PACIFIC JOURNAL OF MATHEMATICS
Vol. 47, No. 2, 1973

A NOTE ON PRIMARY DECOMPOSITIONS OF A PSEUDOVALUATION

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Some connections are established between a primary decomposition of a pseudovaluation v on a commutative ring and a primary decomposition of the zero ideal of the associated graded ring of v. The primary decomposition of a certain pseudovaluation v_q on a one-dimensional local ring Q is described in terms of the extensions of v_q to monoidal transforms of Q.

1. Primary decompositions and the associated graded ring. Let R be a commutative ring with an identity. We consider a *pseudovaluation* v on R. By this we mean that v is a mapping from R to $\boldsymbol{P}$, the set of all real numbers together with ∞, such that

$$v(0) = \infty\ , \quad v(1) = 0\ ,$$

and, for $x,\ y \in R$,

$$v(xy) \geqq v(x) + v(y)\ ,$$

and

$$v(x - y) \geqq \min\{v(x), v(y)\}\ .$$

For each $a \in \boldsymbol{P}$, write $v_a = \{x \in R \mid v(x) \geqq a\}$ and $v_{\bar{a}} = \{x \in R \mid v(x) > a\}$. The *associated graded ring of* v, introduced by Szpiro in [11], is $G = \bigoplus_{a \in R} v_a/v_{\bar{a}}$. We shall use $-$ to denote the natural mapping from R into G.

Let u be a pseudovaluation such that $u \geqq v$ (this means that $u(x) \geqq v(x)$ for all x). We denote by $T(u)$ the set of all x, not in u_∞, such that $u(x^n) = nu(x)$ for all positive integers n, and by $S(u)$ the set of all x, not in u_∞, such that $u(xy) = u(x) + u(y)$ for all $y \in R$. As in [10], we call u *primary* if $T(u) = S(u)$. We denote by $F(u, v)$ the set of all x such that either $u(x) > v(x)$ or $u(x) = \infty$, and we put $T(u, v) = T(u) \backslash F(u, v)$.

Let $I(u, v)$ be the ideal generated in G by $\overline{F(u, v)}$.

LEMMA 1. *$\overline{F(u, v)}$ is the set of all homogeneous elements of $I(u, v)$, and $\overline{T(u, v)}$ is the set of all homogeneous elements of $G \backslash \mathrm{rad}\, I(u, v)$. If the pseudovaluation u is primary then the ideal $I(u, v)$ is primary.*

Proof. Let $r \in R$ and $s \in F(u, v)$. Either $\bar{r}\bar{s} = \bar{0}$ or $\bar{r}\bar{s} = \overline{rs}$. In the latter case either $v(s) = \infty$ or

$$v(rs) = v(r) + v(s) < u(r) + u(s) \leqq u(rs)\ .$$

Thus, in each case, $\bar{r}\bar{s} \in \overline{F(u, v)}$. If we suppose, also, that $r \in F(u, v)$ and that $\bar{r}$ and $\bar{s}$ have the same degree, then either $\bar{r} - \bar{s} = \bar{0}$ or $\bar{r} - \bar{s} = \overline{r - s}$. In the latter case either $v(r) = \infty$ or

$$v(r - s) = v(r) = v(s) < \min\{u(r), u(s)\} \leqq u(r - s) .$$

Hence, in each case, $\bar{r} - \bar{s} \in \overline{F(u, v)}$. It is now clear that $\overline{F(u, v)}$ is the set of homogeneous elements of $I(u, v)$.

Let $r \in T(u, v)$ and let n be a positive integer. Then it is easy to see that $u(r^n) = v(r^n) = nv(r) \neq \infty$; i.e., $r^n \notin F(u, v)$. Therefore, $\bar{r}^n = \overline{r^n} \notin I(u, v)$. Now suppose that $r \notin T(u, v)$. If $r \notin T(v)$ then there exists m such that $\bar{r}^m = \bar{0}$. Suppose that $r \in T(v)$. Then, by 4.1 of [10], there exists n such that $r^n \in F(u, v)$. Hence $\bar{r}^n = \overline{r^n} \in I(u, v)$.

Finally let u be primary, and suppose that r, s are elements of R such that $r \in T(u, v)$, $\bar{s} \neq \bar{0}$, and $\bar{r}\bar{s} \in I(u, v)$. Either $\bar{r}\bar{s} = \bar{0}$ and so $v(r) + v(s) < v(rs) \leqq u(rs) = u(r) + u(s) = v(r) + u(s)$, or $\bar{r}\bar{s} = \overline{rs}$ and so $v(r) + v(s) = v(rs) < u(rs) = u(r) + u(s) = v(r) + u(s)$. In each case $v(s) < u(s)$ and, hence, $\bar{s} \in I(u, v)$. Therefore, $I(u, v)$ is primary.

REMARK. The set $S(u)\backslash F(u, v)$ is contained in the set $S_0(u, v)$ of all $x \notin F(u, v)$ such that $u(xy) = u(x) + u(y)$ for all $y \notin F(u, v)$. These sets and their images in G are multiplicatively closed, and $\overline{S_0(u, v)}$ is the set of all homogeneous elements of G which are relatively prime to $I(u, v)$.

If W is a collection of pseudovaluations the lower envelope $w_0 = \bigwedge W$ is defined by $w_0(x) = \inf\{w(x) \mid w \in W\}$. From Lemma 1 we deduce

THEOREM 1. *If $u_1 \wedge u_2 \wedge \cdots \wedge u_n$ is a primary decomposition of v then $I(u_1, v) \cap I(u_2, v) \cap \cdots \cap I(u_n, v)$ is a primary decomposition of 0_G.*

COROLLARY. *Let $u_1 \wedge u_2 \wedge \cdots \wedge u_n$ be an irredundant primary decomposition of v, and suppose that G is Noetherian. Then, for each i, there exists $r_i \in R$ such that $T(u_i, v)$ is the set of x, not in v_∞, for which $v(xr_i) = v(x) + v(r_i)$.*

Proof. The decomposition $0_G = I(u_1, v) \cap \cdots \cap I(u_n, v)$ is clearly irredundant. It follows that the homogeneous elements of G not in $\overline{T(u_i, v)}$ generate a prime ideal which belongs to 0_G and which, therefore, takes the form $0_G\colon (G\bar{r}_i)$ for some homogeneous element $\bar{r}_i$ in G.

REMARK. For each positive $b \in \boldsymbol{P}$, denote by $F(u, v, b)$ the set of all $r \in R$ such that either $u(r) = \infty$ or $u(r) - v(r) \geqq b$. The proof of Lemma 1 shows that $\overline{F(u, v, b)}$ is the set of homogeneous elements of the ideal $I(u, v, b)$ which it generates in G, and that $\overline{T(u, v)}$ is the set of homogeneous elements of $G \backslash \operatorname{rad} I(u, v, b)$. It is easy to verify that, for a (possibly infinite) collection of pseudovaluations $v_i \geqq v$, $v = \bigwedge_i v_i$ if and only if, for every $b > 0$, $0_G = \bigcap_i I(v_i, v, b)$.

For all $b > 0$ and $c > 0$,

$$I(u, v, b)\, I(u, v, c) \subseteqq I(u, v, b + c) .$$

Hence each $u \geqq v$ naturally induces a (nonnegative) pseudovaluation u' on G. Thus $v = \bigwedge_i v_i$ if and only if $\bigwedge_i v_i'$ is the trivial pseudovaluation on G.

When v is homogeneous the following result may be regarded as a special case of [11, Théorème 1]. Recall that v is said to be *discrete* if $v(R \backslash v_\infty)$ generates a discrete subgroup of $\boldsymbol{R}$.

THEOREM 2. *Suppose that v is a discrete pseudovaluation. If 0_G has a finite primary decomposition without embedded components then v has a primary decomposition.*

Proof. Suppose that $H_1 \cap H_2 \cap \cdots \cap H_k$ is the primary decomposition of 0_G. For each i, write $\operatorname{rad} H_i = P_i$ and denote by S_i the set of elements $r \in R$ such that $\bar{r} \notin P_i$; then $v(ab) = v(a) + v(b)$ for all a and b in S_i, and S_i is multiplicatively closed. Mappings v_i are defined, for all $x \in R$, by

$$v_i(x) = \sup \{v(xa) - v(a) \mid a \in S_i\} .$$

Observe that if $a, b \in S_i$ then

$$v_i(x) \geqq v(xab) - v(ab) \geqq v(xa) - v(a) \geqq v(x) .$$

By 3.1 and 3.2 of [6], v_i is a pseudovaluation.

Let $x \in R \backslash v_\infty$. Then there exists i such that $\bar{x} \notin H_i$. If $c \in S_i$ then $\bar{x}\bar{c} \neq \bar{0}$, and so $v(xc) - v(c) = v(x)$. Thus $v_i(x) = v(x)$, and so $\bigwedge_i v_i = v$.

We shall now show that v_1, being a typical v_i, is primary. Let $x \notin S(v_1)$ and suppose that $v_1(x) \neq \infty$. Then there exists $y \in R$ such that $v_1(xy) > v_1(x) + v_1(y)$. Therefore, we may choose $a \in S_1$ such that

$$v_1(x) = v(xa) - v(a) ,$$

and

$$v_1(xy) \geqq v(xya) - v(a) > v_1(x) + v_1(y) .$$

Now write $\bigcap_{i>1} P_i = K$ and choose $c \in R$ such that $\bar{c} \in K \backslash P_1$. Then $\bar{a}\bar{c} \notin P_1$, and so $\overline{ac} = \bar{a}\bar{c} \in K \backslash P_1$. We may therefore assume (by replacing a by ac) that $\bar{a} \in K \backslash P_1$. This implies that $\overline{a^2} = \bar{a}\bar{a} \in K \backslash P_1$. Since $v_1(x) = v(xa^2) - v(a^2) = v(xa) - v(a)$, it follows that

$$v(xa^2) = v(xa) + v(a) .$$

Therefore, $\overline{xa^2} = \overline{xa}\bar{a} \in K$, and so (replacing a by a^2) we may also assume that $\overline{xa} \in K$. If $\overline{xa} \notin P_1$ then

$$v(xya) - v(a) = v(yxa) - v(xa) + v(xa) - v(a) \leqq v_1(y) + v_1(x) ,$$

which is false. Therefore, $\overline{xa} \in \bigcap_{i \geqq 1} P_i$, and so, for some n, $(\overline{xa})^n = 0_G$. Since $v_1(x) \neq \infty$, we have $v(xa) \neq \infty$ and so $v((xa)^n) > nv(xa)$. Therefore, $v_1(x^n) \geqq v(x^n a^n) - v(a^n) > nv(xa) - v(a^n) = nv_1(x)$. Thus $x \notin T(v_1)$. Therefore, $S(v_1) = T(v_1)$; i.e., v_1 is primary.

2. **Extensions of pseudovaluations.** In this section we introduce some terminology for use in § 3, and we prove a result pertinent to [2].

We suppose the definition of a pseudovaluation u to be modified as follows:

(i) $u \geqq 0$.

(ii) It is not required that $u(1) = 0$ (this facilitates the statement of Lemma 2; moreover, the rings in this section need not contain an identity).

We consider a homomorphism f from a commutative ring R to a commutative ring S. If I is an ideal of S(resp. R) then I^c (resp. I^e) will denote $f^{-1} I$(resp. the ideal generated by $f(I)$ in S). Suppose that v is a pseudovaluation on R. Define v^e to be the mapping from S to $\boldsymbol{P}$ such that, for all $x \in S$,

$$v^e(x) = \sup \{a \in \boldsymbol{P} \mid x \in (v_a)^e\} .$$

LEMMA 2. *The mapping v^e is a pseudovaluation on S.*

Proof. It is clear that $v^e(0) = \infty$.

Let $x, y \in S$, and suppose that $x \in (v_a)^e$ and $y \in (v_b)^e$ where $a, b \in \boldsymbol{P}$. Then $xy \in (v_a)^e(v_b)^e \subseteqq (v_a v_b)^e \subseteqq (v_{a+b})^e$. Thus

$$v^e(xy) \geqq a + b .$$

It follows that $v^e(xy) \geqq v^e(x) + v^e(y)$.

Similarly, assuming that $a \geqq b$, $x - y \in (v_a)^e + (v_b)^e = (v_a + v_b)^e = (v_b)^e$. Thus $v^e(x - y) \geqq b$. It follows that

$$v^e(x - y) \geqq \min\{v^e(x), v^e(y)\} .$$

Let w be a pseudovaluation on S. We shall denote wf by w^c. It is easy to verify that w^c is a pseudovaluation on R which is primary if w is primary.

LEMMA 3. (i) *The pseudovaluation v on R satisfies $v \leqq v^{ec}$.*
(ii) *The pseudovaluation w on S satisfies $w \geqq w^{ce}$.*

Proof. (i) Let x be an element of R such that $v(x) = a$. Then $f(x) \in (v_a)^e$ and so $a \leqq v^e(f(x)) = v^{ec}(x)$.

(ii) Let y be an element of S such that $y \in ((w^c)_a)^e$. Since $(w^c)_a \subseteqq (w_a)^c$, $y \in (w_a)^{ce} \subseteqq w_a$ and so $w(y) \geqq a$. It follows that $w(y) \geqq w^{ce}(y)$.

THEOREM 3. $v = v^{ec}$ *if and only if* $v_a = (v_a)^{ec}$ *for each* $a \in \boldsymbol{R}$.

Proof. If $v = v^{ec}$ then, for each $a \in \boldsymbol{P}$,

$$(v_a)^{ec} \subseteqq \{x \in R \mid v^e(f(x)) \geqq a\} = (v^{ec})_a = v_a ,$$

and so $(v_a)^{ec} = v_a$. Conversely, suppose that $v_a = (v_a)^{ec}$ for each $a \in \boldsymbol{R}$. Let $x \in R$ and let $f(x) \in (v_a)^e$ where $a < \infty$. Then $x \in (v_a)^{ec} = v_a$, that is $v(x) \geqq a$. It follows that $v \geqq v^{ec}$, and hence that $v = v^{ec}$.

We refer to [2, p. 296, Definition 2] for the definition of a *best filtration*. If v has a best filtration $\{A_i\}_{i=0}^{\infty}$ then, by [2, p. 297, Lemma 1], the set of all distinct A_i's is the same as the set of all distinct v_a's where $a < \infty$. Thus, taking f to be an inclusion map, our theorem includes, in the case of nonnegative pseudovaluations, Theorem 2, p. 299, and Theorem 4, p. 301, of [2].

3. An example in a one-dimensional ring. Let $Q, \mathfrak{m}$ be a one-dimensional local ring and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of Q. We shall consider the pseudovaluation $v = v_{\mathfrak{q}}$ determined by the powers of $\mathfrak{q}$ according to the rule

$$v_{\mathfrak{q}}(x) = \sup\{n \mid x \in \mathfrak{q}^n\} .$$

By considering the associated graded ring G of v and proceeding as in Theorem 2, we could show that v decomposes into primary pseudovaluations corresponding to the isolated primary components of 0_G together with an "irrelevant" component. Apart from the irrelevant component this decomposition is unique (by [10]). We shall now show how the theory of monoidal transformations developed by Northcott and Kirby provides an alternative description of this decomposition.

Let A denote the intersection of the primary components of 0_Q of rank nought, and write $Q/A = Q'$ and $\mathfrak{q}Q' = \mathfrak{q}'$. Then not every element of $\mathfrak{m}Q'$ is a zero divisor. Let $\mathfrak{R}$ be the $\mathfrak{q}'$*-resolute of* Q', for the definition of which see p. 136 of [4]; let $Q_1, \cdots, Q_r$ be the *monoidal transforms of* Q' *with respect to* $\mathfrak{q}'$, i.e., the rings of quotients of $\mathfrak{R}$ with respect to the maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ of $\mathfrak{R}$; and, for $i = 1, \cdots, r$, let f_i be the composition of the natural homomorphisms $Q \longrightarrow Q' \longrightarrow Q_i$. Using the symbols e_i and c_i to relate to f_i in the same way that e and c were related to f in § 2, we observe that v^{e_i} is the pseudovaluation on Q_i determined the powers of the ideal $\mathfrak{q}^{e_i}$. However, by [4, Theorems 1 and 8, and Lemma 3] $\mathfrak{q}^{e_i}$ is a principal ideal of Q_i. Therefore, by an example in § 3 of [10], v^{e_i} is primary, and so $v^{e_i c_i}$ is primary.

Now, denoting by $\mathfrak{q}_i$ the restriction to $\mathfrak{R}$ of $\mathfrak{q}^{e_i}$, $\operatorname{rad} \mathfrak{q}_i = \mathfrak{p}_i$ and $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$ is the primary decomposition of $\mathfrak{R}\mathfrak{q}$ (by the corollary on p. 142 of [4] and since $\mathfrak{R}\mathfrak{m} \subseteqq \operatorname{rad} \mathfrak{R}\mathfrak{q}$). Therefore, for all n,

$$\mathfrak{R}\mathfrak{q}^n = \mathfrak{q}_1^n \cap \mathfrak{q}_2^n \cap \cdots \cap \mathfrak{q}_r^n \,.$$

By an argument on p. 88 of [8], $\mathfrak{R}\mathfrak{q}^n = \mathfrak{q}'^n$ for all sufficiently large n. Therefore, for $n \geqq h$ say,

$$\mathfrak{q}^n + A = (v^{e_1 c_1} \wedge \cdots \wedge v^{e_r c_r})_n \,.$$

However, we may choose h such that $A \cap \mathfrak{q}^h = 0_Q$ and, hence, for $n \geqq h$, $\mathfrak{q}^h \cap (\mathfrak{q}^n + A) = q^n$. Therefore, using e_0 and c_0 to relate to the natural map f_0 from Q to $Q/\mathfrak{q}^h$, we have, for all n,

$$\mathfrak{q}^n = (v^{e_0 c_0})_n \cap (v^{e_1 c_1} \wedge \cdots \wedge v^{e_r c_r})_n \,.$$

Finally we show that v^{e_0}, $= w$ say, is primary. If $x \in f_0(\mathfrak{m})$ then, for some k, $w(x^k) = \infty$ and so $x \notin T(w)$. On the other hand, if x is a unit of $Q/\mathfrak{q}^h$ then $w(x) = 0$ and, for any y,

$$w(xy) = w(y) = w(y) + w(x) \,;$$

i.e., $x \in S(w)$. Thus $T(w) = S(w)$.

It is now clear that

THEOREM 4. *In the notation developed above*

$$v^{e_0 c_0} \wedge v^{e_1 c_1} \wedge \cdots \wedge v^{e_r c_r}$$

is a primary decomposition of v.

It is easy to extend this theorem and obtain a primary decomposition of the pseudovaluation v_I determined by an ideal I of rank 1 in a 1-dimensional Noetherian ring R. Let $M_1, \cdots, M_m$ be the associated prime ideals (necessarily maximal) of I, and, for $j = 1, \cdots, m$, let g_j be the natural homomorphism from R to the ring R_j of quotients of R with respect to M_j. For each positive integer n,

$$I^n = \bigcap_j (I^n)^{e_j c_j} ,$$

where e_j, c_j relate to g_j, and so

$$v_I = \bigwedge_j v_I^{e_j c_j} ,$$

which yields a primary decomposition of v_I on application of Theorem 4 to each $v_I^{e_j}$.

We conclude by describing a result, in the same vein as the foregoing, which is implicit, as a special case, in [9]. Suppose that our ring R is a domain; let $\bar{v}_I$ denote the least homogeneous pseudovaluation $\geqq v_I$; and let $\bar{R}_1, \cdots, \bar{R}_h$ be the rings of quotients with respect to the maximal ideals of the integral closure of R which contain I. Then $\bar{v}_I$ decomposes into valuations

$$\bar{v}_I = \bigwedge_i (\overline{v_I^{e_i}})^{c_i}$$

where, for each i, e_i, c_i refer to the natural mapping $R \rightarrow \bar{R}_i$.

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Received May 15, 1972.

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