PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

PROXIMITY CONVERGENCE STRUCTURES

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In this paper the notion of proximity convergence structures is introduced. These constitute a layer between Cauchy structures and uniform convergence structures (in the sense of Cook and Fischer [1]). They are a natural generalization of proximity structures. A study of the relations among these various structures constitutes §§2 and 3. In §4, compact extensions for a special class of proximity convergence spaces are constructed, and a characterization of these is obtained. They satisfy a mapping property with respect to compact T_2 proximity convergence spaces which satisfy a strong regularity condition. One problem left open is the obtaining of a more reasonable definition of regularity for these spaces.

1. Proximity convergence structures. A proximity convergence structure is the natural analogue, in the context of convergence spaces, of a proximity structure. Here convergence space is used in the sense of Fischer [3], and proximity in the sense of Efremovič and Smirnov. A proximity convergence structure is a filter of proximity-like orders on a set X, and satisfies a composition property. If the filter is principal then it corresponds to an ordinary proximity.

The notation used is largely that in Cook and Fischer [1]. By $\mathscr{O}(X)$ is meant the set of all symmetric topogenous orders on X.

So a relation $<$ on the subsets of X is in $\mathscr{O}(X)$ iff it satisfies the following:

(ST 1) $\phi < A < X$ for $A \subseteqq X$;

(ST 2) $A < B \Rightarrow A \subseteqq B$;

(ST 3) if $A' \subseteqq A < B \subseteqq B'$ then $A' < B'$;

(ST 4) if $A < C$ and $B < C$ then $A \cup B < C$; also if $C < A$ and $C < B$ then $C < A \cap B$;

(ST 5) $A < B$ then $X \backslash B < X \backslash A$.

DEFINITION 1. A proximity convergence structure on a set X is a family $\mathscr{P} \subseteqq \mathscr{O}(X)$ satisfying

(P 1) if $<_1, <_2 \in \mathscr{P}$ then $<_1 \cap <_2 \in \mathscr{P}$;

(P 2) if $< \in \mathscr{P}$ then $< \circ < \in \mathscr{P}$

(P 3) if $< \in \mathscr{P}$ and $< \subseteqq <' \in \mathscr{O}(X)$ then $<' \in \mathscr{P}$.

We will call $(X, \mathscr{P})$ a *proximity convergence space*. Both concepts will be abbreviated by p.c.s.

REMARK AND DEFINITION 2. We say one p.c.s. on X is *less* than another if it *contains* it. Under this ordering the set of all p.c.s.'s on X is a complete lattice. The largest member is $\{\subseteqq\}$ and corresponds to the discrete topology on X. The smallest is $\mathscr{O}(X)$, which yields the indiscrete topology. (See Definition 30.) The intersection of any family of p.c.s.'s on X is also a p.c.s., so that suprema are easily described.

DEFINITION 3. If $\mathscr{G}$ is a nonempty subset of $\mathscr{O}(X)$ then clearly there is a smallest p.c.s. $[\mathscr{G}]$ containing G. We will call $\mathscr{G}$ a *base*, provided $[\mathscr{G}]$ consists of refinements of orders in $\mathscr{G}$. In case $[\mathscr{G}]$ consists of refinements of finite intersections of orders in $\mathscr{G}$, we call $\mathscr{G}$ a *subbase* for $[\mathscr{G}]$.

As in the uniform case, ordinary proximity relations on X correspond to "principal" p.c.s's.; i.e., those which have a single element as base.

THEOREM 4. *Let* $\ll \in \mathscr{O}(X)$. *Then* $\ll$ *is a proximity on* X *iff* $\{\ll\}$ *is a base for a* p.c.s. *on* X.

Proof. Let $\mathscr{S}$ denote the set of refinements (in $\mathscr{O}(X)$) of $\ll$. If $\ll$ is a proximity relation then $\ll = \ll \circ \ll$ and hence $\mathscr{S}$ satisfies (P 2). The other properties are clearly satisfied. Conversely, if $\mathscr{S}$ is a p.c.s. then $\ll \circ \ll \in \mathscr{S}$ and so $\ll$ is dense. Clearly then $\ll$ is a proximity relation.

DEFINITION 5. If $\subset\subset$ is a proximity on X we will call $[\subset\subset]$ a *proximity structure*.

2. Relation with uniform convergence structure. As with ordinary proximities, each uniform convergence structure (abbreviated u.c.s.) gives rise to a p.c.s. This allows us to divide the uniform convergence structures into proximity classes. Each class contains a smallest member, which is *strongly bounded*. This last is a condition stronger than total boundedness, and more satisfying in that every proximity class contains a *unique* strongly bounded member. (A class can contain more than one totally bounded member.) Moreover if the p.c.s. is a *proximity* structure than the strongly bounded member in its class is a uniform structure; the other totally bounded uniform convergence structures in the class will not be uniform structures.

DEFINITION 6. A *standard* filter on $X \times X$ is a symmetric filter $\Phi \subseteqq [\Delta]$, the filter generated by the diagonal on X. For Φ a standard filter we define

$$A <_\Phi B \text{ iff } H(A) \subseteq B \text{ for some } H \in \Phi .$$

This imitates the usual way a proximity is obtained from a uniformity. Notice that if Φ is standard, $<_\Phi \in \mathscr{O}(X)$.

If $\mathscr{J}$ is a uniform convergence structure (abbreviated u.c.s.) we define

$$\mathscr{B}_{\mathscr{J}} = \{<_\Phi : \Phi \text{ is a standard filter in } \mathscr{J}\} .$$

It turns out that $\mathscr{B}_{\mathscr{J}}$ is a base for a p.c.s. $\mathscr{P}_{\mathscr{J}}$ on X.

LEMMA 7. *Let Φ and Ψ be standard filters on $X \times X$.*
(i) *If $\theta = \Phi \cap \Psi$ then $<_\theta = <_\Phi \cap <_\Psi$*
(ii) *If $\mathscr{X} = \Phi \circ \Phi$ then $<_{\mathscr{X}} = <_\Phi \circ <_\Phi$.*

Proof. Straightforward.

THEOREM 8. *If $\mathscr{J}$ is a* u.c.s. *on X then $\mathscr{B}_{\mathscr{J}}$ is a base for a p.c.s. $\mathscr{P}_{\mathscr{J}}$ on X. If $\mathscr{J}$ is generated by a uniformity $\mathscr{X}$ then $\mathscr{P}_{\mathscr{J}}$ is a proximity structure generated by $<_{\mathscr{X}}$.*

Proof. From the preceding lemma it is clear that $\mathscr{B}_{\mathscr{P}}$ is a base for a p.c.s. on X. Suppose $\mathscr{X}$ is a uniformity which generates $\mathscr{J}$. Then for $< \in \mathscr{P}_{\mathscr{J}}$ we have $<_{\mathscr{X}} \subseteq <$. Hence $\{<_{\mathscr{X}}\}$ is a base for $\mathscr{P}_{\mathscr{J}}$.

DEFINITION 9. If $\mathscr{C}$ is a cover of X we define $H_{\mathscr{C}} = \bigcup \{C \times C: C \in \mathscr{C}\}$. If $\mathscr{C}$ is finite then any entourage which contains $H_{\mathscr{C}}$ is said to be *strongly bounded*. A *filter* Φ on $X \times X$ is *strongly bounded* iff it consists of strongly bounded entourages. A u.c.s. is *strongly bounded* iff it has a base of strongly bounded filters.

REMARK 10. Notice that for uniform structures strongly bounded is equivalent to totally bounded. However, in the case of a u.c.s. total-boundedness is a weaker condition.

THEOREM 11. *Every strongly bounded* u.c.s. *is totally bounded.*

Proof. Let $\mathscr{J}$ be a strongly bounded u.c.s. on X, and let $\mathscr{U}$ be an ultrafilter on X. Let Φ be any strongly bounded filter in $\mathscr{J}$. We claim that $\Phi \subseteq \mathscr{U} \times \mathscr{U}$.

Let $H \in \Phi$, and let $\mathscr{C}$ be a finite cover of X such that $H_{\mathscr{C}} \subseteq H$. Since $\mathscr{U}$ is an ultrafilter, $\mathscr{U} \cap \mathscr{C} \neq \emptyset$. But if $C \in \mathscr{U} \cap \mathscr{C}$ then $H \supseteq C \times C \in \mathscr{U} \times \mathscr{U}$.

THEOREM 12. *Let $\mathscr{J}$ be a* u.c.s. *on X. The following conditions* *are equivalent:*

(i) $\mathscr{J}$ *is strongly bounded;*

(ii) $\mathscr{J}$ *has at least one strongly bounded member;*

(iii) *The filter* $[\Delta]^*$ *of all strongly bounded entourages is in* $\mathscr{J}$.

Proof. Since $\mathscr{J} \neq \emptyset$, clearly (i) $\Rightarrow$ (ii). Now suppose (ii) holds. Note $[\Delta]^*$ is a filter. If Φ is any strongly bounded filter in $\mathscr{J}$ then $\Phi \subseteqq [\Delta]^*$.

Finally, assume $[\Delta]^* \in \mathscr{J}$. If $\Phi \in \mathscr{J}$ then $\Phi \cap [\Delta]^*$ is a strongly bounded filter in $\mathscr{J}$, and is contained in Φ.

LEMMA 13. *A strongly bounded* u.c.s. *is the smallest member of its proximity class.*

Proof. Let $\mathscr{J}$ and $\mathscr{K}$ be u.c.s.'s on X and suppose $\mathscr{J}$ is strongly bounded, with $\mathscr{P}_{\mathscr{J}} = \mathscr{P}_{\mathscr{K}}$. We wish to show $\mathscr{K} \subseteqq \mathscr{J}$. Let $\Phi \in \mathscr{K}$ and let $\Psi = \Phi \cap \Phi^{-1} \cap [\Delta]$. Then $<_{\Psi} \in \mathscr{P}_{\mathscr{K}} = \mathscr{P}_{\mathscr{J}}$, so we can choose $\theta \in \mathscr{J}$ so that $<_{\theta} \subseteqq <_{\Psi}$. Let $\theta^* = \theta \cap [\Delta]^*$. We claim that $\theta^* \circ \theta^* \subseteqq \Phi$.

Let $H \in \theta^*$, and let $\mathscr{C}$ be a finite cover of X with $H_{\mathscr{C}} \subseteqq H$. Then for $C \in \mathscr{C}$ we have $C <_{\theta} H(C)$. Let $K_C \in \Psi$ such that $K_C(C) \subseteqq H(C)$, and define K to be the intersection of the K_C's. Then $K \in \Phi$. We claim $K \subseteqq H \circ H$.

Let $(x, y) \in K$. Choose $C \in \mathscr{C}$ so $x \in C$. Then $y \in K_C(C) \subseteqq H(C)$. Set $c \in C$ with $(c, y) \in H$. Then $(x, c) \in C \times C \subseteqq H$. Hence $(x, y) \in H \circ H$.

THEOREM 14. *Let* $\mathscr{P}$ *be a* p.c.s. *on* X, *and define*

$$\mathscr{B}_{\mathscr{P}} = \{\Phi\colon \Phi \text{ is standard and } <_{\Phi} \in \mathscr{P}\}\,.$$

Then $\mathscr{B}_{\mathscr{P}}$ *is a base for a strongly bounded* u.c.s. $\mathscr{J}_{\mathscr{P}}$ *in the proximity class of* $\mathscr{P}$.

Proof. If $\Phi = [\Delta]$ then $<_{\Phi} = \subseteqq$, so $[\Delta] \in \mathscr{B}_{\mathscr{P}}$. From Lemma 7 it is clear that $\mathscr{B}_{\mathscr{P}}$ is a base for a u.c.s. $\mathscr{J}_{\mathscr{P}}$.

(1) $\mathscr{J}_{\mathscr{P}}$ is strongly bounded.

Let $\theta = [\Delta]^*$. We will show $<_{\theta} = \subseteqq$, so that $\theta \in \mathscr{B}_{\mathscr{P}}$. Let $A \subseteqq B$, and define $\mathscr{C} = \{B, X\backslash A\}$. Then $H_{\mathscr{C}} \in [\Delta]^*$ and $H_{\mathscr{C}}(A) \subseteqq B$. Thus $A <_{\theta} B$.

(2) $\mathscr{J}_{\mathscr{P}}$ is in the proximity class of $\mathscr{P}$.

Clearly the p.c.s. determined by $\mathscr{J}_{\mathscr{P}}$ is contained in $\mathscr{P}$. Now let $< \in \mathscr{P}$. We define

$$\mathscr{A} = \{H \subseteqq X \times X\colon A < H(A) \text{ if } A \subseteqq X\}$$
$$\mathscr{B} = \{H_{\mathscr{C}}\colon \exists A, B \subseteqq X \text{ with } A <^2 B \text{ and } \mathscr{C} = \{B, X\backslash A\}\,.$$

Notice $\mathscr{B} \subseteqq [\Delta]$, so $\mathscr{B}$ is a subbase for a proper filter Φ on $X \times X$. Since each member of $\mathscr{B}$ is symmetric, clearly Φ is symmetric; hence Φ is standard. We will show $<^2 \subseteqq <_\Phi \subseteqq <$. If this holds, then $\Phi \in \mathscr{J}_\mathscr{P}$ and hence $<$ is in the p.c.s. induced by $\mathscr{J}_\mathscr{P}$.

If $A <^2 B$ we define $\mathscr{C} = \{B, X\backslash A\}$. Then $H_\mathscr{C} \in \Phi$, and $H_\mathscr{C}(A) \subseteqq B$. Thus $<^2 \subseteqq <_\Phi$. To show that $<_\Phi \subseteqq <$ it is sufficient to establish that $\Phi \subseteqq A$.

Let $H_i \in \mathscr{B}$ for $1 \leqq i \leqq n$ and suppose $\bigcap_i H_i \subseteqq H$. For each i, let $A_i <^2 B_i$ such that $H_i = H_{\mathscr{C}_i}$, where $\mathscr{C}_i = \{B_i, X\backslash A_i\}$. Choose D_i so $A_i < D_i < B_i$, and define $\mathscr{D}_i = \{D_i, X\backslash D_i\}$. Set $\mathscr{K} = \prod_i \mathscr{D}_i$, and for $k \in \mathscr{K}$ let $C_k = \bigcap_i k(i)$. Note the C_k's cover X.

Now let $E \subseteqq X$. We must show $E < H(E)$. This holds, provided $E \cap C_k < H(E)$ for $k \in \mathscr{K}$: we will actually show that if $E \cap C_k \neq \varnothing$ then $C_k < H(E)$.

Let $k \in \mathscr{K}$, with $E \cap C_k \neq \varnothing$. Define $h(i)$ to be B_i if $k(i) = D_i$, and $X\backslash A_i$ otherwise. Then $k(i) < h(i)$ for $1 \leqq i \leqq n$, and so $C_k < \bigcap_i h(i)$. We claim $\bigcap h(i) \subseteqq H(E)$.

Let $x \in \bigcap_i h(i)$, and pick $x_0 \in E \cap C_k$. We will show $(x_0, x) \in H$. Choose i, and suppose $k(i) = D_i$. Then $h(i) = B_i$, and so $(x_0, x) \in D_i \times B_i \subseteqq H_i$. Similarly if $k(i) = X\backslash D_i$ then x_0 and x are both in $X\backslash A_i$, and hence $(x_0, x) \in H$.

THEOREM 15. *If $\mathscr{P}$ is a proximity structure then $\mathscr{J}_\mathscr{P}$ is a uniform structure.*

Proof. Suppose $\ll$ generates $\mathscr{P}$. Let $\Phi \in \mathscr{J}_\mathscr{P}$ so $<_\Phi \subseteqq \ll$ and Φ is strongly bounded. We claim Φ^2 generates $\mathscr{J}_\mathscr{P}$.

Let $\Psi \in \mathscr{J}_\mathscr{P}$ and assume Ψ is standard. Then $<_\Psi \in \mathscr{P}$, so $<_\Phi \subseteqq <_\Psi$. Let $H \in \Phi$. Then we can choose $\mathscr{C}$ a finite cover of X such that $H_\mathscr{C} \subseteqq H$. For $C \in \mathscr{C}$ we have $C <_\Psi H(C)$. Pick $K \in \Psi$ so $K(C) \subseteqq H(C)$ for all C in $\mathscr{C}$. Then $K \subseteqq H^2$, so $H^2 \in \Psi$. This establishes that $\Phi^2 \subseteqq \Psi$.

EXAMPLE 16. We conclude this section with an example to show that a totally bounded u.c.s. need not be strongly bounded. Let τ be a compact T_2 convergence structure on a set X, and suppose that every finite intersection of convergent filters has a member with an infinite complement. For example, we would let τ be the usual topology on the closed unit interval. Let $\mathscr{J}$ be the u.c.s. generated by $\{\mathscr{F} \times \mathscr{F}: \mathscr{F}$ is convergent$\}$. Clearly $\mathscr{J}$ is totally bounded. We claim it is not strongly bounded.

Let $\Phi \in \mathscr{J}$. We will exhibit a member of Φ which is not strongly bounded. Let $\mathscr{F}_1, \cdots, \mathscr{F}_n$ be convergent filters with $(\bigcap_i \mathscr{F}_i \times \mathscr{F}_i) \cap [\Delta] \subseteqq \Phi$. Pick $F \in \bigcap_i \mathscr{F}_i$ so that X. F is infinite. Define $H = (F \times$

$F) \cup \Delta$. Note $H \in \Phi$.

Now let $\mathscr{C}$ be any cover X with $H_{\mathscr{C}} \subseteq H$. For $x \in X \backslash F$ let $C_x \in \mathscr{C}$ such that $x \in C_x$. Since $C_x \times C_x \subseteq H$, clearly $C_x = \{x\}$ for $x \notin F$. Thus $\mathscr{C}$ is infinite, and H is not strongly bounded.

3. Relation with Cauchy structures. In contrast to the classical case, a totally bounded Cauchy structure $\mathscr{C}$ can be induced by several different p.c.s.'s. However there always exist a smallest and a largest p.c.s. which induce $\mathscr{C}$. If $\mathscr{C}$ is uniform, the smallest p.c.s. associated with it is a proximity structure, but the largest need not be. We call the smallest p.c.s. yielding $\mathscr{C}$ a *saturated* p.c.s.

DEFINITION 17. A Cauchy structure on X is a family $\mathscr{C}$ of proper filters on X such that

(C 1) if $x \in X$ then $\dot{x} \in \mathscr{C}$;

(C 2) if $\mathscr{F}$ is a proper filter which contains a member of $\mathscr{C}$ then $\mathscr{F} \in \mathscr{C}$;

(C 3) if $\mathscr{F}, \mathscr{G} \in \mathscr{C}$ with $\mathscr{F} \vee \mathscr{G} \neq [\varnothing]$ then $\mathscr{F} \cap \mathscr{G} \in \mathscr{C}$.

Keller [4] has shown that $\mathscr{C}$ is a Cauchy structure on X iff it is the set of Cauchy filters for some u.c.s. on X. If $\mathscr{C}$ is induced by a *uniformity* we call $\mathscr{C}$ a *uniform* Cauchy structure. We say $\mathscr{C}$ is totally bounded iff every ultrafilter on X is in $\mathscr{C}$.

DEFINITION 18. For $\mathscr{F}$ a filter on X we define a relation $<_{\mathscr{F}}$ on X by $A <_{\mathscr{F}} B$ iff $A \subseteq B$ and B or $X \backslash A$ is in $\mathscr{F}$.

REMARK 19. Notice that $<_{\mathscr{F}}$ is in $\mathscr{O}(X)$. Also if $\Phi = (\mathscr{F} \times \mathscr{F}) \cap [\Delta]$ then $<_{\Phi} = <_{\mathscr{F}}$.

THEOREM 20. *Let $\mathscr{C}_{\mathscr{P}} = \{\mathscr{F} \colon <_{\mathscr{F}} \in \mathscr{P}\}$, where $\mathscr{P}$ is a* p.c.s. *on X. If $\mathscr{J}$ is any totally bounded* u.c.s. *in the proximity class of $\mathscr{P}$ then $\mathscr{C}_{\mathscr{P}}$ is the set of $\mathscr{J}$-Cauchy filters.*

Proof. Let $\mathscr{F}$ be a filter on X and define $\Phi = (\mathscr{F} \times \mathscr{F}) \cap [\Delta]$. If $\mathscr{F}$ is $\mathscr{J}$-Cauchy then $\Phi \in \mathscr{J}$, and so $<_{\mathscr{F}} = <_{\Phi} \in \mathscr{P}$. Hence $\mathscr{F} \in \mathscr{C}_{\mathscr{P}}$.

Conversely, suppose $\mathscr{F} \in \mathscr{C}_{\mathscr{P}}$. Then $<_{\Phi} = <_{\mathscr{F}} \in \mathscr{P} = \mathscr{P}_{\mathscr{J}}$ and so we can choose $\Psi \in \mathscr{J}$ with $<_{\Psi} \subseteq <_{\Phi}$. Let $\mathscr{U}$ be an ultrafilter containing $\mathscr{F}$. Then $\mathscr{U}$ is $\mathscr{J}$-Cauchy, and therefore $\Psi(\mathscr{U})$ is also $\mathscr{J}$-Cauchy. (By $\Psi(\mathscr{U})$ is meant the filter generated by all sets of the form $H(U)$, where $H \in \Psi$ and $U \in \mathscr{U}$. It is easy to check that $[\Psi \cap (\mathscr{U} \times \mathscr{U})]^3 \subseteq \Psi(\mathscr{U}) \times \Psi(\mathscr{U})$.)

We claim that $\Psi(\mathscr{U}) \subseteq \mathscr{F}$. Let $H \in \Psi$ and $U \in \mathscr{U}$. Then $U <_{\Psi} H(U)$, and since $<_\Psi \subseteqq <_\Phi$ we can choose $K \in \Phi$ with $K(U) \subseteqq H(U)$. Now pick $F \in \mathscr{F}$ so $F \times F \subseteqq K$. Then since $\mathscr{F} \subseteqq \mathscr{U}$ we have $F \cap U \neq \varnothing$, and so $F \subseteqq K(U)$. This establishes that $H(U) \in \mathscr{F}$, as desired.

REMARK 21. This theorem tells us that the totally bounded u.c.s.'s in the same proximity class all induce the same Cauchy structure.

DEFINITION 22. A p.c.s. $\mathscr{P}$ is compatible with a totally bounded Cauchy structure $\mathscr{C}$ iff $\mathscr{C} = \mathscr{C}_{\mathscr{P}}$.

NOTATION 23. Let $\mathscr{F}$ be a filter on X and let $< \varepsilon \mathscr{O}(X)$. Then

$$r_<(\mathscr{F}) = \{A : F < A \text{ for some } F \in \mathscr{F}\}\,.$$

Notice $r_<(\mathscr{F})$ is a filter contained in $\mathscr{F}$.

DEFINITION 24. Let $\mathscr{C}$ be a totally bounded Cauchy structure on C.

(1) $\mathscr{P}_L(\mathscr{C}) = [\{<_{\mathscr{F}} : \mathscr{F} \in \mathscr{C}\}]$;

(2) $\mathscr{P}_S(\mathscr{C}) = \{< \in \mathscr{O}(X) : \mathscr{F} \in \mathscr{C} \Longrightarrow r_<(\mathscr{F}) \in \mathscr{C}\}$.

THEOREM 25. *If $\mathscr{C}$ is a totally bounded Cauchy structure on X then $\mathscr{P}_L(\mathscr{C})$ is the largest* p.c.s *on X compatible with $\mathscr{C}$, and $\mathscr{P}_S(\mathscr{C})$ is the smallest. Moreover, $\mathscr{S} = \{<_{\mathscr{F}} : \mathscr{F} \in \mathscr{C}\}$ is a subbase for $\mathscr{P}_L(\mathscr{C})$.*

Proof.

(1) $\mathscr{S}$ is a subbase for $\mathscr{P}_L(\mathscr{C})$.

Let $\mathscr{R}$ be the set of refinements of finite intersections of orders in $\mathscr{S}$. We need $\mathscr{R} = \mathscr{P}_L(\mathscr{C})$. It is sufficient to show that $\mathscr{R}$ is a p.c.s. Clearly $\mathscr{R}$ satisfies (P 1) and (P 3).

Let $\mathscr{F}_1, \cdots, \mathscr{F}_n \in \mathscr{C}$ with $<_i = <_{\mathscr{F}i}$. Suppose $\bigcap_i <_i \subseteqq < \in \mathscr{O}(X)$. We wish to show $< \circ < \in \mathscr{R}$. We may assume the $\mathscr{F}_i$'s are pairwise disjoint; i.e., $\mathscr{F}_i \vee \mathscr{F}_j = [\varnothing]$ for $i \neq j$. This follows by induction from (C 3), since if $\mathscr{F}_i \vee \mathscr{F}_j \neq [\varnothing]$ we replace $<_i \cap <_j$ by $<_{\mathscr{F}}$, where $\mathscr{F} = \mathscr{F}_i \cap \mathscr{F}_j$. Choose $F_i \in \mathscr{F}_i$ so that the F_i's are pairwise disjoint.

Suppose now that $A <_i B$ for $1 \leqq i \leqq n$. We will show $A <^2 B$. For each i, define

$$D_i = \begin{cases} F_i \cap B \text{ if } B \in \mathscr{F}_i \\ F_i \backslash A \text{ if } B \notin \mathscr{F}_i\,. \end{cases}$$

Note $D_i \in \mathscr{F}_i$ for all i. Let $H = (\bigcup_i D_i \times D_i) \cup \Delta$. We claim $A < H(A) < B$.

Clearly $A \subseteqq H(A)$. To see that $H(A) \subseteqq B$, let $a \in A$ with $(a, x) \in H$. If $x = a$ then $x \in B$. If $x \neq a$ then for some i, a and x are both in D_i. Since $a \notin \mathscr{F}_i \backslash A$, clearly $D_i = F_i \cap B$, and so $x \in B$.

Now fix i. We wish to show $A <_i H(A) <_i B$. It is sufficient to show that either $H(A)$ or $X\backslash H(A)$ or $B\backslash A$ is in $\mathscr{F}_i$. If $D_i = F_i\backslash A$ it is not difficult to prove that $D_i \cap H(A) = \emptyset$, so that $X\backslash H(A) \in \mathscr{F}_i$. (Recall the D_j's are pairwise disjoint.) If $D_i = F_i \cap B$ and $D_i \cap A = \emptyset$ then clearly $B\backslash A \in \mathscr{F}_i$. If $D_i \cap A \neq \emptyset$ then $D_i \subseteqq H(A)$ and so $H(A) \in \mathscr{F}_i$.

(2) $\mathscr{P}_S(C)$ is a p.c.s.

If $< \, = \, <_1 \cap <_2$ and $\mathscr{F} \in \mathscr{C}$ then $r_<(F) = r_{<_1}(\mathscr{F}) \cap r_{<_2}(\mathscr{F})$. Using this and (C 3), we conclude $\mathscr{P}_S(\mathscr{C})$ is closed under finite intersections. Similarly $r_{<^2}(\mathscr{F}) = r_<(r_<(\mathscr{F}))$, so $\mathscr{P}_S(\mathscr{C})$ is closed under "squaring". Since $r_<(\mathscr{F}) \subseteqq r_{<'}(\mathscr{F})$ whenever $< \subseteqq <'$ clearly (P 3) holds.

(3) $\mathscr{P}_L(\mathscr{C}) \subseteqq \mathscr{P}_S(\mathscr{C})$.

It is sufficient to show that $<_{\mathscr{F}} \in \mathscr{P}_S(\mathscr{C})$ for $\mathscr{F} \in \mathscr{C}$. Let $< \, = \, <_{\mathscr{F}}$ and let $\mathscr{G} \in \mathscr{C}$. If $\mathscr{G} \vee \mathscr{F} = [\emptyset]$ then $r_<(\mathscr{G}) = \mathscr{G}$; and if $\mathscr{G} \vee \mathscr{F} \neq [\emptyset]$ then $r_<(\mathscr{G}) \supseteqq \mathscr{F} \cap \mathscr{G}$. Thus in either case $r_<(\mathscr{G}) \in \mathscr{C}$.

(4) $\mathscr{P}_S(\mathscr{C})$ and $\mathscr{P}_L(\mathscr{C})$ are both compatible with $\mathscr{C}$. Let $\mathscr{C}_S$ denote the Cauchy structure induced by $\mathscr{P}_S(\mathscr{C})$; and similarly for $\mathscr{C}_L$. Suppose $\mathscr{F} \in \mathscr{C}$. Then by definition of $\mathscr{P}_L(\mathscr{C})$ we have $<_{\mathscr{F}} \in \mathscr{P}_L(\mathscr{C})$ and hence $\mathscr{F} \in \mathscr{C}_L$. Therefore $\mathscr{C} \subseteqq \mathscr{C}_L \subseteqq \mathscr{C}_S$.

Now suppose $\mathscr{G} \in \mathscr{C}_S$. Then $<_{\mathscr{G}} \in \mathscr{P}_S(\mathscr{C})$. Let $< \, = \, <_{\mathscr{G}}$ and let $\mathscr{U}$ be an ultrafilter containing $\mathscr{G}$. Then $\mathscr{U} \in \mathscr{C}$, and so by definition of $\mathscr{P}_S(\mathscr{C})$ we have $r_<(\mathscr{U}) \in \mathscr{C}$. But $r_<(\mathscr{U}) \subseteqq \mathscr{G}$, and so $\mathscr{C}_S \subseteqq \mathscr{C}$.

(5) If $\mathscr{P}$ is a p.c.s. compatible with $\mathscr{C}$ then $\mathscr{P}_S(\mathscr{C}) \leqq \mathscr{P} \leqq \mathscr{P}_L(\mathscr{C})$.

For $\mathscr{F} \in \mathscr{C} = \mathscr{C}_{\mathscr{P}}$ we have $<_{\mathscr{F}} \in \mathscr{P}$. Thus $\mathscr{P}_L(\mathscr{C}) \subseteqq \mathscr{P}$. Now let $< \in \mathscr{P}$ and choose $\mathscr{F} \in \mathscr{C}$. Let $\mathscr{G} = r_<(\mathscr{F})$. We must show $\mathscr{G} \in \mathscr{C}$; i.e., $<_{\mathscr{G}} \in \mathscr{P}$. It is straightforward to check that $(<_{\mathscr{F}} \cap <)^3 \subseteqq <_{\mathscr{G}}$.

REMARK 26. This theorem tells us that each totally bounded Cauchy structure has a largest and smallest p.c.s. compatible with it. Since an intersection of proximity convergence structures is also a p.c.s., we see that the set of proximity convergence structures compatible with a given totally bounded Cauchy structure is a complete lattice.

THEOREM 27. *If $\mathscr{C}$ is a totally bounded Cauchy structure and $\mathscr{P}$ is a proximity structure compatible with $\mathscr{C}$ then $\mathscr{P} = \mathscr{P}_S(\mathscr{C})$.*

Proof. Let $\mathscr{P}$ be a p.c.s. compatible with $\mathscr{C}$ and suppose $\{\ll\}$ is a base for $\mathscr{P}$. We will show $\mathscr{P}_S(\mathscr{C}) \subseteqq \mathscr{P}$.

Let $< \in \mathscr{P}_S(\mathscr{C})$ and suppose $A \not< B$. We wish to show $A </< B$. For this it is sufficient to produce a filter $\mathscr{F}$ in $\mathscr{C}$ with $A \not<_{\mathscr{F}} B$. (Recall if $\mathscr{F} \in \mathscr{C}$ then $<_{\mathscr{F}} \in \mathscr{P}$ and so $\ll \subseteqq <_{\mathscr{F}}$.)

Set $\mathscr{S} = \{D: A < D\} \cup \{X \backslash E: E < B\}$. Then since $A \not< B$, $\mathscr{S}$ has the finite intersection property. Let $\mathscr{U}$ be an ultrafilter containing $\mathscr{S}$. Then $\mathscr{U} \in \mathscr{C}$. Since $< \varepsilon \mathscr{P}_S(\mathscr{C})$ we have $r_<(\mathscr{U}) \in \mathscr{C}$. Clearly neither B nor $X \backslash A$ is in $r_<(\mathscr{U})$.

REMARK AND DEFINITION 28. From this theorem it follows easily that if $\mathscr{C}$ is uniform (and totally bounded) then $\mathscr{P}_S(\mathscr{C})$ is the unique proximity structure compatible with $\mathscr{C}$. We will call $\mathscr{P}_S(\mathscr{C})$ a *saturated* p.c.s. (whether or nor $\mathscr{C}$ is uniform). Obviously then every proximity structure is saturated.

EXAMPLE 29. Even if $\mathscr{C}$ is uniform, $\mathscr{P}_L(\mathscr{C})$ need not be a proximity structure. For example let $\mathscr{X}$ be a totally bounded uniformity with Cauchy family $\mathscr{C}$. Assume that no finite intersection of Cauchy filters equals $\{X\}$. This is the case as long as $\mathscr{X} \neq \{X \times X\}$, but the proof is somewhat involved and will not be given. Certainly it is true for the usual uniformity on the closed unit interval. Assume also that if $A <_{\mathscr{X}} A$ then $A = \varnothing$ or X. This is true if the associated topology is connected, for example.

Suppose $<_{\mathscr{X}} \in \mathscr{P}_L(\mathscr{C})$. By Theorem 25, there are Cauchy filters $\mathscr{F}_1, \cdots, \mathscr{F}_n$ such that $\bigcap_i <_{\mathscr{F}_i} \subseteqq <_{\mathscr{X}}$. Therefore if $F \in \bigcap_i \mathscr{F}_i$ then $F <_{\mathscr{X}} F$, and so $F = X$. Hence $\bigcap_i \mathscr{F}_i = \{X\}$, which is impossible. Therefore $<_{\mathscr{X}} \notin \mathscr{P}_L(\mathscr{C})$, and so $\mathscr{P}_L(\mathscr{C}) \neq \mathscr{P}_S(\mathscr{C})$. By Theorem 27, $\mathscr{P}_L(\mathscr{C})$ is not a proximity structure.

4. The Σ-compactification.

A p.c.s. is compact, provided the associated convergence structure is compact. A *compactification* of p.c.s. is a compact p.c.s. in which the given space can be densely embedded. In general a p.c.s has many compactifications. We will confine ourselves to one, called the Σ-compactification. This works at least for relatively round spaces, and has a nice characterization. Using it we can obtain a generalization of the classical one-to-one correspondence between proximity structures and T_2 compactifications of a given topological space.

Continuous maps to compact T_2 spaces can be extended to this compactification, provided the range spaces satisfy a strong regularity condition. We leave open the problem of obtaining the "right" definition of regularity for a p.c.s.

DEFINITION 30. Let $\mathscr{P}$ be a p.c.s. on X. For $x \in X$ we define $\tau_{\mathscr{P}}(x)$ to be the intersection ideal generated by the filters of the form $r_<(\dot{x})$, where $< \in \mathscr{P}$.

THEOREM 31. *If $\mathscr{J}$ is in the proximity class of $\mathscr{P}$ then $\tau_{\mathscr{J}} = \tau_{\mathscr{P}}$.*

Proof. Notice that $\{r_<(\dot{x})\colon < \varepsilon \mathscr{P}\}$ is a base for $\tau_{\mathscr{P}}(x)$. Thus if $\mathscr{F} \in \tau_{\mathscr{P}}(x)$ then for some $< \in \mathscr{P}$ we have $r_<(\dot{x}) \subseteqq \mathscr{F}$. Let $\Psi \in \mathscr{J}$ with $<_\Psi \subseteqq <$. Now, $\Psi \subseteqq \dot{x} \times \Psi(\dot{x})$, so $\Psi(\dot{x}) \in \tau_{\mathscr{J}}(x)$. But $\Psi(\dot{x}) \subseteqq r_<(\dot{x})$, since for $H \in \Psi$ we have $\{x\} <_\Psi H(x)$.

Now suppose $\mathscr{F} \in \tau_{\mathscr{J}}(x)$. Let $\mathscr{G} = \mathscr{F} \cap \dot{x}$ and let $\Phi = \mathscr{G} \times \mathscr{G} \cap [\Delta]$. Then $\Phi \in \mathscr{J}$ and so $<_\Phi \in \mathscr{P}$. Set $< = <_\Phi = <_{\mathscr{G}}$. Then $r_<(\dot{x}) \subseteqq \mathscr{F}$.

REMARK 32. We can also describe $\tau_{\mathscr{P}}$ as follows: $\mathscr{F} \in \tau_{\mathscr{P}}(x)$ iff for some $\mathscr{G} \in \mathscr{F} \cap \dot{x}$ we have $<_{\mathscr{G}} \in \mathscr{P}$.

Next we will describe the construction of the Σ-extension of a p.c.s.

DEFINITION 33. Let $\mathscr{C}$ be a Cauchy structure on X. Two filters in $\mathscr{C}$ are *equivalent* iff their intersection is in $\mathscr{C}$. We denote the associated partition by $X^*(\mathscr{C})$, or just X^*. The map which assigns to a point x in X the equivalence class of $\dot{x}$ is denoted by j. If $(X, \mathscr{C})$ is T_2 then j is an injection of X into X^*.

We define Σ to be the set of all maps σ which assign to each equivalence class p in X^* a filter in p; we further require for $x \in X$ and $\sigma \in \Sigma$ that $\sigma(j(x)) = \dot{x}$.

For each σ in Σw e obtain a map from $\mathscr{P}(X)$ to $\mathscr{P}(X^*)$; namely,

$$A^\sigma = \{p \in X^*\colon A \in \sigma(p)\}\,.$$

This allows us to define a map from $\mathscr{O}(X)$ to the set of relations on X^*. For $< \in \mathscr{O}(X)$ we define $A <^\sigma B$ iff there are subsets C and D of X with $A \subseteqq C^\sigma$, $D^\sigma \subseteqq B$, and $C < D$.

Now suppose $\mathscr{C}$ is totally bounded, and let $\mathscr{P}$ be a compatible p.c.s. We define $\mathscr{P}_\Sigma = \{<' \in \mathscr{O}(X^*)\colon \text{for } \sigma \in \Sigma,\ \exists < \in \mathscr{P} \text{ with } <^\sigma \subseteqq <'\}$. It is easy to check that $\mathscr{P}_\Sigma$ is a p.c.s. on X. We will call $(j, (X^*, \mathscr{P}_\Sigma))$ the *Σ-extension* of $(X, \mathscr{P})$. It is closely related to the Kowalsky completion of $(X, \mathscr{C})$, described in [5] and in [7].

DEFINITION 34. Let $k\colon (X, \mathscr{P}) \to (Y, \mathscr{Q})$. For $< \in \mathscr{O}(X)$ we define $k(<) \in \mathscr{O}(Y)$ by $A\ k(<)\ B$ iff $A \subseteqq B$ and $k^{-1}(A) < k^{-1}(B)$. We say k is a *dense embedding* of $(X, \mathscr{P})$ into $(Y, \mathscr{Q})$, provided k is one-to-one and for $< \in \mathscr{O}(X)$ we have $< \in \mathscr{P}$ iff $k(<) \in \mathscr{Q}$.

Next we will establish that j is a dense embedding of $(X, \mathscr{P})$ into $(X^*, \mathscr{P}_\Sigma)$.

LEMMA 35. *Let $(X, \mathscr{P})$ be T_2 and let τ' denote the convergence structure induced by $\mathscr{P}_\Sigma$.*

(i) *If $p \in X^*$ and $\mathscr{F} \in p$ then $j(\mathscr{F}) \in \tau'(p)$.*

(ii) *If $\mathscr{G} \in \tau'(p)$ and $\sigma \in \Sigma$ then the filter $\mathscr{G}_\sigma = \{A\colon A^\sigma \in \mathscr{G}\}$ is in p.*

Proof. Suppose $\mathscr{F} \in p$ and define $\mathscr{G} = j(\mathscr{F}) \cap \dot{p}$. To show $j(\mathscr{F}) \to p$ it is sufficient to establish $<_{\mathscr{G}}$ is in $\mathscr{P}_{\Sigma}$.

Pick $\sigma \in \Sigma$ and set $\mathscr{H} = \mathscr{F} \cap \sigma(p)$. Now $\mathscr{H}$ is Cauchy, and so $<_{\mathscr{H}} \in \mathscr{P}$. Observe that $\mathscr{H} \subseteqq \mathscr{G}_{\sigma}$, so that $<^{\sigma}_{\mathscr{H}} \subseteqq <_{\mathscr{G}}$.

Now assume $\mathscr{U} \in \tau'(p)$, and let $\sigma \in \Sigma$. Pick $< \in \mathscr{P}_{\Sigma}$ with $r_{<}(\dot{p}) \subseteqq \mathscr{U}$, and choose $<_1 \in \mathscr{P}$ so that $<^{\sigma}_1 \subseteqq <$. Then $r_{<_1}(\sigma(p)) \subseteqq \mathscr{U}_{\sigma}$. For if $A \in \sigma(p)$ and $A <_1 B$ then $A^{\sigma} <^{\sigma}_1 B^{\sigma}$ and hence $A^{\sigma} < B^{\sigma}$. Since $p \in A^{\sigma}$ we have $B^{\sigma} \in r_{<}(\dot{p}) \subseteqq \mathscr{U}$.

Now $\sigma(p) \in p$ and $<_1 \in \mathscr{P}$. Therefore $r_{<_1}(\sigma(p)) \in p$. (Use Theorem 25 and (C 3)).

THEOREM 36. *Let $(X, \mathscr{P})$ be T_2. Then $(X^*, \mathscr{P}_{\Sigma})$ is T_2 and j is a dense embedding of $(X, \mathscr{P})$ into $(X^*, \mathscr{P}_{\Sigma})$.*

Proof. Suppose $\mathscr{G}$ converges to both p and q. Let $\sigma \in \Sigma$. By the preceding lemma $\mathscr{G}_{\sigma} \in p \cap q$. Thus $p = q$, and $\mathscr{P}_{\Sigma}$ is T_2.

Notice that for $\sigma \in \Sigma$ and $A \subseteqq X$ we have $j^{-1}(A^{\sigma}) = A$. Here strong use is made of the fact that $\sigma(j(x)) = \dot{x}$ for $x \in X$. From this it is easy to see that for $< \in \mathscr{P}$ and $\sigma \in \Sigma$ we have $<^{\sigma} \subseteqq j(<)$. Thus $j(<) \in \mathscr{P}_{\Sigma}$.

Now suppose $< \in \mathscr{O}(X)$ and $j(<) \in \mathscr{P}_{\Sigma}$. Let $\sigma \in \Sigma$ and choose $<_1 \in \mathscr{P}$ with $<^{\sigma}_1 \subseteqq j(<)$. Using the same fact as before, we see that $<_1 \subseteqq <$. This establishes that j is an embedding.

It is easy to check that $j(X)$ is dense in X^*, since for $\mathscr{F} \in p$ we have $j(\mathscr{F}) \to p$. (Lemma 35).

Next we will give conditions under which the Σ-extension is actually a compactification.

DEFINITION 37. Let $(X, \mathscr{P})$ be a p.c.s. For $\sigma \in \Sigma$ we define

$$<_{\sigma} = \bigcap \{<_{\mathscr{F}} : \mathscr{F} = \sigma(p) \text{ for some } p \in X^*\} .$$

Then $\mathscr{P}$ is *relatively round* iff each $<_{\sigma}$ is in $\mathscr{P}$.

Notice that every proximity structure is relatively round. In fact if $\subset\subset$ is a proximity on X then $\subset\subset = \bigcap \{<_{\mathscr{F}} : \mathscr{F} \in \mathscr{C}(\subset\subset)\}$.

THEOREM 38. *If $(X, \mathscr{P})$ is relatively round and T_2 then $(j, (X^*, \mathscr{P}_{\Sigma}))$ is a compactification of $(X, \mathscr{P})$.*

Proof. In view of Theorem 36, we need only establish that $\mathscr{P}_{\Sigma}$ is compact. Let $\mathscr{U}$ be an ultrafilter on X^*.

Notice that for $\sigma \in \Sigma$, if $A <_{\sigma} B$ then $(X^* \backslash B^{\sigma}) \subseteqq (X \backslash A)^{\sigma}$; thus either B^{σ} or $(X \backslash A)^{\sigma}$ is in $\mathscr{U}$. This yields $<_{\sigma} \subseteqq <_{\mathscr{U}_{\sigma}}$. Since $\mathscr{P}$ is relatively round, we conclude $\mathscr{U}_{\sigma}$ is Cauchy for $\sigma \in \Sigma$.

Moreover, the $\mathscr{U}_{\sigma}$'s are all in the same equivalence class. To see this, suppose σ and μ are in Σ and let $\eta(p) = \sigma(p) \cap \mu(p)$ for $p \in X^*$. Then $\eta \in \Sigma$, and also $\mathscr{U}_\eta \subseteq \mathscr{U}_\sigma \cap \mathscr{U}_\mu$. Thus $\mathscr{U}_\sigma$ and $\mathscr{U}_\mu$ are equivalent.

Let q be the equivalence class of the $\mathscr{U}_\sigma$'s. We claim $\mathscr{U} \to q$. Let $\sigma \in \Sigma$ and define $\mathscr{F} = \mathscr{U}_\sigma \cap \sigma(q)$. Then $\mathscr{F} \in q$, so $<_{\mathscr{F}} \in \mathscr{P}$. Let $\mathscr{V} = \mathscr{U} \cap \dot{q}$. Then it is simple to check that $<^o_{\mathscr{F}} \subseteq <_{\mathscr{V}}$.

Next we wish to characterize the Σ-compactification of $(X, \mathscr{P})$ as its unique relatively round T_2 compactification. This will be done by using the corresponding fact for uniform convergence spaces, established in [7].

DEFINITION 39. Let $f: (X, \mathscr{P}) \to (Y, \mathscr{Q})$. Then f is *p-continuous* iff $f(<) \in \mathscr{Q}$ whenever $< \in \mathscr{P}$.

LEMMA 40. *Let* $f: (X, \mathscr{P}) \to (Y, \mathscr{Q})$

(i) *f is p-continuous iff it is uniformly continuous with respect to* $\mathscr{J}_{\mathscr{P}}$ and $\mathscr{J}_{\mathscr{Q}}$.

(ii) *f is an embedding of $(X, \mathscr{P})$ into $(Y, \mathscr{Q})$ iff it embeds $(X, \mathscr{J}_{\mathscr{P}})$ in $(Y, \mathscr{J}_{\mathscr{Q}})$.*

Proof. Notice that if Φ is a standard filter on $X \times X$ and $\Psi = (f \times f)(\Phi) \cap [\Delta]$ then $<_\Psi = f(<_\Phi)$. Clearly then (i) holds. Also if $\Psi \in \mathscr{J}_{\mathscr{Q}}$ and f is a p-embedding then $\Phi \in \mathscr{J}_{\mathscr{P}}$. Therefore every p-embedding is a uniform embedding.

Now assume f is a uniform embedding. Suppose $< \in \mathscr{O}(X)$ with $f(<) \in \mathscr{Q}$. Pick $\theta \in \mathscr{J}_{\mathscr{Q}}$ with $<_\theta \subseteq f(<)$. Set $\theta_1 = (f \times f)^{-1}(\theta)$. We claim $\theta_1 \in \mathscr{J}_{\mathscr{P}}$ and $<_{\theta_1} \subseteq <$.

Since $<_\theta$ is defined, θ is standard; therefore θ_1 is standard, and in particular it is proper. Note $\theta \subseteq (f \times f)(\theta_1)$, so that $\theta_1 \in \mathscr{J}_{\mathscr{P}}$. Now if $A <_{\theta_1} B$ then $f(A) <_\theta Y \backslash f(X \backslash B)$. Since $<_\theta \subseteq f(<)$ we conclude $A < B$.

DEFINITION 41. Let $f: (X, \mathscr{P}) \to (Y, \mathscr{Q})$. By $\Sigma(f)$ we mean the set of all maps σ which assign to each point y in Y a filter converging to y. We further require that for $y \in f(X)$ and $\sigma \in \Sigma(f)$ we have $\sigma(y) = \dot{y}$.

We define $(f, (Y, \mathscr{Q}))$ to be *relatively round* provided $<_\sigma \in \mathscr{Q}$ for each σ in $\Sigma(f)$. We say $(f, (Y, \mathscr{J}_{\mathscr{Q}}))$ is relatively round iff for $\sigma \in \Sigma(f)$ the filter $\bigcap \{\sigma(y) \times \sigma(y): y \in Y\}$ is in $\mathscr{J}_{\mathscr{Q}}$.

LEMMA 42. *If $(k, (Y, \mathscr{Q}))$ is a relatively round compactification of $(X, \mathscr{P})$ then $(k, (Y, \mathscr{J}_{\mathscr{Q}}))$ is a relatively round completion of $(X, \mathscr{J}_{\mathscr{P}})$.*

Proof. From the preceding lemma we know that k is an embedding of $(X, \mathscr{J}_{\mathscr{P}})$ into $(Y, \mathscr{J}_{\mathscr{Q}})$. Since $\mathscr{J}_{\mathscr{Q}}$ and $\mathscr{Q}$ induce the same convergence structure τ', clearly this embedding is dense. Since τ' is compact, $\mathscr{J}_{\mathscr{Q}}$ is complete.

Now let $\sigma \in \Sigma(f)$. Then $<_\sigma \in \mathscr{Q}$. Set $\theta = \bigcap \{\sigma(y) \times \sigma(y): y \in Y\}$. We claim $<_\theta = <_\sigma$, so that $\theta \in \mathscr{J}_{\mathscr{Q}}$. To see that $<_\sigma \subseteqq <_\theta$ notice that if $A <_\sigma B$ then $(B \times B) \cap (X\backslash A \times X\backslash A) \in \theta$.

THEOREM 43. *If $(X, \mathscr{P})$ is relatively round and T_2 then $(j, (X^*, \mathscr{P}_\Sigma))$ is the unique relatively round T_2 compactification of $(X, \mathscr{P})$.*

Proof. In [7], Theorem 19, it was shown that any two relatively round T_2 completions of a u.c.s. are equivalent. From this, and from the two preceding lemmas, it follows that $(X, \mathscr{P})$ can have at most one relatively round T_2 compactification.

By Theorem 38 we know $(j, (X^*, \mathscr{P}_\Sigma))$ is a compactification of $(X, \mathscr{P})$. To see that it is relatively round pick $\sigma \in \Sigma(j)$ and let $\mu \in \Sigma$. Set $\eta(p) = \sigma(p)_\mu$ for $p \in X^*$. By Lemma 35, $\eta(p) \in p$ for $p \in X^*$. It is easy to check that if $p = j(x)$ then $(\dot{p})_\mu = \dot{x}$. Thus $\eta \in \Sigma$, and $<_\eta \in \mathscr{P}$. Notice that $<_\eta^\mu \subseteqq <_\sigma$, so that $<_\sigma \in \mathscr{P}_\Sigma$.

THEOREM 44. *If $(X, \mathscr{P})$ is a relatively round saturated T_2* p.c.s. *then $(X^*, \mathscr{P}_\Sigma)$ is saturated.*

Proof. Suppose $<' \in \mathscr{O}(X^*)$, and $r_{<'}(\mathscr{F})$ is Cauchy whenever $\mathscr{F}$ is. Let $\sigma \in \Sigma$ and define

$$A < B \text{ iff } X^*\backslash(X\backslash A)^\sigma <' B^\sigma \,.$$

Then $< \in \mathscr{O}(X)$ and $<^\sigma \subseteqq <'$. We claim $< \in \mathscr{P}$.

Let $\mathscr{F} \in \mathscr{C}_{\mathscr{P}}$, and let p be its equivalence class. Then $j(\mathscr{F}) \to p$ (Lemma 35). Define $\mu \in \Sigma(j)$ by $q \to j(\sigma(q)) \cap \dot{q}$. Since $\mathscr{P}$ is relatively round, so is $(j, (X^*, \mathscr{P}_\Sigma))$ (Theorem 43). Thus $<_\mu \in \mathscr{P}_\Sigma$, and $\mathscr{G}_1 = r_{<_\mu}(j(\mathscr{F}))$ converges to p. Let $\mathscr{G} = r'_<(\mathscr{G}_1)$. Then $\mathscr{G} \to p$, and so $\mathscr{G}_\sigma \in p$.

It is not difficult to check that $\mathscr{G}_\sigma \subseteqq r_<(\mathscr{F})$ so that $r_<(\mathscr{F})$ is Cauchy. Since $\mathscr{P}$ is saturated we conclude $< \in \mathscr{P}$, and $<' \in \mathscr{P}_\Sigma$.

REMARK 45. There is a one-to-one correspondence between certain T_2 compactifications of a given T_2 convergence space (X, τ) and certain of its compatible p.c.s.'s. If $\mathscr{P}$ is relatively round then $(j, (X^*, \tau(\mathscr{P}_\Sigma)))$ is a T_2 compactification of $(X, \tau_{\mathscr{P}})$. It is also a *relatively round* compactification meaning that if $\mathscr{F} \to p$ and $\sigma \in \Sigma(j)$ then $r_{<_\sigma}(\mathscr{F}) \to p$. Thus the map $\mathscr{P} \to (j, (X^*, \tau(\mathscr{P}_\Sigma)))$ takes relatively round p.c.s.'s on (X, τ) to relatively round T_2 compactifications of (X, τ).

This map is one-to-one, provided we limit ourselves to *saturated* structures. This follows from the preceding theorem and from the fact that a homeomorphism is p-continuous with respect to the largest compatible saturated structures.

The above map is also a surjection. Given a relatively round T_2 compactification $(k, (Y, \tau'))$ we define $\mathscr{P}'$ to be the (unique) compatible saturated p.c.s. Set $\mathscr{P} = \{<: k(<) \in \mathscr{P}'\}$. Then $\mathscr{P}$ is relatively round, saturated and compatible with τ. Moreover, $(k, (Y, \mathscr{P}'))$ is a compactification of $(X, \mathscr{P})$. Using Theorem 43, we can establish that the given compactification is equivalent to $(j, (X^*, \tau(\mathscr{P}_\Sigma)))$.

If $\mathscr{P}_1 \geqq \mathscr{P}_2$ then $\kappa_1 \geqq \kappa_2$. (κ_i is the compactification associated with $\mathscr{P}_i$.) However it is not clear the converse holds.

In the final part of this section we will show that a certain class of p-continuous functions on $(X, \mathscr{P})$ extend to its Σ-compactification.

DEFINITION 46. For any convergence space (X, τ) we define an order $<^c$ on X by $A <^c B$ iff $\bar{A} \subseteqq B^i$. A compatible p.c.s. $\mathscr{P}$ is *c-regular* iff $<^c \in \mathscr{P}$. A compatible u.c.s. $\mathscr{J}$ is c-regular iff it is regular in the sense of Pervin and Biesterfeldt [6]. In their notation, this means if $\Phi \in \mathscr{J}$ then $\Phi^c \in \mathscr{J}$.

REMARK 47. Both these definitions of regularity seem too strong. If $\mathscr{P}$ is c-regular then $\tau_{\mathscr{P}}$ is a *regular topological* structure. The same is true if $\mathscr{J}$ is c-regular and strongly bounded. Finding a better definition of regularity has proved unexpectedly difficult.

THEOREM 48. *Let $\mathscr{J}$ be the strongly bounded* u.c.s. *in the proximity class of $\mathscr{P}$. Then $\mathscr{P}$ is c-regular iff $\mathscr{J}$ is c-regular.*

Proof. Let Φ be a standard, strongly bounded member of $\mathscr{J}$, and set $\Psi = \Phi^c \cap (\Phi^c)^{-1}$. We will establish that $<^c \in \mathscr{P}$ iff $\Psi \in \mathscr{J}$. Since the standard strongly bounded members of $\mathscr{J}$ are a base for $\mathscr{J}$ this is sufficient to establish the desired equivalence.

(1) $(<^c \cap <_\Phi)^2 \subseteqq <_\Psi$.

This is established by the following observations.

(i) If $H \subseteqq X \times X$ then $H^c(A) \subseteqq H(A)^-$ for $A \subseteqq X$.

If $(a, x) \in H^c$ with $a \in A$ then $x \in H(a)^- \subseteqq H(A)^-$.

(ii) If $H = H^{-1}$ then $(H^c)^{-1}(A^i) \subseteqq H(A)$. Let $a \in A^i$ with $(x, a) \in H^c$. Then $a \in H(x)^-$ and so $A \cap H(x) \neq \varnothing$. For $z \in A \cap H(x)$ we have $x \in H(z) \subseteqq H(A)$.

(2) $<_\Psi^2 \subseteqq <^c$.

We will show first that if K is strongly bounded then $A^- \subseteqq K^c(A)$ for $A \subseteqq X$. Let $\mathscr{C}$ be a finite cover of X such that $H_{\mathscr{C}} \subseteqq K$. Pick $x \in A^-$. Then there is a set C in $\mathscr{C}$ with $x \in C^-$ and $C \cap A \neq \varnothing$. To see this, let $\mathscr{F} \to x$ such that $A \in \mathscr{F}$, and let $\mathscr{U}$ be an ultrafilter containing $\mathscr{F}$. Then $\mathscr{U} \cap \mathscr{C} \neq \emptyset$, and for C in $\mathscr{U} \cap \mathscr{C}$ the desired conditions hold.

Now pick $u \in C \cap A$. Then $C \subseteqq K(u)$ and so $x \in K(u)^-$. This means $(u, x) \in K^c$ and thus $x \in K^c(A)$.

From this it follows that if $A <_\varPsi B$ then $A^- \subseteqq B$. Moreover, $A \subseteqq B^i$; note $X \backslash B <_\varPsi X \backslash A$, so that $(X \backslash B)^- \subseteqq X \backslash A$. Therefore if $A <_\varPsi^2 B$ then $A^- \subseteqq B^i$.

THEOREM 49. *Let* $(X, \mathscr{P})$ *be* T_2. *Every p-continuous function from* $(X, \mathscr{P})$ *to a c-regular compact* T_2 p.c.s. *has a unique extension to* $(X^*, \mathscr{P}_\Sigma)$.

Proof. Let f be a p-continuous function from $(X, \mathscr{P})$ to a c-regular compact T_2 space $(Y, \mathscr{Q})$. It is easy to check that f is Cauchy-continuous. Since Y is compact and T_2, the image of a filter in $\mathscr{C}_\mathscr{P}$ has a unique limit in Y. Moreover, the images of equivalent filters have the same limit. This defines a map $h: X^* \to Y$; namely, $h(p)$ is the limit of the f-image of any filter in p. Notice $hj = f$. We need to establish that h is p-continuous. This is where c-regularity is used.

Let $< \in \mathscr{P}_\Sigma$ and select $\sigma \in \Sigma$. Choose $<_1 \in \mathscr{P}$ so $<_1^\sigma \subseteqq <$ and set $<_2 = f(<_1) \cap <^c$. We claim $<_2^3 \subseteqq h(<)$. This is based on the following observations.

(i) If $A \subseteqq B^i$ then $h^{-1}(A) \subseteqq f^{-1}(B)^\sigma$.

(ii) If $C^- \subseteqq D$ then $f^{-1}(C)^\sigma \subseteqq h^{-1}(D)$.

(iii) If $B\ f(<_1)\ C$ then $f^{-1}(B)^\sigma < f^{-1}(C)^\sigma$.

Note h is unique, since every continuous extension of f must agree with h on the dense subset $j(X)$.

REFERENCES

1. C. H. Cook and H. R. Fischer, *Uniform convergence structures*, Math. Ann., **173** (1967), 290-306.
2. Á. Császár, *Foundations of General Topology*, New York: Macmillan, 1963.
3. H. R. Fischer, *Limesräume*, Math. Ann., **137** (1959), 269-303.
4. H. H. Keller, *Die Limes—Uniformisierbarkeit der Limesräume*, Math. Ann., **176** (1968), 334-341.
5. H. J. Kowalsky, *Limesräume und Komplettierung*, Math. Nachr., **12** (1954), 301-340.
6. W. J. Pervin and H. J. Biesterfeldt, *Uniformization of convergence spaces II*, Math. Ann., **177** (1968), 43-48.
7. Ellen E. Reed, *Completions of uniform convergence spaces*, Math. Ann., **194** (1971), 83-108.
8. Wolfgang J. Thron, *Topological Structures*, New York: Holt, Rinehart and Winston, 1966.

Received February 24, 1972.

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