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STRUCTURE HYPERGROUPS FOR MEASURE ALGEBRAS

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An abstract measure algebra A is a Banach algebra of measures on a locally compact Hausdorff space X such that the set of probability measures in A is mapped into itself under multiplication, and if μ is a finite regular Borel measure on X and $\mu << \nu \in A$ then $\mu \in A$. If A is commutative then the spectrum of A, Δ_A, is a subset of the dual of A, A^*, which is a commutative W^*-algebra. In this paper conditions are given which insure that the weak-* closed convex hull of Δ_A, or of some subset of Δ_A, is a subsemigroup of the unit ball of A^*. This statement implies the existence of certain byper-group structures. An example is given for which the conditions fail.

The theory is then applied to the measure algebra of a compact P^*-hypergroup, for example, the algebra of central measures on a compact group, or the algebra of measures on certain homogeneous spaces. A further hypothesis, which is satisfied by the algebra of measures given by ultraspherical series, is given and it is used to give a complete description of the spectrum and the idempotents in this case.

A hypergroup is a locally compact space on which the space of finite regular Borel measures has a commutative convolution structure preserving the probability measures. The spectrum of the measure algebra of a locally compact abelian group is the semigroup of all continuous semicharacters of a commutative compact topological semi-group (Taylor [7], or see [2, Ch. 1]). In this paper we consider the spectrum of an abstract measure algebra and investigate the question of whether the spectrum or some subset of it has a hypergroup structure.

Section 1 of the paper contains a general theorem on the existence of hypergroup structures on the spectrum of an abstract measure algebra. The fact that the dual space of an appropriate space of measures is a commutative W^*-algebra is of basic importance in the proof of this theorem. This section also contains an example of a compact hyper-group whose measure algebra does not satisfy the hypotheses of the theorem.

In §2 we recall the definition of a compact P^*-hypergroup from a previous paper [1] and apply the main theorem of §1 to this situation. The result is that the closure of the set of characters of the hypergroup in the spectrum is a compact semitopological hypergroup and is a set of characters on another compact semitopological hyper-group.

Section 3 defines a class of P^*-hypergroups of which ultraspherical series form a particular example. A complete description of the spectrum and the idempotents of the measure algebra is given. The results are much like those which Ragozin [6] obtained for the algebra of central measures on a compact simple Lie group.

1. The general situation. We will use the following notation; for a locally compact Hausdorff space X, $C^B(X)$ is the space of bounded continuous functions on X, $C_0(X)$ is the space $\{f \in C^B(X): f \text{ tends to } 0 \text{ at } \infty\}$, $M(X)$ is the space of finite regular Borel measures on X, $M_p(X)$ is the set $\{\mu \in M(X): \mu \geqq 0, \mu X = 1\}$ (the probability measures), δ_x is the unit point mass at $x \in X$, and $M(X)^*$ is the dual space of $M(X)$. If X is compact we write $C(X)$ for $C^B(X)$. We let w^* denote either of the topologies $\sigma(M(X), C_0(X))$ or $\sigma(M(X)^*, M(X))$.

Note that $M(X)^*$ may be interpreted as the space of generalized functions on X, (the projective limit of the spaces $\{L^\infty(X, \mu): \mu \in M_p(X)\}$ ordered by absolute continuity) and is thus seen as a commutative W^*-algebra (see [2, p. 9]). We will write $f \rightarrow \bar{f} (f \in M(X)^*)$ for the involution, $f \cdot \mu$ for the action of $M(X)^*$ on $M(X)$, and $\langle \mu, f \rangle$ for the pairing of $M(X)$ and $M(X)^*$, $(\mu \in M(X), f \in M(X)^*)$. Note $\langle f \cdot \mu, g \rangle = \langle \mu, fg \rangle$ for $f, g \in M(X)^*$, $\mu \in M(X)$, and $\langle \mu, 1 \rangle = \int_X d\mu$. The unit ball B (the set $\{f: \|f\| \leqq 1\}$) of $M(X)^*$ is w^*-compact and is a commutative semitopological semigroup under multiplication and the w^*-topology. We will be concerned with compact convex subsemigroups of B.

Suppose there is given for each $x, y \in X$ a measure $\lambda(x, y) \in M_p(X)$ such that for each $f \in C_0(X)$ the map $(x, y) \mapsto \int_X f d\lambda(x, y)$ is separately continuous. Then for each $\mu, \nu \in M(X)$ the function

$$x \mapsto \int_X \int_X f d\lambda(x, y) d\nu(y)$$

is continuous and

$$\int_X d\mu(x) \int_X d\nu(y) \int_X f d\lambda(x, y) = \int_X d\nu(y) \int_X d\mu(x) \int_X f d\lambda(x, y) \,.$$

This fact was proved by Glicksberg [3]. We will use this to define semitopological hypergroups.

DEFINITION 1.1. A locally compact space H is called a semitopological hypergroup if there is a map $\lambda: H \times H \rightarrow M_p(H)$ with the following properties:

(1) $\lambda(x, y) = \lambda(y, x)$, $(x, y \in H)$, (commutativity);

(2) for each $f \in C_0(H)$ the map $(x, y) \mapsto \int_H f d\lambda(x, y)$ is separately continuous, $(x, y \in H)$;

(3) the convolution on $M(H)$ defined implicitly by

$$\int_H fd(\mu*\nu) = \int_H d\mu(x)\int_H d\nu(y)\int_H fd\lambda(x, y),\ (\mu, \nu \in M(H), f \in C_0(H))$$

is associative, (note $\delta_x*\delta_y = \lambda(x, y)$, $(x, y \in H)$).

If there is a point $e \in H$ such that $\lambda(e, x) = \delta_x$, $(x \in H)$, then e is called the identity of H. A bounded continuous function ϕ on H such that $\int_H \phi d\lambda(x, y) = \phi(x)\phi(y)$, $(x, y \in H)$, is called a character of H.

If H is a compact semitopological hypergroup then it is easily shown that convolution on $M(H)$ is separately w^*-continuous, and that $M_p(H)$ is a compact commutative semitopological affine semigroup ("affine" means $\mu*(s_1\nu_1 + s_2\nu_2) = s_1(\mu*\nu_1) + s_2(\mu*\nu_2)$ for $s_1, s_2 \geqq 0$, $s_1 + s_2 = 1$, $\mu, \nu_1, \nu_2 \in M_p(H)$). The converse to the latter holds (Pym [4] proved a form of this statement; we will give a proof of it in the present context).

PROPOSITION 1.2. *Let H be a compact space and suppose $M_p(H)$ is a commutative semitopological affine semigroup (in the w^*-topology), then H can be given the structure of a compact semitopological hypergroup, so that convolution restricted to $M_p(H)$ gives the original semigroup structure.*

Proof. Let $*$ denote the semigroup operation on $M_p(H)$. This operation extends uniquely to $M(H)$, and $M(H)$ becomes a commutative Banach algebra. For each $x, y \in H$ let $\lambda(x, y) = \delta_x * \delta_y \in M_p(H)$. Now we must show that λ satisfies Definition 1.1, and the convolution induced by λ is the same as the given. By hypothesis, the function $Tf(x, y) = \int_H fd\lambda(x, y) = \int_H fd(\delta_x*\delta_y)$ is separately continuous $(x, y \in H)$. Glicksberg's result [3] shows that $x \mapsto \int_H Tf(x, y)d\mu(y)$ is continuous for each $\mu \in M(H)$. Let μ, ν be finitely supported (discrete) measures in $M_p(H)$, then by an easy computation we have

$$\int_H\int_H Tf(x, y)d\mu(x)d\nu(y) = \int_H fd\mu*\nu\ , \qquad (f \in C(H))\ .$$

For fixed ν the set of μ for which this identity holds is w^*-closed. Thus the identity holds for all $\mu \in M_p(H)$, all finitely supported $\nu \in M_p(H)$. Repeat the argument to show the identity holds for all $\nu \in M_p(H)$.

It is convenient to isolate the following situation as a lemma.

LEMMA 1.3. *Suppose X is a locally compact space, S is a completely regular Hausdorff space, and there is a bounded linear map $j: M(X) \to C^B(S)$ with the following properties (we will write $\|\mu\|_S$ for* $\sup\{|j\mu(s)|: s \in S\}$:

(1) $\|j\| = 1$;

(2) *there exists $\iota \in M_p(X)$ such that $j\iota = 1$ (the constant function);*

(3) $\|j_1 s \cdot \mu\|_S \leqq \|\mu\|_S$, *where $j_1 s \in M(X)^*$ is defined by $\langle \mu, j_1 s\rangle = j\mu(s)$, $(s \in S, \mu \in M(X))$.*

Then the w^*-closed convex hull of $j_1 S$, denoted by $w^* \operatorname{co}(j_1 S)$, is a compact (semitopological) subsemigroup of B, the unit ball in $M(X)^*$. Each map $f \mapsto \langle \delta_x, f\rangle$, $(x \in H)$, is an affine semicharacter on $w^* \operatorname{co}(j_1 S)$. Further, if S is compact and $jM(X)$ is sup-norm dense in $C(S)$, then S has a semitopological hypergroup structure, and the functions $\{j\delta_x: x \in X\}$ are characters of S.

Proof. Let S_1 be a compactification of S such that $jM(X) \subset C(S_1)$, and let j^* denote the adjoint map: $M(S_1) \to M(X)^*$,

$$\left(\text{given by } \langle \mu, j^*\lambda\rangle = \int_{S_1} j\mu d\lambda,\ \mu \in M(X), \lambda \in M(S_1)\right).$$

Denote $w^* \operatorname{co}(j_1 S)$ by S_c. We claim $j^* M_p(S_1) = S_c$. The map j^* is w^*-continuous $M(S_1) \to M(X)^*$ thus j^* maps $w^* \operatorname{co}\{\delta_s: s \in S_s\}$ (in $M(S_1)$) into S_c. That is, $j^* M_p(S_1) \subset S_c$. Conversely let $f \in S_c$, then there exists a net $\{f_\alpha\} \subset \operatorname{co}(j_1 S)$, (the convex hull of $j_1 S$) so that $f_\alpha \xrightarrow[\alpha]{} f(w^*)$. But for each α there exists a finitely supported $\lambda_\alpha \in M_p(S_1)$ so that $j^*\lambda_\alpha = f_\alpha$. By the w^*-compactness of $M_p(S_1)$ there exists $\lambda \in M_p(S_1)$ so that $j^*\lambda = f$. Thus $j^* M_p(S_1) = S_c$.

We observe for $g \in M(X)^*$ that $g \in S_c$ if and only if $|\langle \mu, g\rangle| \leqq \|\mu\|_S$, $(\mu \in M(X))$ and $\langle \iota, g\rangle = 1$. The latter condition and the Hahn-Banach and Riesz theorems imply that there exists $\lambda \in M_p(S_1)$ so that $j^*\lambda = g$. We now show for $s \in S$, $\lambda \in M_p(S_1)$ that $(j_1 s)(j^*\lambda) \in S_c$. Indeed for $\mu \in M(X)$, |

$$\begin{aligned}\langle \mu, (j_1 s)(j^*\lambda)\rangle| &= |\langle j_1 s \cdot \mu, j^*\lambda\rangle| \\ &= \left|\int_{S_1} j(j_1 s \cdot \mu) d\lambda\right| \leqq \|j_1 s \cdot \mu\|_S \leqq \|\mu\|_S .\end{aligned}$$

Also $\langle \iota, (j_1 s)(j^*\lambda)\rangle = \langle j_1 s \cdot \iota, j^*\lambda\rangle = \langle \iota, j^*\lambda\rangle = 1$, (note $j_1 s \cdot \iota = \iota$, since $\|j_1 s\| \leqq 1$, $\langle \iota, j_1 s\rangle = j\iota(s) = 1$ and $\iota \in M_p(X)$). Thus $(j_1 s)(j^*\lambda) \in S_c$ and we conclude from the separate w^*-continuity of multiplication that $S_c S_c \subset S_c$; so S_c is a subsemigroup of B.

For each $x \in X, f \in M(X)^*$ we have that $f \cdot \delta_x = \langle \delta_x, f\rangle \delta_x$ so the maps $f \mapsto \langle \delta_x, f\rangle$ are affine semicharacters of S_c.

Now suppose that S is a compact and $jM(X)$ is norm dense in $C(S)$. Then j^* maps $M_p(S)$ one-to-one, w^*-continuous, and onto S_c. Thus $M_p(S)$ with the w^*-topology is homeomorphic to S_c. We define a semigroup structure on $M_p(S)$ by using this isomorphism (that is, for $\lambda, \nu \in M_p(S)$ define $\lambda * \nu = (j^*)^{-1}((j^*\lambda)(j^*\nu)))$. Thus $M_p(S)$ is a commutative affine w^*-semitopological semigroup. By Proposition 1.2 S is a compact semitopological hypergroup. Further for $x \in X, \lambda \in M(S)$, $\int_S (j\delta_x)d\lambda = \langle \delta_x, j^*\lambda \rangle$, which shows that $j\delta_x$ is a character of S.

Note that in the lemma $M(X)$ may be replaced by an L-subspace A of $M(X)$, (that is, A is a closed subspace of $M(X)$ and $\mu \in M(X)$ and $\mu << \nu \in A$ implies $\mu \in A$). The dual of A is a w^*-closed ideal in $M(X)^*$ and so is itself a commutative W^*-algebra. However, the point masses δ_x may not be in A.

DEFINITION 1.4. Suppose X is a locally compact Hausdorff space and A is an L-subspace of $M(X)$. Say A is an abstract measure algebra if it is a Banach algebra in the measure norm, and $A_pA_p \subset A_p$ (where $A_p = A \cap M_p(X)$). We say A has an identity if there exists an algebra identity $\iota \in A_p$. If A is commutative we let Δ_A denote the spectrum (maximal ideal space) of A, considered as a subset of the unit ball of the dual A^* of A. Further $\tilde{\mu}$ denotes the Gelfand transform of $\mu \in A$, so $\tilde{\mu} \in C_0(\Delta_A)$.

THEOREM 1.5. *Suppose A is a commutative abstract measure algebra with identity ι, and E is a w^*-closed subset of Δ_A with the following properties*: (1) $1 \in E$; (2) $f \in E$ *implies* $\bar{f} \in E$; (3) $g \in E, \mu \in A$ *imply* $\|(g \cdot \mu)^\sim\|_E \leqq \|\tilde{\mu}\|_E$, (*where* $\|\tilde{\mu}\|_E = \sup\{|\tilde{\mu}(f)| : f \in E\}$). *Then the norm-closed linear span of $w^* \operatorname{co} E$ is isomorphic to $C(Y)$, where Y is a compact semitopological hypergroup with an identity, and the natural map $\sigma: A \to M(Y)$ is a homomorphism with w^*-dense range. Further $\sigma\iota = \delta_e$, where e is the identity in Y. If A contains a point mass δ_x, then $\sigma\delta_x$ is a point mass in Y. The set E considered as a subset of $C(Y)$ consists of characters of Y.*

Proof. The Gelfand transform maps $A \to C(E)$. By Lemma 1.3 $w^* \operatorname{co}(E)$ is closed under multiplication. Thus the norm closure of $\operatorname{sp}(w^* \operatorname{co}(E))$ is a self-adjoint closed subalgebra of A^*, hence is isomorphic to $C(Y)$, (Y is its spectrum). We define the natural map $j: M(E) \to C(Y)$ so that $\langle \mu, j\lambda \rangle = \int_E \tilde{\mu} d\lambda$, ($\mu \in A, \lambda \in M(E)$); note $j\lambda \in C(Y) \subset A^*$. Observe $j\delta_1 = 1$, and $jM_p(E) = w^* \operatorname{co}(E)$. We show that j satisfies the hypotheses of Lemma 1.3. Note that $\|j\lambda\|_Y$ is given by

$$\begin{aligned} \|j\lambda\|_Y &= \sup\{|\langle\mu, j\lambda\rangle| : \mu \in A, \|\mu\| \leqq 1\} \\ &= \sup\left\{\left|\int_E \tilde{\mu} d\lambda\right| : \mu \in A, \|\mu\| \leqq 1\right\}. \end{aligned}$$

Let $y \in Y$ and define $j_1: Y \to M(E)^*$ by $\langle\lambda, j_1 y\rangle = j\lambda(y)$, $(\lambda \in M(E))$. For $\mu \in A, \lambda \in M(E)$ we have

$$\langle\mu, j(j_1 y \cdot \lambda)\rangle = \int_E \tilde{\mu} d(j_1 y \cdot \lambda) = \langle\tilde{\mu} \cdot \lambda, j_1 y\rangle = j(\tilde{\mu} \cdot \lambda)(y) .$$

Thus

$$\|j_1 y \cdot \lambda\|_Y \leqq \sup\{\|j(\tilde{\mu} \cdot \lambda)\|_Y : \mu \in A, \|\mu\| \leqq 1\} .$$

Now

$$\begin{aligned} \|j(\tilde{\mu} \cdot \lambda)\|_Y &= \sup\{|\langle\nu, j(\tilde{\mu} \cdot \lambda)\rangle| : \nu \in A, \|\nu\| \leqq 1\} \\ &= \sup\left\{\left|\int_E \tilde{\nu}\tilde{\mu} d\lambda\right| : \nu \in A, \|\nu\| \leqq 1\right\} \\ &\leqq \sup\{\|\nu\| \, \|\mu\| \, \|j\lambda\|_Y : \nu \in A, \|\nu\| \leqq 1\} \\ &= \|\mu\| \, \|j\lambda\|_Y , \end{aligned}$$

(since $\tilde{\nu}\tilde{\mu} = (\nu\mu)^\sim$ and $\|\nu\mu\| \leqq \|\nu\| \, \|\mu\|$). Thus $\|j_1 y \cdot \lambda\|_Y \leqq \|\lambda\|_Y$. Further $jM(E) = \mathrm{sp}\,(w^* \mathrm{co}\, E)$ is dense in $C(Y)$, so by Lemma 1.3 Y is a compact semitopological hypergroup. Note that $E \subset C(Y)$ consists of characters of Y.

Let σ be the natural map $A \to M(Y)$. Clearly σA is w^*-dense in $M(Y)$. Further the convolution on $M(Y)$ is defined in terms of multiplication in $M(E)^*$, but the map $A \to C(E) \subset M(E)^*$ is a homomorphism, so σ is a homomorphism.

Since $\tilde{\iota} = 1$ on E we have $\langle\iota, f\rangle = 1$ for all $f \in w^* \mathrm{co}\, E$. For $f, g \in w^* \mathrm{co}\,(E)$, $\langle\iota, fg\rangle = 1 = \langle\iota, f\rangle\langle\iota, g\rangle$ (since $fg \in w^* \mathrm{co}\, E$) thus $f \to \langle\iota, f\rangle$ is multiplicative and norm bounded on $\mathrm{sp}\,(w^* \mathrm{co}\,(E))$, so there exists a unique point $e \in Y$ so that $\langle\iota, f\rangle = f(e)$, $(f \in C(Y))$. Thus $\sigma\iota = \delta_e$ and e is the identity of Y. If there is a point mass $\delta_x \in A$ then $f \to \langle\delta_x, f\rangle$ is multiplicative on A^*, so $\sigma\delta_x$ is a point mass in Y.

It would be interesting to know whether Y has any characters other than the elements of E, but the answer is presently unknown to the author. If Δ_A has the properties specified for E, then the set characters of Y is Δ_A, since σA is w^* dense in $M(Y)$ and characters of Y give multiplicative linear functionals on $M(Y)$.

This line of investigation was motivated partly by Taylor's work [7] on structure semigroups of convolution measure algebras. Pym [5] has a result similar to Theorem 1.5 for the spectrum of a commutative Banach measure algebra $M(X)$ in which multiplication is separately w^*-continuous and the map $\mu \mapsto f \cdot \mu$ is bounded in the spectral norm ($\mu \mapsto \|\tilde{\mu}\|_\infty$), for each $f \in \Delta_{M(X)}$.

A compact hypergroup H is defined by Definition 1.1 with "separately continuous" in condition (2) replaced by "jointly continuous". We write $\hat{H}$ for the set of characters of H, and Δ_H for the spectrum of $M(H)$. For $\mu \in M(H)$, $\phi \in \hat{H}$, let $\hat{\mu}(\phi) = \int_H \bar{\phi} d\mu$. In the sequel we will refer to [1] for necessary details.

We will now construct a compact hypergroup H for which neither Δ_H nor the closure of $\hat{H}$ in Δ_H satisfy the hypotheses of Theorem 1.5.

EXAMPLE 1.6. There exists a compact hypergroup H and $\psi \in \kappa\hat{H}$ (the closure of $\hat{H}$ in Δ_H) such that $\mu \mapsto \psi \cdot \mu$, ($\mu \in M(H)$), is bounded in neither the $\|\tilde{\ }\|_\infty$ nor the $\|\hat{\ }\|_\infty$ norm.

Proof. Let H_1 be the finite hypergroup described in Example 4.6 of [1]. Briefly the points of H_1 correspond to rows of the matrix

$$\begin{array}{c c} & \begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array} \\ \begin{array}{c} e \\ r_1 \\ r_2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1/2 & 0 \\ 1 & 1/4 & 0 \end{bmatrix} \end{array}$$

and multiplication is pointwise. That is, the columns correspond to the characters of H_1. Note that $\phi_1^2 = (1/8)(\phi_0 - 2\phi_1 + 9\phi_2)$. Let ν be the measure $\delta_e + \delta_{r_1} - 2\delta_{r_2}$ on H_1, then $\tilde{\nu}(\phi_0) = 0$, $\tilde{\nu}(\phi_1) = 0$, and $\tilde{\nu}(\phi_2) = 1$.

Let H be the Tikhonov product $\prod_{n=1}^{\infty} H_1$, so H is a compact hypergroup. For $n = 1, 2, \cdots$, let $H_n = \prod_{i=1}^{n} H_1$. We identify $M(H_n)$ with a subalgebra of $M(H)$ under the map

$$\int_H f d\sigma\mu = \int_{H_n} f(x_1, \cdots, x_n, e, e, \cdots) d\mu(x_1, \cdots, x_n) ,$$

($f \in C(H)$, $\mu \in M(H_n)$). By a multi-index I we mean a sequence $I = (i_1, i_2, \cdots)$ where $i_s = 0, 1, 2$ and $i_s = 0$ for all but finitely many s. For a multi-index I let $\phi_I(x) = \phi_{i_1}(x_1)\phi_{i_2}(x_2) \cdots$, then $\phi_I \in \hat{H}$. Let $\nu_n = \nu \times \cdots \times \nu$ (n times), an element of $M(H_n)$, and let $\mu_n = \sigma\nu_n \in M(H)$. The spectrum of $M(H_n)$ is isomorphic to $S_n = \{\phi_I : I \text{ multi-index}, i_s = 0 \text{ for } s > n\}$. Thus the spectral norm of a measure in $M(H_n)$ (or $\sigma M(H_n)$) is realized on S_n. Let $\psi_n^m \in \hat{H}$ be given by $\psi_n^{(m)}(x) = \phi_m(x_1) \cdots \phi_m(x_n)$ ($x \in H$, $m = 1, 2$). We claim $\|\tilde{\mu}_n\|_\infty = \|\hat{\mu}_n\|_\infty = 1$, in fact for $\phi_I \in S_n$, $\langle \mu_n, \phi_I \rangle = \prod_{s=1}^{n} \langle \nu, \phi_{i_s} \rangle = 0$ if $\phi_I \neq \psi_n^{(2)}$, and $\langle \mu_n, \psi_n^{(2)} \rangle = 1$. Let $m \geqq n$, then $\langle \psi_m^{(1)} \cdot \mu_n, \psi_n^{(1)} \rangle = (9/8)^n$. Indeed $\langle \psi_m^{(1)} \cdot \mu_n, \psi_n^{(1)} \rangle = \int_H \psi_m^{(1)} \psi_n^{(1)} d\mu_n = \prod_{s=1}^{n} \langle \nu, \phi_1\phi_1 \rangle = (9/8)^n$. Let ψ be a w^*-cluster point of $\{\psi_n^{(1)}\}$ in Δ_H.

Then $\langle \psi \cdot \mu_n, \psi_n^{(1)} \rangle = (9/8)^n$ and $\|\tilde{\mu}_n\|_\infty = \|\hat{\mu}_n\|_\infty = 1$, but $\|(\psi \cdot \mu_n)^\sim\|_\infty \geqq \|(\psi \cdot \mu_n)^\wedge\|_\infty \geqq (9/8)^n$.

2. P^*-hypergroups.

See [1] for a reference for this section.

DEFINITION 2.1. A compact hypergroup H is called a P^*-hypergroup if:

(1) there exists an invariant measure $m_H \in M_p(H)$ and a continuous involution $x \mapsto x'$, $(x \in H)$ such that

$$\int_H (R(x)f)\bar{g}\,dm_H = \int_H f(R(x')g)^- dm_H ,$$

and such that $e \in \text{support } \lambda(x, x')$, $(f, g \in C(H), x \in H)$, $(R(x)\colon C(H) \to C(H)$ is defined by $R(x)f(y) = \int_H f d\lambda(x, y), f \in C(H), x \in H)$;

(2) $\hat{H}\hat{H} \subset \text{co}\,\hat{H}$, that is, for each $\phi, \psi \in \hat{H}$ there exists a nonnegative function $n(\phi, \psi; \cdot)$ on $\hat{H}$ with only finitely many nonzero values such that $\phi(x)\psi(x) = \sum_{\omega \in \hat{H}} n(\phi, \psi; \omega)\omega(x)$, $(x \in H)$.

Recall from [1] that each subhypergroup K of H is, by definition, closed and is normal ($x \in K$ implies $x' \in K$), if H is P^*. Furthermore, K is itself a P^*-hypergroup with invariant measure m_K.

DEFINITION 2.2. Let H be a compact P^*-hypergroup and let $\mu \in M(H)$. Define $\mu^* \in M(H)$ by

$$\int_H f d\mu^* = \left(\int_H (f(x'))^- d\mu(x)\right)^-, (f \in C(H)) .$$

Then $\mu \to \mu^*$ is an algebra involution and $(\mu^*)^\wedge(\phi) = (\hat{\mu}(\phi))^-$, $(\phi \in \hat{H})$ (see Theorem 3.5 [1]).

DEFINITION 2.3. The set $B(\hat{H}) = \{\hat{\mu}\colon \mu \in M(H)\} \subset C^B(\hat{H})$ is a self-adjoint separating algebra of continuous functions on $\hat{H}$ and contains the constants. Let $\kappa\hat{H}$ be the compactification of $\hat{H}$ induced by this algebra. Equivalently $\kappa\hat{H}$ is the spectrum of the sup-norm closure of $B(\hat{H})$, and $\hat{H}$ is a dense open subset.

THEOREM 2.4. *$\kappa\hat{H}$ is a compact semitopological hypergroup, and $\hat{H}$ is a discrete subhypergroup. Further $\kappa\hat{H}$, as a subset of Δ_H (the spectrum of $M(H)$), is w^*-closed, contains* 1, *and is self-adjoint.*

Proof. Let j be the bounded linear map: $M(H) \to C(\kappa\hat{H})$ which is determined by $(j\mu)(\phi) = \hat{\mu}(\phi) = \int_H \bar{\phi}\,d\mu$, $(\mu \in M(H), \phi \in \hat{H})$. Observe $\|j\mu\|_\infty = \|\hat{\mu}\|_\infty$. Also $j\delta_e = 1$. For $\phi, \psi \in \hat{H}, \mu \in M(H)$ we have

$$j(\bar{\phi}\cdot\mu)(\psi) = \int_H \overline{\phi\psi}\,d\mu = \sum_{\omega\in\hat{H}} n(\phi, \psi; \omega)\int_H \bar{\omega}\,d\mu = \sum_{\omega\in\hat{H}} n(\phi, \psi; \omega)\hat{\mu}(\omega) .$$

But $|j(\bar{\phi}\cdot\mu)(\psi)| \leqq \sum_{\omega\in\hat{H}} n(\phi, \psi; \omega)|\hat{\mu}(\omega)| \leqq \|\hat{\mu}\|_\infty = \|j\mu\|_\infty$. Thus we can apply Lemma 1.3 and obtain that $\kappa\hat{H}$ is a semitopological hypergroup. Further $M_p(\kappa\hat{H})$ is isomorphic to $w^*\operatorname{co}(\hat{H}) \subset M(H)^*$, and the functions $\{j\delta_x : x \in H\}$ are characters of $\kappa\hat{H}$.

We now apply Theorem 1.5 to $\kappa\hat{H}$ and obtain the following:

THEOREM 2.5. *Suppose H is a compact P^*-hypergroup, then there exists a compact semitopological hypergroup Y such that $\kappa\hat{H}$ is a set of characters of Y, the norm-closed span of $w^*\operatorname{co}(\hat{H})$ is isomorphic to $C(Y)$, and there is a monomorphism $\sigma: M(H) \to M(Y)$ with w^*-dense range.*

3. **Simple P^*-hypergroups.** In this section H will always denote a compact P^*-hypergroup. We will describe an additional hypothesis which allows a complete description of Δ_H. This hypothesis is realized in the algebra of ultraspherical series (see Example 4.3 [1]). The author suspects that the algebra of central measures on a compact simple Lie group also satisfies the hypothesis.

Recall from [1] that the center of H, $Z(H)$, is $\{x \in H: y \in H$ implies that $\lambda(x, y)$ is a point mass$\}$. Further $Z(H)$ is a compact subgroup of H and is the set $\{x \in H: |\phi(x)| = 1, (\phi \in \hat{H})\}$.

DEFINITION 3.1. Let n be a positive integer. Say H has property S_n if for each compact set $K \subset H\backslash Z(H)$ the sum $\sum_{\phi\in\hat{H}} c(\phi)(\sup_K|\phi|)^{2n} < \infty$, $\left(\text{where } c(\phi) = \left(\int_H |\phi|^2 dm_H\right)^{-1}\right)$. (The letter "$S$" suggests "simple" in the sense that if K is a subhypergroup of H such that $K \not\subset Z(H)$ then K is open; see 3.4.) Say H is an SP-* hypergroup if it has property S_n for some n.

DEFINITION 3.2. Let $M_h(H) = \{\mu \in M(H): |\mu|Z(H) = 0\}$, an L-subspace of $M(H)$. Note $M(H) = M(Z(H)) \oplus M_h(H)$. Let π be the norm-bounded projection: $M(H) \to M(Z(H))$. For $\mu \in M(H)$ we write $\mu = \pi\mu + \mu_h$, so $\mu_h \in M_h(H)$.

We will show that if H is an SP-* hypergroup and $m_H(Z(H)) = 0$ then $M_h(H)$ is an ideal in $M(H)$ and its annihilator in Δ_H is $\Delta_H\backslash\hat{H}$. Thus $\Delta_H\backslash\hat{H}$ is isomorphic to $\Delta_{Z(H)}$. The case $m_H(Z(H)) > 0$ will also be discussed.

PROPOSITION 3.3. *Suppose H is an SP-* hypergroup with property S_n for some positive integer n and $\mu \in M_h(H)$, then $\mu^n \in L^1(H)$, (note $\mu^n = \mu * \mu \cdots * \mu$ (n times)).*

Proof. First suppose $\mu \in M_h(H)$ has compact support K with $Z(H) \cap K = \varnothing$. Then for $\phi \in \hat{H}$, $|\hat{\mu}(\phi)| = \left|\int_K \bar{\phi} d\mu\right| \leqq ||\mu|| \sup_K |\phi|$. We claim $\mu^n \in L^2(H) \subset L^1(H)$; indeed $\sum_{\phi \in \hat{H}} c(\phi) |(\mu^n)\hat{}(\phi)|^2 = \sum_\phi c(\phi) |\hat{\mu}(\phi)|^{2n} \leqq ||\mu||^{2n} \sum_\phi c(\phi) (\sup_K |\phi|)^{2n} < \infty$. The set of such μ is norm-dense in $M_h(H)$ and the map $\mu \mapsto \mu^n$ is norm-continuous taking a dense subset of $M_h(H)$ into $L^1(H)$, a closed subspace of $M(H)$.

For $M_h(H)$ to be a nontrivial ideal it is necessary that $L^1(H) \subset M_h(H)$. We present a lemma which gives several equivalent characterizations of this.

LEMMA 3.4. *Let K be a subhypergroup of a compact P^*-hypergroup H. The following statements are equivalent:*
(Recall $K^\perp = \{\phi \in \hat{H} : \phi | K = 1\}$)

(1) *K is open;*
(2) *$m_H(K) > 0$;*
(3) *each hypercoset of $K^\perp$ is finite;*
(4) *some hypercoset of $K^\perp$ is finite;*
(5) *m_K is a nonzero multiple of $m_H | K$.*

Proof. We first observe that each of (3) and (4) is equivalent to $K^\perp$ being finite. It $K^\perp$ is finite then each hypercoset $\phi \cdot K^\perp$, $(\phi \in \hat{H})$, is finite, since $\phi\psi$ has finite support in $\hat{H}$, $(\psi \in \hat{H})$. Further $K^\perp$ is contained in the support of $\bar{\phi} \cdot (\phi \cdot K^\perp)$ for each $\phi \in \hat{H}$, so if some hypercoset is finite then $K^\perp$ is finite (for more details see 3.16 [1]).

(1) implies (2): Note that the support of m_H is H, (3.2 [1]).

(2) implies (3): The characteristic function $\chi_K \in L^2(H)$ and $(\chi_K)\hat{}(\phi) = \int_K \bar{\phi} dm_H = m_H(K) > 0$ for $\phi \in K^\perp$. But $\sum_{\phi \in \hat{H}} c(\chi_K) |(\phi)\hat{}(\phi)|^2 < \infty$, thus $K^\perp$ is finite, (since $c(\phi) \geqq 1$).

(3) implies (1) and (5): Recall $(m_K)\hat{}$ is 1 on $K^\perp$ and 0 off $K^\perp$ (3.14 [1]). Since $K^\perp$ is finite we have $m_K = f \cdot m_H$ where $f \in C(H)$; in fact $f \in \operatorname{sp} \hat{H}$. Since the support of m_H is H we see that $f \geqq 0$ and $f = 0$ off K. We will show that f is constant on K, which implies that K is open and m_K is a nonzero multiple of $m_H | K$. Since $f \cdot m_H$ is the invariant measure on K, the identity $(f \cdot m_H) * \mu = f \cdot m_H$ holds for each $\mu \in M_p(K)$, (1.12 [1]). By Proposition 3.4 [1] this implies that

$$f(x) = \int_K R(x) f(y') d\mu(y) , \qquad (x \in K) .$$

Thus $f(x) = R(x) f(y')$ for each $x, y \in K$. Let $a = \sup_K f$ and let $K_1 = \{x \in K : f(x) = a\}$. For $x \in K_1$, $y \in K$, $a = f(x) = R(x) f(y') = \int_K f d\lambda(x, y')$, but this implies that f is constant with value a on the support of $\lambda(x, y')$. Thus K_1 is a nonempty (closed) ideal in K, but K is normal so $K_1 = K$ and f is constant on K.

(5) implies (2): Clear.

Note if H is an SP-* hypergroup and $x \in H \backslash Z(H)$ then

$$\{\phi \in \hat{H}: |\phi(x)| = 1\}$$

is finite, so if K is a subhypergroup of H with $K \not\subset Z(H)$ then $K^{\perp}$ is finite implying K is open (by 3.4).

The following will be needed for the case where $Z(H)$ is open in H.

LEMMA 3.5. *Suppose K is an open subhypergroup of a compact P^*-hypergroup H, $\psi \in \hat{K}$ and $\mu \in M(H)$ with $|\mu|K = 0$, then*

$$\sum \{c(\phi)\hat{\mu}(\phi): \phi \in \hat{H}, \phi|K = \psi\} = 0 ,$$

(note this is a sum over a (finite) hypercoset of $K^{\perp}$).

Proof. We will show that $\sum_{\phi|K=\psi} c(\phi)\phi$ is equal to a multiple of ψ on K and is zero off K. By Lemma 3.4 there exists $d \geqq 1$ such that $m_K = dm_H|K$. Let $f \in C(H)$ be defined by $f = \psi$ on K and $f = 0$ off K. Then $\hat{f}(\phi) = \int_K \bar{\phi}\psi dm_H = (1/d)\int_K \bar{\phi}\psi dm_K$, so $\hat{f}(\phi) = (dc(\psi))^{-1}$ for $\phi|K = \psi$ and $\hat{f}(\phi) = 0$ otherwise, $\left(\text{note } c(\psi) = \left(\int_K |\psi|^2 dm_K\right)^{-1}\text{, see 3.17 [1]}\right)$.

Thus $f \in \text{sp}\hat{H}$ and is given by the series $(dc(\psi))^{-1} \sum_{\phi|K=\psi} c(\phi)\phi$. Now

$$\begin{aligned} 0 = \int_H \bar{f} d\mu &= (dc(\psi))^{-1} \sum_{\phi|K=\psi} c(\phi) \int_H \bar{\phi} d\mu \\ &= (dc(\psi))^{-1} \sum_{\phi|K=\psi} c(\phi)\hat{\mu}(\phi) . \end{aligned}$$

For the following H will be an SP-* hypergroup, and for notational convenience we will write G for $Z(H)$.

PROPOSITION 3.6. *If $m_H G = 0$ then the projection $\pi: M(H) \to M(G)$ is a homomorphism and is bounded in the $\hat{H}$-sup-norm ($\|\hat{\mu}\|_\infty$).*

Proof. For $\mu \in M(H)$ we set $\mu = \pi\mu + \mu_h$. By 3.3 there exists an integer n so that $\mu_h^n \in L^1(H)$. Thus $\hat{\mu}_h \to 0$ at ∞ on $\hat{H}$. Let $\gamma \in \hat{G}$, then $E_\gamma = \{\phi \in \hat{H}: \phi|G = \gamma\}$ is a hypercoset of $G^{\perp}$ and is infinite (see 3.17 [1]). Let $\psi \in \kappa\hat{H} \backslash \hat{H}$ ($\kappa\hat{H}$ is the closure of $\hat{H}$ in Δ_H) be the limit of an infinite convergent net $\{\phi_\alpha\} \subset E_\gamma$. Then $\tilde{\mu}(\psi) = \lim_\alpha \tilde{\mu}(\phi_\alpha) =$

$\lim_\alpha ((\pi\mu)\hat{}(\gamma) + (\mu_h)\hat{}(\phi_\alpha)) = (\pi\mu)\hat{}(\gamma)$. Note also $|\tilde{\mu}(\psi)| \leqq \|\hat{\mu}\|_\infty$. Thus $\|(\pi\mu)\hat{}\|_\infty \leqq \|\hat{\mu}\|_\infty$ and the functional $\mu \mapsto (\pi\mu)\hat{}(\gamma)$ is multiplicative for each $\gamma \in \hat{G}$. Hence π is a homomorphism.

The following is now evident, (note for $\mu_h \in M_h(H)$ that $\tilde{\mu}_h = 0$ off $\hat{H}$).

THEOREM 3.7. *If $m_H G = 0$ then each element of $\Delta_H \backslash \hat{H}$ is of the form $\mu \mapsto (\pi\mu)\tilde{}(\psi)$ for some $\psi \in \Delta_G$. This correspondence is an isomorphism (of compact semitopological semigroups) of $\Delta_H \backslash \hat{H}$ with Δ_G. The hypergroup $\kappa\hat{H}$ is isomorphic to $\hat{H} \cup \kappa\hat{G}$ (where $\kappa\hat{G}$ is the closure of $\hat{G}$ in Δ_G), and $\hat{H}$ is attached to $\kappa\hat{G}$ so that an unbounded net $\{\phi_\alpha\} \subset \hat{H}$ clusters at a point $\psi \in \kappa\hat{G}$ if $\{\phi_\alpha | G\} \subset \hat{G}$ clusters at ψ.*

In this particular situation, $\operatorname{co} \Delta_H$ is already a semigroup. Let S be the spectrum of the norm-closed span of Δ_G in $M(G)^*$, then S is a compact semitopological semigroup (Taylor [7], or see [2, Ch. 1]). Let σ_1 be the canonical homomorphism: $M(G) \to M(S)$. Let Y be the spectrum of the norm-closed span of $\operatorname{co}(\Delta_H)$ in $M(H)^*$. Then Y is the disjoint union of H and S. The homomorphism $\sigma: M(H) \to M(Y)$ is given by $\sigma\mu = \sigma_1(\pi\mu) + \mu_h$; recall $\pi\mu \in M(G)$ so $\sigma_1(\pi\mu) \in M(S)$ and $\mu_h \in M(H)$. Since σ has w^*-dense range we see that H is an ideal in Y.

THEOREM 3.8. *Suppose $m_H G = 0$ and μ is an idempotent in $M(H)$, then $\pi\mu$ is an idempotent in $M(G)$ and $\hat{\mu}_h$ has finite support in $\hat{H}$. Thus $\{\phi \in \hat{H}: \hat{\mu}(\phi) = 1\}$ is in the hypercoset ring of $\hat{H}$.*

Proof. Since π is a homomorphism, $\pi\mu$ is idempotent in $M(G)$. Thus $(\mu_h)\hat{} = \hat{\mu} - (\pi\mu)\hat{}$ is integer-valued, but tends to zero at ∞ on $\hat{H}$, so is zero for all but finitely many points in $\hat{H}$. By Cohen's theorem [2, Ch. 5], $S = \{\gamma \in \hat{G}: (\pi\mu)\hat{}(\gamma) = 1\}$ is in the coset ring of $\hat{G}$. The set $\{\phi \in \hat{H}: (\pi\mu)\hat{}(\phi) = 1\} = \{\phi \in \hat{H}: \phi | G \in S\}$, which is in the hypercoset ring of $\hat{H}$ (see 3.18 [1]).

If G is open in H then each hypercoset of $G^\perp$ is finite. In this case $M_h(H)$ is not an ideal (unless $H = G$), but $\mu \in M_h(H)$ does imply $\tilde{\mu} = 0$ off $\hat{H}$. Each element of $\Delta_H \backslash \hat{H}$ is of the form $\mu \mapsto (\pi\mu)\tilde{}(\psi)$, $(\mu \in M(H))$ for some $\psi \in \Delta_G \hat{G}$. (Note if $\pi\mu \in L^1(G) \subset L^1(H)$ then $(\pi\mu)\tilde{}$ is zero off $\hat{G} \subset \Delta_G$ and is zero off $\hat{H} \subset \Delta_H$.) Thus $\Delta_H \backslash \hat{H}$ is isomorphic to $\Delta_G \backslash \hat{G}$. It can be shown that Δ_H is isomorphic to $(\Delta_G \backslash \hat{G}) \cup \hat{H}$ with $\hat{H}$ attached to $\kappa\hat{G} \backslash \hat{G}$ in the obvious way.

THEOREM 3.7. *If G is open in H and μ is an idempotent in $M(H)$ then $\{\phi \in \hat{H}: \hat{\mu}(\phi) = 1\}$ is in the hypercoset ring of $\hat{H}$.*

Proof. Set $\mu = \pi\mu + \mu_h$. We will show $\hat{\mu}_h$ is finitely supported on $\hat{H}$, thus $\pi\mu$ differs from an idempotent in $M(G)$ by a trig polynomial on G (an element of $\operatorname{sp} \hat{G} \subset C(G)$). Since $\hat{\mu}_h \to 0$ at ∞ on $\hat{H}$, the set $F = \{\phi \in \hat{H}: |(\mu_h)^\wedge(\phi)| > 1/3\}$ is finite. Let $F_1 = \bigcup_{\phi \in F} \phi \cdot G^\perp$, a finite union of hypercosets of $G^\perp$, then F_1 is finite since $G^\perp$ is finite (see 3.4). We claim $(\mu_h)^\wedge = 0$ off F_1. Indeed, let $\phi \in \hat{H}\backslash F_1$ and suppose $\phi_1 \in \hat{H}$ with $\phi | G = \phi_1 | G$, then $\phi_1 \notin F_1$ and $(\pi\mu)^\wedge(\phi) = (\pi\mu)^\wedge(\phi_1)$. Thus $|\hat{\mu}(\phi_1) - \hat{\mu}(\phi)| = |(\mu_h)^\wedge(\phi_1) - (\mu_h)^\wedge(\phi)| \leqq 2/3$. But $\hat{\mu}$ is integer valued so $\hat{\mu}(\phi_1) = \hat{\mu}(\phi)$ and $(\mu_h)^\wedge(\phi_1) = (\mu_h)^\wedge(\phi)$. Thus $\hat{\mu}_h$ is constant on $\phi \cdot G^\perp$ and by Lemma 3.5 we have $(\mu_h)^\wedge = 0$ on $\phi \cdot G^\perp$.

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