}$$

$$q_i = \frac{\prod_{j=1}^{i}(j - 2\alpha - 2)j^{-1}}{(2i+2)(2i+N)} . \tag{9}$$

Proof. Substitution of (5) into (4′) gives

$$\begin{aligned} &-\sum_{i=0}^{\infty}(1-r^2)(2i+2)(2i+1)b_i r^{2i+2} - \sum_{i=0}^{\infty}(N-1)(2i+2)b_i r^{2i+2} \\ &+ \sum_{i=0}^{\infty}[(N-1) + 2(N-2)\alpha](2i+2)b_i r^{2i+4} + r^2 \\ &+ r^2\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{j-2\alpha-2}{j}\right)r^{2i} = 0 . \end{aligned}$$

On changing by unity the summation index in the coefficients of r^{2i+4} we obtain

$$\sum_{i=0}^{\infty}(2i+2)(2i+N)b_i r^{2i+2} = \sum_{i=1}^{\infty} 2i[2i+(N-2)(2\alpha+1)]b_{i-1}r^{2i+2}$$
$$+ r^2 + \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i}\frac{j-2\alpha-2}{j}\Big)r^{2i+2} = 0 .$$

We equate the coefficient of r^2 to 0 and have (6). The coefficient of r^{2+2i} for $i > 0$ gives in notation (8), (9)

$$b_i = p_i b_{i-1} + q_i , \tag{10}$$

which by induction yields (7).

4. We recall that we are only interested in $\alpha \in (-1, 1/(N-2))$, and we shall at this point introduce the condition $\alpha < 1/(N-2)$. To estimate b_i given by (7), we start with Πp_j. Let i_0 be any integer such that

$$i_0 \geqq 1 - \alpha(N-2) - \frac{N}{2} . \tag{11}$$

Further conditions on i_0 will be imposed in the course of our reasoning.

LEMMA 3. *For $\alpha < 1/(N-2)$ and $i > i_0$,*

$$\prod_{j=i_0+1}^{i} p_j < \frac{i_0+1}{i+1}\Big(\frac{2i_0+N+2}{2i+N+2}\Big)^{1-\alpha(N-2)} . \tag{12}$$

Proof. In p_i, consider first the factor

$$\delta_i = \frac{2i+(N-2)(2\alpha+1)}{2i+N} = 1 - \frac{2[1-\alpha(N-2)]}{2i+N} .$$

For $\alpha < 1/(N-2)$ and $i > i_0$, we have $0 < \delta_i < 1$ and

$$\log \delta_i < -\frac{2[1-\alpha(N-2)]}{2i+N} < 0 .$$

Therefore

$$\log \prod_{j=i_0+1}^{i} \delta_j < -2[1-\alpha(N-2)]\int_{i_0+1}^{i+1}\frac{dx}{2x+N}$$

and

$$\prod_{j=i_0+1}^{i} \delta_j < \Big(\frac{2i_0+N+2}{2i+N+2}\Big)^{1-\alpha(N-2)} .$$

In view of

$$p_i = \frac{i}{i+1}\delta_i ,$$

the lemma follows.

5. To proceed with the estimation of b_i we now utilize also the condition $\alpha > -1$ and impose on i_0 the additional requirement

$$i_0 \geqq 2(\alpha + 1) . \tag{13}$$

In the sequel c will stand for a positive constant, not always the same.

LEMMA 4. *For* $\alpha \in (-1, 1/(N-2))$ *and* $i > i_0$,

$$|q_i| < \frac{c}{(2i+2)(2i+N)}\left(\frac{i_0+1}{i+1}\right)^{2(\alpha+1)} . \tag{14}$$

Proof. For $j > i_0$

$$0 < 1 - \frac{2(\alpha+1)}{j} < 1 ,$$

and therefore

$$\log \prod_{j=i_0+1}^{i}\left(1 - \frac{2(\alpha+1)}{j}\right) < -2(\alpha+1)\sum_{i_0+1}^{i}\frac{1}{j} < -2(\alpha+1)\int_{i_0+1}^{i+1}\frac{dx}{x} .$$

This gives

$$\prod_{j=i_0+1}^{i}\frac{j-2\alpha-2}{j} < \left(\frac{i_0+1}{i+1}\right)^{2(\alpha+1)} ,$$

hence (14).

6. We now come to the main step in estimating b_i. It will be necessary to consider separately the cases $\alpha \in (-1/N, 1/(N-2))$, $\alpha = -1/2$, and $\alpha \in (-1, 0) - \{-1/2\}$.

LEMMA 5. *For* $\alpha \in (-1/N, 1/(N-2))$, *and* $i > i_0$,

$$|b_i| < c\left(\frac{1}{i}\right)^{2-\alpha(N-2)} + d\left(\frac{1}{i}\right)^{(3/2)-(1/2)\alpha(N-2)} + e\left(\frac{1}{i}\right)^{2(\alpha+2)} , \tag{15}$$

where c, d, e *are positive constants.*

Proof. By (7)

$$b_i = b_{i_0} \prod_{j=i_0+1}^{i} p_j + \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^{i} p_k + q_i \,, \tag{16}$$

and by (12)

$$\left| b_{i_0} \prod_{j=i_0+1}^{i} p_j \right| < \frac{c}{i+1} \left(\frac{1}{2i+N+2} \right)^{1-\alpha(N-2)} < c \left(\frac{1}{i} \right)^{2-\alpha(N-2)} . \tag{17}$$

In view of (12) and (14) we have

$$\begin{aligned} \left| q_j \prod_{k=j+1}^{i} p_k \right| &< \frac{c}{(2j+2)(2j+N)} \left(\frac{1}{j+1} \right)^{2(\alpha+1)} \\ &\quad \cdot \frac{j+1}{i+1} \left(\frac{2j+N+2}{2i+N+2} \right)^{1-\alpha(N-2)} . \end{aligned} \tag{18}$$

For $\alpha \in (-1/N, 1/(N-2))$,

$$1 - \alpha(N-2) < 2(1+\alpha) .$$

We therefore may and do require of i_0 further that for $j > i_0$

$$\frac{(2j+N+2)^{1-\alpha(N-2)}}{(j+1)^{2(1+\alpha)}} < 2^{1-\alpha(N-2)} .$$

We obtain

$$\left| \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^{i} p_k \right| < \frac{c}{2i+2} \left(\frac{1}{2i+N+2} \right)^{1-\alpha(N-2)} \sum_{j=i_0+1}^{i-1} \frac{1}{2j+N}$$

where

$$\sum_{j=i_0+1}^{i-1} \frac{1}{2j+N} < \int_{i_0}^{i-1} \frac{dx}{2x+N} = \frac{1}{2} \log \frac{2i+N-2}{2i_0+N} .$$

Accordingly, we set on i_0 the additional condition that for $i > i_0$

$$\left(\frac{1}{i} \right)^{(1/2)[1-\alpha(N-2)]} \log \frac{2i+N+2}{2i_0+N} < 1 .$$

Then

$$\left| \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^{i} p_k \right| < c \left(\frac{1}{i} \right)^{(3/2)-(1/2)\alpha(N-2)} . \tag{19}$$

A bound for the last term in (16) is immediate by (14):

$$|q_i| < c \left(\frac{1}{i} \right)^{2(\alpha+2)} . \tag{20}$$

We combine (16), (17), (19), and (20), and obtain (15).

7. We are ready to state:

LEMMA 6. *For $\alpha \in (-1, 1/(N-2))$, the function $s(r)$ of Lemma 2 is bounded quasiharmonic.*

In fact, for $\alpha \in (-1/N, 1/(N-2))$, all three exponents in (15) are > 1, and therefore

$$|s(r)| = \left|\sum_{i=0}^{\infty} b_i r^{2i+2}\right| < \sum_{i=0}^{\infty} |b_i| < \infty \,. \tag{21}$$

The case $\alpha = -1/2$ is simple, as all $q_i = 0$, and by (8)

$$|b_i| = |b_{i_0}| \prod_{j=i_0+1}^{i} p_j < |b_{i_0}| \frac{(2i_0+2)^2}{(2i+2)(2i+N)} < c\left(\frac{1}{i}\right)^2 ,$$

whence $\sum_0^\infty |b_i| < \infty$.

It remains to consider the case $\alpha \in (-1, 0) - \{-1/2\}$. We obtain at once

$$p_k < \frac{2k(2k+N-2)}{(2k+2)(2k+N)} ,$$

$$\prod_{j+1}^{i} p_k < \frac{(2j+2)(2j+N)}{(2i+2)(2i+N)} ,$$

and by (14)

$$|q_j| < \frac{c}{(2j+2)(2j+N)} \cdot \left(\frac{1}{j+1}\right)^{2(\alpha+1)}$$

for $j > i_0$. Therefore

$$\sum_{j=i_0+1}^{i-1} \left| q_j \prod_{k=j+1}^{i} p_k \right| < \frac{c}{(2i+2)(2i+N)} \int_{i_0}^{i-1} \frac{dx}{(x+1)^{2(\alpha+1)}} ,$$

where the integral has the value

$$\frac{1}{-2\alpha-1}[i^{-2\alpha-1} - (i_0+1)^{-2\alpha-1}]$$

since $\alpha \neq -1/2$. As a consequence

$$\sum_{j=i_0+1}^{i-1} \left| q_j \prod_{k=j+1}^{i} p_k \right| < c\left(\frac{1}{i}\right)^{3+2\alpha} + d\left(\frac{1}{i}\right)^2 . \tag{22}$$

Similarly

$$\left| b_{i_0} \prod_{k=i_0+1}^{i} p_k \right| < c\left(\frac{1}{i}\right)^2 \tag{23}$$

and

$$|q_i| < c\left(\frac{1}{i}\right)^{2\alpha+4} . \tag{24}$$

Since all exponents in (22)-(24) are > 1, it follows again by (16) and (21) that the function $s(r)$ is bounded.

8. We have established our result:

THEOREM. *There exist bounded quasiharmonic functions on the Riemannian ball B_α if and only if the function $s(r)$ of Lemma 2 is bounded.*

In fact, we know that there exist bounded quasiharmonic functions on B_α if and only if $\alpha \in (-1, 1/(N-2))$ (Sario-Wang [16]). This together with Lemma 6 gives the theorem.

A simple consequence is perhaps worth stating. Let R be the family of *radial* functions, characterized by the dependence on r only. Denote by O_{QBR} and O_{QB} the classes of Riemannian manifolds which do not carry bounded radial quasiharmonic functions, or bounded quasiharmonic functions, respectively, and set $B = \{\cup B_\alpha | \alpha \in \boldsymbol{R}\}$.

COROLLARY 1. $B \cap O_{QBR} = B \cap O_{QB}$.

That is, there exist bounded radial quasiharmonic functions on B_α if and only if there exist bounded quasiharmonic functions.

COROLLARY 2. $B \cap O_{QNR} \neq \varnothing$.

For $\alpha \in (-1, 1/(N-2))$, we have $s - \sup_{B_\alpha} |s| \in QNR$. For all α, it is readily seen that the function

$$-1 - \int_0^r \int_0^\sigma \left(\frac{\rho}{\sigma}\right)^{N-1} \frac{(1-\rho^2)^{N\alpha}}{(1-\sigma^2)^{(N-2)\alpha}} d\rho d\sigma$$

is radial, negative, and quasiharmonic.

REFERENCES

1. Y. K. Kwon, L. Sario and B. Walsh, *Behavior of biharmonic functions on Wiener's and Royden's compactifications*, Ann. Inst. Fourier (Grenoble), **21** (1971), 217-226.

2. M. Nakai, *Dirichlet finite biharmonic functions on the plane with distorted metric*, (to appear).

3. M. Nakai and L. Sario, *Biharmonic classification of Riemannian manifolds*, Bull. Amer. Math. Soc., **77** (1971), 432-436.

4. ———, *Quasiharmonic classification of Riemannian manifolds*, Proc. Amer. Math. Soc., **31** (1972), 165-169.

5. ———, *Dirichlet finite biharmonic functions with Dirichlet finite Laplacians*, Math. Z., **122** (1971), 203-216.
6. ———, *A property of biharmonic functions with Dirichlet finite Laplacians*, Math. Scand., **29** (1971), 307-316.
7. ———, *Existence of Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn., (to appear).
8. ———, *Existence of bounded biharmonic functions*, J. Reine Angew. Math., (to appear).
9. ———, *Existence of bounded Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn. A. I., **505** (1972), 1-12.
10. ———, *Biharmonic functions on Riemannian manifolds*, Continuum Mechanics and Related Problems of Analysis, Nauka, Moscow, (1972), 329-335.
11. H. O'Malla, *Dirichlet finite biharmonic functions on the unit disk with distorted metrics*, Proc. Amer. Math. Soc., **32** 1972, 521-524.
12. L. Sario, *Lectures on biharmonic and quasiharmonic functions on Riemannian manifolds*, Duplicated lecture notes, 1969-70, University of California, Los Angeles.
13. L. Sario and C. Wang, *The class of (p, q)-biharmonic functions*, Pacific J. Math., **41** (1972), 799-808.
14. ———, *Counterexamples in the biharmonic classification of Riemannian 2-manifolds*, (to appear).
15. ———, *Generators of the space of bounded biharmonic functions*, Math. Z., **127** (1972), 273-280.
16. ———, *Quasiharmonic functions on the Poincaré N-ball*, Rend. Mat. (to appear).
17. ———, *Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball*, Pacific J. Math., (to appear).
18. L. Sario, C. Wang and M. Range, *Biharmonic projection and decomposition*, Ann. Acad. Sci. Fenn. A. I., **494** (1971), 1-14.
19. C. Wang and L. Sario, *Polyharmonic classification of Riemannian manifolds*, Kyoto Math. J., **12** (1972), 129-140.

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