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ON k-SPACES, k_R-SPACES AND $k(X)$

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Two examples of k_R-spaces which are not k-spaces are constructed; one of them is a σ-compact cosmic space, and the other is an $\aleph_0$-space. On the positive side, a theorem is proved which implies that every σ-compact $\aleph_0$-space which is a k_R-space must be a k-space.

1. Introduction. In this paper, we prove and extend some results which were announced in [8].

Recall that a topological space X is called a *k-space* if every subset of X, whose intersection with every compact $K \subset X$ is relatively open in K, is open in X. (For example, locally compact spaces and first-countable spaces are k-spaces.) Analogously, a space X is a *k_R-space* if it is completely regular and if every $f: X \to R$, whose restriction to every compact $K \subset X$ is continuous, is continuous on X. Clearly every completely regular k-space is a k_R-space. The converse is false, as was first shown by an example of M. Katětov which appeared in a paper by V. Pták [15, p. 357]. That example, however, was not normal[1], and our first purpose in this note is to construct two examples which *are* normal,—in fact, regular Lindelöf and thus paracompact. Both our examples are modifications of Katětov's example, which had, in turn, been previously introduced (for a different purpose) by J. Novák in [12].

Before giving more details, let us review some definitions. A covering $\mathscr{A}$ of a space X is a *network* (resp. *pseudobase*) for X if, wherever $C \subset U$ with C a singleton (resp. compact) and U open in X, then $C \subset A \subset U$ for some $A \in \mathscr{A}$. A regular space with a countable network (resp. pseudobase) is called *cosmic* (resp. an *$\aleph_0$-space*). It is shown in [7, Proposition 10.2 and Corollary 11.5] that a regular space is a continuous (resp. quotient) image of a separable metric space if and only if it is cosmic (resp. an $\aleph_0$-space and a k-space). We clearly have

$$\text{separable metric} \to \aleph_0 \to \text{cosmic} \to \text{regular Lindelöf},$$

and none of these implications is reversible.

We can now describe the principal features of our examples, as follows.

EXAMPLE 1.1. There exists a σ-compact, cosmic k_R-space which

[1] It has a countable dense subset and a closed, discrete subset of cardinality c, and is thus not normal by a result of F. B. Jones [4].

is not a k-space.

EXAMPLE 1.2. There exists an $\aleph_0$-space which is a k_R-space but not a k-space.

By Corollary 1.4 below, it is not possible to combine Examples 1.1 and 1.2 by constructing a σ-compact $\aleph_0$-space which is a k_R-space but not a k-space.

Let us now take a slightly different point of view. Recall that, if X is a topological space with topology $\mathscr{T}$, then $k(X)$ is the set X with the finest topology $k(\mathscr{T})$ which agrees with $\mathscr{T}$ on every $\mathscr{T}$-compact subset of X. Thus a set $U \subset X$ is $k(\mathscr{T})$-open if and only if $U \cap K$ is relatively $\mathscr{T}$-open in K for every $\mathscr{T}$-compact subset K of X. Clearly $k(X)$ is always a k-space, and X is a k-space if and only if $X = k(X)$. Moreover, if X is a k_R-space, than X is a k-space if and only if $k(X)$ is completely regular.

We will prove that X is not a k-space in Examples 1.1 and 1.2 by showing that $k(X)$ is not regular. In contrast to this, we have the following positive result.

THEOREM 1.3. *Suppose X is an $\aleph_0$-space which is the union of countably many closed subsets which are k-spaces. Then $k(X)$ is regular, and thus also an $\aleph_0$-space.*[2)]

As we shall see in § 5, Theorem 1.3 can be extended to $\aleph$-spaces, an interesting generalization of $\aleph_0$-spaces which was recently introduced by P. O'Meara [13] [14].

Since every $\aleph_0$-space is completely regular, Theorem 1.3 and the preceding discussion imply the following result.

COROLLARY 1.4. *If X satisfies the hypotheses of Theorem 1.3, and if X is a k_R-space, then X is a k-space.*

We conclude this introduction with a brief discussion of products. It is known that the product of two k-spaces need not be a k-space (see [9] for more details). In fact, N. Noble has shown [11, p. 189-190] that there exist two normal k-spaces X and Y such that $k(X \times Y)$ is not completely regular. I do not know whether there is such an example with X and Y both paracompact or even $\aleph_0$-spaces. One can, however, conclude from Theorem 1.3 (and known results) that, if X and Y are k-spaces and $\aleph_0$-spaces, then $k(X \times Y)$ is also an $\aleph_0$-space — and thus completely regular — under either of the

2) It follows from [7, Proposition 8.2] that, if X is an $\aleph_0$-space and $k(X)$ is regular, then $k(X)$ is also an $\aleph_0$-space.

following two (possibly superfluous) assumptions.

1.5(a). X and Y are both unions of countably many closed, metrizable subsets.

1.5(b). X or Y is σ-compact.

It should be remarked that there exist k-and-$\aleph_0$-spaces (in fact, continuous, closed images of separable metric spaces) which do not satisfy condition 1.5(a); see B. Fitzpatrick, Jr. [3, Example 2].

Section 2 contains some preliminary material needed in §§ 3 and 4. Sections 3 and 4 are devoted to the proofs of Examples 1.1 and 1.2, while our extension of Theorem 1.3 is formulated and proved in § 5.

2. Notation and a lemma. Examples 1.1 and 1.2 are both defined by retopologizing a subset of the plane R^2. If $x \in R^2$, then x_1, x_2 will, as usual, denote the coordinates of x. We thus cannot use subscripts to denote the terms of a sequence, and will therefore denote sequences by functional notation, such as $x(n)$. The ith coordinate of $x(n)$ will thus be denoted by $x_i(n)$.

Let R be the real line with its usual topology, and let $\mathscr{T}_0$ denote the usual topology on R^2. If $X \subset R^2$ and $x \in X$, then a function $f\colon X \to R$ is called *separately continuous* at x provided $f|(L \cap X)$ is $\mathscr{T}_0$-continuous at x if L is either the horizontal or the vertical line through x in R^2.

LEMMA 2.1. *Let $X \subset (R^2, \mathscr{T}_0)$, and let x and $x(n)$ $(n = 1, 2, \cdots)$ be elements of X. Suppose $f(x(n)) \to f(x)$ for every $f\colon X \to R$ which is separately continuous at x and continuous at every other point of X. Then there is an n such that $x_1(n) = x_1$ or $x_2(n) = x_2$.*

Proof. Suppose not. Let $Y = X - \{x\}$, and let

$$A = \{x(n)\colon n \in N\} ,$$

$$B = \{y \in Y\colon y_1 = x_1 \quad \text{or} \quad y_2 = x_2\} .$$

Our assumption implies that $x(n) \to x$, so A and B are disjoint closed subsets of Y and hence there is a continuous $g\colon Y \to R$ such that $g(A) = 1$ and $g(B) = 0$. Extend g to a function $f\colon X \to R$ by letting $f(x) = 0$. Then clearly f is separately continuous at x and continuous elsewhere. But $f(x(n)) = 1$ for all n, while $f(x) = 0$, so $f(x(n)) \nrightarrow f(x)$. This contradiction completes the proof.

3. Proof of Example 1.1. Let X be the plane, and $\mathscr{T}_0$ its usual topology. Let $A \subset X$ be the x-axis. Let F be the set of all

$f\colon X \to R$ which are $\mathscr{T}_0$-continuous on $X - A$ and separately continuous at every $x \in A$, and let $\mathscr{T}$ be the coarsest topology making every $f \in F$ continuous. Clearly $(X, \mathscr{T})$ is completely regular. Let us now show (in Lemmas 3.1, 3.2, and 3.8) that $(X, \mathscr{T})$ is a σ-compact, cosmic k_R-space which is not a k-space.

Observe that, on every horizontal and every vertical line in X, $\mathscr{T}$ agrees with $\mathscr{T}_0$ and hence (being metrizable) with $k(\mathscr{T})$.

LEMMA 3.1. *$(X, \mathscr{T})$ is σ-compact and cosmic.*

Proof. Let M be the disjoint union of A and $X - A$ (on both of which $\mathscr{T}_0$ and $\mathscr{T}$ agree), and let $g\colon M \to X$ be the obvious map. Then M is a σ-compact metric space, and g is $\mathscr{T}$-continuous. Since $(X, \mathscr{T})$ is completely regular, this implies that it is a σ-compact cosmic space.

LEMMA 3.2. *$(X, \mathscr{T})$ is a k_R-space.*

Proof. We have already observed that $(X, \mathscr{T})$ is completely regular. Suppose $f\colon X \to R$ and $f \mid K$ is $\mathscr{T}$-continuous for every $\mathscr{T}$-compact $K \subset X$, and let us prove that f is $\mathscr{T}$-continuous by showing that $f \in F$.

Since $\mathscr{T}_0$ and $\mathscr{T}$ agree on $X - A$, it is clear that f is $\mathscr{T}_0$-continuous on $X - A$. To prove that f is separately continuous at every $x \in A$, it will surely suffice to show that, if L is a horizontal or vertical line in X, then $f \mid L$ is $\mathscr{T}_0$-continuous. But $\mathscr{T}_0$ and $\mathscr{T}$ coincide on L, so $f \mid L$ is $\mathscr{T}_0$-continuous by our assumption on f and the fact that $(L, \mathscr{T}_0)$ is metrizable and therefore a k-space.

LEMMA 3.3. *Let $x(n) \to x$ in $(X, \mathscr{T})$, with $x \in A$. Then there exists an n such that $x_1(n) = x_1$ or $x_2(n) = x_2$.*

Proof. Immediate from Lemma 2.1.

LEMMA 3.4. *If $C \subset X$ is $\mathscr{T}$-compact, then there exists an $\varepsilon > 0$ and a finite $A' \subset A$ such that, if $y \in C$ and $0 < |y_2| < \varepsilon$, then $y_1 = x_1$ for some $x \in A'$.*

Proof. Let $K = C \cap A$. We begin by showing that, if $x \in K$, then x has a $\mathscr{T}_0$-neighborhood $U(x)$ in X such that $y_1 = x_1$ for every $y \in (C - A) \cap U(x)$. Suppose there were no such $U(x)$. Then there exists a sequence $x(n)$ in $C - A$ such that $x(n) \to x$ for $\mathscr{T}_0$ and $x_1(n) \neq x_1$ for all n. Since $\mathscr{T}$ is finer than $\mathscr{T}_0$, they coincide on the

$\mathscr{T}$-compact set C, so $x(n) \to x$ for $\mathscr{T}$. By Lemma 3.3, that is impossible.

Since $\mathscr{T}$ is finer than $\mathscr{T}_0$, the set C is $\mathscr{T}_0$-compact, and hence so is K. Hence there is a finite $A' \subset K$ such that $\{U(x): x \in A'\}$ covers K. Let $U = \bigcup \{U(x): x \in A'\}$. Then $C - U$ is $\mathscr{T}_0$-compact and disjoint from A, so there exists an $\varepsilon > 0$ such that $|y_2| > \varepsilon$ wherever $y \in C - U$. This choice of A' and ε satisfies our requirements.

The proof of the following lemma can be left to the reader.

LEMMA 3.5. *There exists a* $B \subset X - A$ *such that*:
(a) *If* $\varepsilon > 0$, *then* $\{x \in B: |x_2| > \varepsilon\}$ *is finite.*
(b) B *intersects each vertical line at most once.*
(c) *If* $x \in A$, *each* $\mathscr{T}_0$*-neighborhood of* x *intersects* B.

LEMMA 3.6. *The set* B *of Lemma* 3.5 *is closed in* $(X, k(\mathscr{T}))$.

Proof. This follows from Lemma 3.4 and parts (a) and (b) of Lemma 3.5.

LEMMA 3.7. *If* $y \in A$, *and if* U *is an open* $k(\mathscr{T})$*-neighborhood of* y *in* X, *then* $\bar{U}$, *the* $k(\mathscr{T})$*-closure of* U *in* X, *is a* $\mathscr{T}_0$*-neighborhood in* X *of some* $x \in A$.

Proof. Recall first that $\mathscr{T}_0$ agrees with $k(\mathscr{T})$ on each horizontal and each vertical line L, so $U \cap L$ is $\mathscr{T}_0$-open in L and $\bar{U} \cap L$ is $\mathscr{T}_0$-closed.

Let $V = \{s \in R: (s, 0) \in U\}$. Then V is open in R, and $V \neq \varnothing$ since $y \in V$. For each n, let

$$E_n = \{s \in R: (s, t) \in U \text{ whenever } |t| < 1/n\} .$$

Then $\bigcup_{n=1}^{\infty} E_n = V$. Since V is open in R, the Baire category theorem implies that there is an m such that $\bar{E}_m$ has an interior point s_0 in R. Let $x = (s_0, 0)$, and let $W = \bar{E}_m \times (-1/m, 1/m)$. Then W is a $\mathscr{T}_0$-neighborhood of x in X, and to complete the proof we will show $W \subset \bar{U}$.

Let $|t| < 1/m$. Then $E_m \times \{t\} \subset U$. Let $L = R \times \{t\}$. Since $\bar{U} \cap L$ is $\mathscr{T}_0$-closed in X, it follows that $\bar{E}_m \times \{t\} \subset \bar{U} \cap L$. This implies that $W \subset \bar{U}$, and that completes the proof.

LEMMA 3.8. $(X, k(\mathscr{T}))$ *is not regular, and thus* $(X, \mathscr{T})$ *is not a k-space.*

Proof. This follows from Lemmas 3.6, 3.7, and 3.5(c).

4. Proof of Example 1.2. Let R^2 be the plane, and $\mathscr{T}_0$ its usual topology. Let A (resp. B) be the set of all $x \in R^2$ both of whose coordinates are rational (resp. irrational), and let $X = A \cup B$. Let F be the set of all $f: X \to R$ which are separately continuous at every $x \in A$ and $\mathscr{T}_0$-continuous at every $x \in B$, and let $\mathscr{T}$ be the coarsest topology on X making every $f \in F$ continuous. Clearly $(X, \mathscr{T})$ is completely regular. Let us now show (in Lemmas 4.2, 4.6, and 4.9) that $(X, \mathscr{T})$ is an $\aleph_0$-space which is a k_R-space but not a k-space.

Observe that, on the intersection of X with every horizontal and every vertical line, $\mathscr{T}$ coincides with $\mathscr{T}_0$ and hence (being metrizable) with $k(\mathscr{T})$.

LEMMA 4.1. *If $x \in B$, then every $\mathscr{T}$-neighborhood of x in X is also a $\mathscr{T}_0$-neighborhood, and conversely.*

Proof. Immediate from the definition.

LEMMA 4.2. *$(X, \mathscr{T})$ is a k_R-space.*

Proof. We have already observed that $(X, \mathscr{T})$ is completely regular. Suppose now that $f: X \to R$ and $f \mid K$ is $\mathscr{T}$-continuous for every $\mathscr{T}$-compact $K \subset X$, and let us prove that f is $\mathscr{T}$-continuous by showing that $f \in F$.

First, let us show that f is $\mathscr{T}_0$-continuous at every $x \in B$. Let $x(n) \to x$ in $(X, \mathscr{T}_0)$. Then $x(n) \to x$ for $\mathscr{T}$ by Lemma 4.1. Hence the set S consisting of x and all the $x(n)$ is $\mathscr{T}$-compact, so $f \mid S$ is $\mathscr{T}$-continuous. Hence $f(x(n)) \to f(x)$, which is what we had to show.

To prove that f is separately continuous at every $x \in A$, it suffices to prove that $f \mid M$ is $\mathscr{T}_0$-continuous whenever M is the intersection with X of a horizontal or vertical line in R^2. But $\mathscr{T}_0$ and $\mathscr{T}$ coincide on M, and $(M, \mathscr{T}_0)$ is a k-space (being metrizable), so our assumptions imply that $f \mid M$ is indeed $\mathscr{T}_0$-continuous.

LEMMA 4.3. *Let $x(n) \to x$ in $(X, \mathscr{T})$, with $x \in A$. Then there exists an n such that $x_1(n) = x_1$ or $x_2(n) = x_2$.*

Proof. Immediate from Lemma 2.1.

LEMMA 4.4. *If $C \subset X$ is $\mathscr{T}$-compact, then $B \cap C$ is $\mathscr{T}$-closed in C (and hence in X).*

Proof. Since $\mathscr{T}$ is finer than $\mathscr{T}_0$, they coincide on the $\mathscr{T}$-compact set C, so C is $\mathscr{T}$-metrizable. If $B \cap C$ were not $\mathscr{T}$-closed in

C, there would exist a sequence $x(n)$ in $B \cap C$ which $\mathscr{T}$-converges to some $x \in A$. This is impossible by Lemma 4.3.

We now introduce the following notation. For each rational $r \in R$ and for $i = 1, 2$, let

$$A_{r,i} = \{x \in A\colon\ x_i = r\} .$$

Note that $\mathscr{T}$ and $\mathscr{T}_0$ agree on each $A_{r,i}$.

LEMMA 4.5. *If $K \subset A$ is $\mathscr{T}$-compact, then K is contained in the union of finitely many $A_{r,i}$.*

Proof. Suppose not. Then one can choose a sequence $x(n) \in K$ such that $x_1(m) \neq x_1(n)$ and $x_2(m) \neq x_2(n)$ whenever $m \neq n$. Since $\mathscr{T}$ and the coarser topology $\mathscr{T}_0$ coincide on the $\mathscr{T}$-compact set K, some subsequence of $x(n)$ must $\mathscr{T}$-converge to some $x \in K$. Since $K \subset A$, that contradicts Lemma 4.3.

LEMMA 4.6. *X is an $\aleph_0$-space.*

Proof. As already observed, X is completely regular. Let us construct a countable pseudobase $\mathscr{P}$ for X.

Let $\mathscr{D}$ be a countable base for $(X, \mathscr{T}_0)$, and let $\mathscr{A}_{r,i}$ be a countable base for $A_{r,i}$. Let

$$\mathscr{A} = \bigcup \{\mathscr{A}_{r,i}\colon\ r \text{ rational},\ i = 1,\ 2\} ,$$

and let $\mathscr{P}$ be the collection of finite unions of elements of $\mathscr{D} \cup \mathscr{A}$.

To show that $\mathscr{P}$ is a pseudobase for X, let $C \subset U$, with C compact and U open in $(X, \mathscr{T})$. Since $B \cap C$ is compact (for $\mathscr{T}$ and hence for $\mathscr{T}_0$) by Lemma 4.4, and since U is a $\mathscr{T}_0$-neighborhood of $B \cap C$ by Lemma 4.1, we have

$$(B \cap C) \subset D \subset U ,$$

where D is a finite union of elements of $\mathscr{D}$. Now $C - D$ is a $\mathscr{T}$-compact subset of A, and hence, by Lemma 4.5, is contained in the union of finitely many $A_{r,i}$. Hence

$$(C - D) \subset E \subset U ,$$

where E is the union of finitely many elements of $\mathscr{A}$. But now

$$C \subset (D \cup E) \subset U ,$$

and $D \cup E \in \mathscr{P}$, which completes the proof.

LEMMA 4.7. *B is $k(\mathscr{T})$-closed in X.*

Proof. Immediate from Lemma 4.4.

LEMMA 4.8. *Let $x(n) \to x$ for $\mathscr{T}_0$ in X, with $x \in B$. Then $x(n) \to x$ for $\mathscr{T}$ and $k(\mathscr{T})$.*

Proof. That $x(n) \to x$ for $\mathscr{T}$ follows from Lemma 4.1. Hence the set S consisting of x and all $x(n)$ is $\mathscr{T}$-compact, so $k(\mathscr{T})$ and $\mathscr{T}$ coincide on S, and thus $x(n) \to x$ also for $k(\mathscr{T})$.

LEMMA 4.9. *$(X, k(\mathscr{T}))$ is not regular, so $(X, \mathscr{T})$ is not a k-space.*

Proof. Let $y \in A$, and U be any $k(\mathscr{T})$-open set in X containing y. We will construct a sequence $x(n)$ in U which converges to some $x \in B$ for $\mathscr{T}_0$ and hence also (by Lemma 4.8) for $k(\mathscr{T})$. This implies that the $k(\mathscr{T})$-closure of U intersects B. By Lemma 4.7, it follows that $(X, k(\mathscr{T}))$ is not regular.

Before choosing the $x(n)$, recall that $\mathscr{T}_0$ and $k(\mathscr{T})$ agree on every $A_{r,i}$, so $U \cap A_{r,i}$ is $\mathscr{T}_0$-open in $A_{r,i}$. Now let $r(n)$ be an enumeration of the rationals in R. Let $x(1) = y$, and choose $x(n) \in U \cap A$ inductively so that, for all n and for $i = 1, 2$,

(a) $\quad x_i(n+1) > x_i(n) \geqq r(n), \quad \text{or} \quad x_i(n+1) < x_i(n) \leqq r(n)\,,$

(b) $\quad |x_i(n+2) - x_i(n+1)| < 1/2\,|x_i(n+1) - x_i(n)|\,.$

Then (b) implies that, for $i = 1, 2$, the sequence $x_i(n)$ $(n = 1, 2, \cdots)$ is Cauchy, and thus converges to some real number x_i. By (a) and (b), $x_i \neq r(n)$ for all n, so x_i is irrational. Thus $x = (x_1, x_2)$ is in B, and $x(n) \to x$ for $\mathscr{T}_0$, which completes the proof.

5. A positive result. In this section we prove Theorem 1.3 in a generalized form. Following P. O'Meara [13] [14], a cover $\mathscr{P}$ of a space X is a *k-network* for X if, whenever $C \subset U$ with C compact and U open in X, then $C \subset \bigcup \mathscr{F} \subset U$ for some finite subcollection $\mathscr{F}$ of $\mathscr{P}$. (Note that this is a weaker concept than that of a pseudobase, but a space has a countable k-network if and only if it has a countable pseudobase.)[3] An *$\aleph$-space*, according to O'Meara, is a

[3] The term "k-network" has been used by A. V. Arhangel'skiĭ and some other authors to mean what we have called a pseudobase. Since O'Meara's concept seems to be the more basic one (being useful even in the uncountable case), it seems to me that the elegant term "k-network" should be a reserved for it. Perhaps "pseudobase", which is not a good term, should then be replaced by something like "special k-network".

regular space with a σ-locally finite k-network. Clearly every metrizable space is an $\aleph$-space, and O'Meara showed [14, Theorem 2.3] that X is an $\aleph_0$-space if and only if it is a Lindelöf $\aleph$-space. Unlike $\aleph_0$-spaces, an $\aleph$-space need not be paracompact or even normal [14, Example 8.4].

We now have the following generalization of Theorem 1.3.

THEOREM 5.1. *Suppose X is a paracompact $\aleph$-space which is the union of countably many closed subsets which are k-spaces. Then $k(X)$ is also a paracompact $\aleph$-space.*

The proof of Theorem 5.1 is based on the following four lemmas. (Only the first two are needed to show that $k(X)$ is an $\aleph$-space.) The first lemma, due to O'Meara [14, Theorem 6.1], generalizes an analogous result for $\aleph_0$-spaces [7, Proposition 8.2].

LEMMA 5.2 (*P. O'Meara*). *If $\mathscr{A}$ is a σ-locally finite k-network, closed under finite intersections, for a space $(X, \mathscr{T})$, and if $\mathscr{T}'$ is any topology on X which is finer than $\mathscr{T}$ and agrees with $\mathscr{T}$ on all $\mathscr{T}$-compact subsets of X, then $\mathscr{A}$ is also a k-network for $(X, \mathscr{T}')$.*[5]

LEMMA 5.3. *Let Y be a k-space. Let (P_n) be an increasing sequence of subsets of Y such that every compact $C \subset Y$ is a subset of some P_n. Suppose that $V_n \supset P_n$ and that V_n is open in Y for all n. Then $\bigcap_{n=1}^{\infty} V_n$ is open in Y.*

Proof. Let $V = \bigcap_{n=1}^{\infty} V_n$. We need only show that, if $C \subset Y$ is compact, then $V \cap C$ is open in C. Pick n so that $C \subset P_n$. Then $C \subset P_i \subset V_i$ for all $i \geqq n$, so

$$V \cap C = \left(\bigcap_{i=1}^{n-1} V_i\right) \cap C .$$

Hence $V \cap C$ is open in C, and that completes the proof.

REMARK. The assumption in Lemma 5.3 that Y be a k-space is essential. This is easily seen by considering a nondiscrete, countable space Y, all of whose compact subsets are finite. (For example, take $Y = N \cup \{y\}$, where $y \in \beta N - N$.)

The next lemma should be compared to Lemma 5.2.

[4] O'Meara states this result for Hausdorff spaces, but it is true without that assumption.

LEMMA 5.4. *If $\mathscr{A}$ is a point-countable network for a space $(X, \mathscr{T})$ which is closed under finite intersections, and if $\mathscr{T}'$ is a topology on X which agrees with $\mathscr{T}$ on every $\mathscr{T}$-compact subset of X, then $\mathscr{A}$ is also a network for $(X, \mathscr{T}')$.*

Proof. Let $x \in U$, where U is $\mathscr{T}'$-open in X. Let $\{A_n: n \in N\}$ be the elements of $\mathscr{A}$ containing x, and let $A_n^* = A_1 \cap \cdots \cap A_n$ for all n. Then $A_n^* \in \mathscr{A}$ for all n, and it suffices to show that $A_n^* \subset U$ for some n.

Suppose not. Then there is an $x_n \in A_n^* - U$ for all n. Since $\mathscr{A}$ is a network for $(X, \mathscr{T})$ and since $x_n \in A_n^*$ for all n, we have $x_n \to x$ for $\mathscr{T}$. Let $K = \{x\} \cup \{x_n: n \in N\}$. Then K is $\mathscr{T}$-compact, so $\mathscr{T}$ and $\mathscr{T}'$ agree on K. Hence $U \cap K$ is relatively $\mathscr{T}$-open in K, so $A_m^* \cap K \subset U$ for some m. But then $x_m \in U$, a contradiction.

For our next lemma, recall that a space X is a *σ-space* in the sense of A. Okuyama if it has a σ-locally finite network. Thus every $\aleph$-space is a σ-space.

LEMMA 5.5. *If $(X, \mathscr{T})$ is a paracompact σ-space, and if $\mathscr{T}'$ is a regular topology on X which is finer than $\mathscr{T}$ and agrees with $\mathscr{T}$ on every $\mathscr{T}$-compact subset of X, then $(X, \mathscr{T}')$ is also a paracompact σ-space.*

Proof. That $(X, \mathscr{T}')$ is a σ-space follows from Lemma 5.4, so let us prove that it is paracompact. Since $\mathscr{T}'$ is regular, it suffices, by [6, Theorem 1], to show that every open cover of $(X, \mathscr{T}')$ has a σ-locally finite open refinement.

Let $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n$ be a network for $(X, \mathscr{T})$, with each $\mathscr{A}_n$ locally finite in $(X, \mathscr{T})$. Since $\mathscr{T}$ is regular, we may suppose that the elements of $\mathscr{A}$ are $\mathscr{T}$-closed. We may also suppose that $\mathscr{A}$ is closed under finite intersections.

Since $(X, \mathscr{T})$ is paracompact, every locally finite covering can be expanded to a locally finite open covering.[5] Hence for each n there is a locally finite open cover $\{V_n(A): A \in \mathscr{A}_n\}$ of $(X, \mathscr{T})$ such that $A \subset V_n(A)$ for all $A \in \mathscr{A}_n$.

Now let $\mathscr{U}$ be an open cover of $(X, \mathscr{T}')$, and let us find a σ-locally finite open refinement in $(X, \mathscr{T}')$. Let

$$\mathscr{A}^* = \{A \in \mathscr{A}: A \subset U \text{ for some } U \in \mathscr{U}\},$$

and for each $A \in \mathscr{A}^*$ pick some $U(A) \in \mathscr{U}$ such that $A \subset U(A)$. Let

[5] This is easily verified, as in the proof of [6, Lemma 1]. A more precise result was obtained by C. H. Dowker in [2, (2, 3)], and subsequently by M. Katětov in [5].

$\mathscr{A}_n^* = \mathscr{A}_n \cap \mathscr{A}^*$, and for each $A \in \mathscr{A}_n^*$, let

$$W_n(A) = V_n(A) \cap U(A) .$$

Let $\mathscr{W}_n = \{W_n(A) : A \in \mathscr{A}_n^*\}$. Then each $\mathscr{W}_n$ is a collection of open subsets of $(X, \mathscr{T}')$ which is locally finite in $(X, \mathscr{T})$ and thus surely in $(X, \mathscr{T}')$. Let $\mathscr{W} = \bigcup_{n=1}^\infty \mathscr{W}_n$. Clearly each $W \in \mathscr{W}$ is a subset of some $U \in \mathscr{U}$. It remains to show that $\mathscr{W}$ covers X. But if $x \in X$, then $x \in U$ for some $U \in \mathscr{U}$, so $x \in A \subset U$ for some $A \in \mathscr{A}$ by Lemma 5.4, hence $A \in \mathscr{A}_n^*$ for some n, so

$$x \in W_n(A) \in \mathscr{W} .$$

That completes the proof.

Proof of Theorem 5.1. We must establish (a), (b), and (c) below.

(a) $k(X)$ has a σ-locally finite k-network: This follows from Lemma 5.2.

(b) $k(X)$ is regular: Let $\mathscr{T}$ and $k(\mathscr{T})$ be the topologies of X and $k(X)$, respectively.

Let $\mathscr{A} = \bigcup_{n=1}^\infty \mathscr{A}_n$ be a k-network for X, with each $\mathscr{A}_n$ locally finite in X. Since X is regular, we may suppose that each $A \in \mathscr{A}$ is closed in X, and we may also assume that $\mathscr{A}$ is closed under finite intersections. By Lemma 5.2, $\mathscr{A}$ is also a k-network for $k(X)$.

Let $x \in U$, with U open in $k(X)$. We must find a $k(\mathscr{T})$-neighborhood V of x whose $k(\mathscr{T})$-closure is a subset of U.

By assumption, $X = \bigcup_{n=1}^\infty X_n$, with each X_n closed in X and a k-space. Hence $\mathscr{T}$ and $k(\mathscr{T})$ agree on every X_n. Let

$$B_n = X_n - U .$$

Then B_n is $k(\mathscr{T})$-closed in X_n, and hence $\mathscr{T}$-closed in X_n and thus in X. For each n, let

$$P_n = \bigcup \left\{A \subset U :\ A \in \bigcup_{i=1}^{n} \mathscr{A}_i\right\} \cup \{x\} .$$

Then P_n and B_n are disjoint closed subsets of the normal space X, so there is a $\mathscr{T}$-open $W_n \subset X$ whose $\mathscr{T}$-closure $\bar{W}_n$ does not meet B_n. Let $V_n = W_n \cap U$, and let $V = \bigcap_{n=1}^\infty V_n$. We must check that this works.

Clearly $x \in V$, and the $\mathscr{T}$-closure — and thus surely the $k(\mathscr{T})$-closure — of V meets no B_n and is therefore a subset of U. To see, finally, that V is $k(\mathscr{T})$-open, we can apply Lemma 5.3 to the space

$$Y = (U, k(\mathscr{T})) .$$

In fact, since Y is an open subset of the Hausdorff k-space $k(X)$, it is a k-space by [10, Proposition 6.E.2]. Also (P_n) is increasing, and each compact $C \subset Y$ is a subset of some P_n because $\mathscr{A}$ is a k-network for Y. Thus Lemma 5.3 is applicable, and we conclude that V is open in Y and hence in $k(X)$. Thus $k(X)$ is regular.

(c) $k(X)$ is paracompact: This follows immediately from (b) and Lemma 5.5.

That completes the proof of Theorem 5.1.

REMARK. Theorem 5.1 remains true if "paracompact" is replaced (both times) by "normal", or by "monotonically normal" in the sense of P. Zenor [16], or by "stratifiable" in the sense of C. J. R. Borges [1]. For normal spaces, this is proved (without Lemmas 5.4 and 5.5) by applying Lemma 5.3 twice; essentially the same proof works for monotonically normal spaces, and the validity for stratifiable spaces then follows from [16, Theorem B]. Since these results seem less interesting than Theorem 5.1, we omit the details.

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