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ORDERS WITH FINITE GLOBAL DIMENSION

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Let Λ be a left noetherian ring of finite left global dimension. Assume that Λ is quasi-local, i.e., that Λ modulo its Jacobson radical $J(\Lambda)$ is a simple artin ring, and suppose that for some left Ore denominator set S contained in Λ, $\Sigma = S^{-1}\Lambda$ is a left artin ring. Then Σ is a simple artin ring. More precisely, if $\Lambda/J(\Lambda) \cong M_n(K)$, the ring of n by n matrices over a division ring K, then there exists an integer m dividing n such that $\Lambda = M_m(\Delta)$ and $\Sigma = M_m(L)$, where L is a division ring and Δ is an order in L.

A corollary of this is that if the above Λ is *local*, then Σ is a division ring. Another corollary is that if F is a field and Λ is a left noetherian left order in $M_p(F)$, and if Λ is quasi-local and l. gl. dim. $\Lambda < \infty$, then $\Lambda = M_p(R)$, where $R \subset F$ is the center of Λ.

These results were originally established by the author in a less general setting (some restrictions were placed upon the center of Λ). The present form of Theorem 4 and its proof are due to George Bergman, whose help is greatly appreciated.

It seems natural to ask whether it is necessary to *assume* that Λ is an order in an artin ring. It is well-known that a left noetherian left hereditary ring has a left artinian left quotient ring [1]. Small [2] has given an example of a two-sided noetherian ring of global dimension two which is not an order in an artin ring. His ring is not quasi-local. We show that Λ has an artinian left quotient ring if Λ is quasi-local, left noetherian, and has left and right global dimension two. For rings of global dimension greater than two, the question is open.

Preliminaries. All rings under consideration have units and all modules are unital finitely generated left modules. By gl. dim. Λ we mean the left global dimension of Λ. We now recall some standard definitions. A multiplicatively closed subset S, all of whose members are *regular* (i.e., neither left nor right zero-divisors) in Λ, is called a *left Ore denominator set* if the *left Ore condition* is satisfied: If $a \in \Lambda$ and $s \in S$ then there exist $a_1 \in \Lambda$ and $s_1 \in S$ such that $s_1 a = a_1 s$. It is well-known that under this condition, $S^{-1}\Lambda = \{s^{-1}a \mid s \in S, a \in \Lambda\}$ is a ring containing Λ, and in which (clearly) the elements of S are invertible. $S^{-1}\Lambda$ is called the *left quotient ring* (or *left ring of fractions*) of Λ with respect to S, and Λ is called a *left order* in $S^{-1}\Lambda$.

1. Some lemmas.

LEMMA 1. *Let Γ be a quasi-local artin ring of finite global dimension. Then* gl. dim. $\Gamma = 0$, *and thus Γ is simple.*

Proof. Since the global dimension of Γ is finite, it equals the finitistic global dimension of Γ. We claim the latter is zero since Γ is quasi-local and artin. (Though this fact is perhaps well-known, we include a proof for the sake of completeness.) It clearly suffices to show that if the projective dimension of a module M is $\leqq 1$, then M is projective. Since Γ is artin, projective covers exist, and so we may assume an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ with the P_i's projective and $\operatorname{Im}(P_1) \subset J(\Gamma)P_0$. Now since Γ is artin, $J(\Gamma)$ is nilpotent, and in particular has nonzero left annihilator. So P_1 is not a faithful R-module. But over a quasi-local ring a nonzero projective module is a projective generator and thus faithful. Hence P_1 is zero, and M is projective.

LEMMA 2. *Let Γ be a left noetherian ring of finite global dimension. Let S be a left Ore denominator set for Γ, and let $\Omega = S^{-1}\Gamma$. Then if P is a projective Ω-module, there exist finitely generated projective Γ-modules G and H such that $P \oplus (\Omega \otimes G) \approx \Omega \otimes H$.*

Proof. If M is *any* finitely generated Ω-module, there is a finitely generated Γ-module N such that $M = S^{-1}N = \Omega \otimes N$. So let $P = \Omega \otimes Q$, and let

$$0 \longrightarrow F_n \longrightarrow \cdots F_0 \longrightarrow Q \longrightarrow 0$$

be a finite Γ-projective resolution of Q. Tensoring with Ω (which is Γ-flat) we obtain an Ω-projective resolution of P, and since P is projective, this sequence splits. Thus $P \oplus \Sigma (\Omega \otimes F_{2i+1}) \approx \oplus \Sigma (\Omega \otimes F_{2i})$. Letting $G = \oplus \Sigma F_{2i+1}$ and $H = \oplus \Sigma F_{2i}$ yields the desired conclusion.

REMARK. In the language of Grothendieck groups, this lemma just says that the natural map $K_0(\Gamma) \to K_0(\Omega)$ is surjective. (If R is a ring, $K_0(R)$ is the abelian group defined by one generator $[P]$ for each isomorphism class of finitely generated projective left R-modules P, and the relations $[P \oplus Q] = [P] + [Q]$.)

The next lemma is a well-known consequence of Nakayama's Lemma and we omit the proof.

LEMMA 3. *Let Γ be a ring. Let $\bar{\Gamma} = \Gamma/J(\Gamma)$.*

(i) *If P and Q are finitely generated projective Γ-modules such that $\bar{\Gamma} \otimes P \approx \bar{\Gamma} \otimes Q$, then $P \approx Q$.*

(ii) *If G is a finitely generated projective $\bar{\Gamma}$-module such that*

$G \oplus (\bar{\Gamma} \otimes Q) \approx \bar{\Gamma} \otimes P$, where P and Q are finitely generated projective Γ-modules, then there exists a projective Γ-module Q' such that $G \approx \bar{\Gamma} \otimes Q'$ and $P \approx Q \oplus Q'$.

2. The main results. We can now prove the main theorem of this paper.

THEOREM 4. *Suppose Λ is a quasi-local left noetherian ring with $\Lambda/J(\Lambda) \approx M_n(K)$, K a division ring, such that* gl. dim. $\Lambda < \infty$. *Suppose S is a left Ore denominator set for Λ such that the left ring of fractions $\Sigma = S^{-1}\Lambda$ is artinian.*

Then Σ is simple artinian. In fact, there exists an integer m dividing n such that Λ has the form $M_m(\Delta)$ and Σ has the form $M_m(L)$, L a division ring and Δ an order in L.

Proof. Let P denote the minimal projective module of $\Lambda/J(\Lambda)$. Then P^n (the direct sum of n copies of P) is free of rank 1. If Q is a finitely generated projective Λ-module, then $\Lambda/J(\Lambda) \otimes Q \approx P^r$ for some r. Choose $Q \neq 0$ so as to minimize r. Using Lemma 3, part (ii) and the fact that every finitely generated projective $\Lambda/J(\Lambda)$-module is a direct sum of copies of P, it is now easy to see that every finitely generated projective Λ-module is a direct sum of copies of Q. In particular, $\Lambda \approx Q^m$ for some m. Since $p^n \approx \Lambda/J(\Lambda) \approx (P^r)^m$, it follows that $n = rm$. Now let us, for the moment at least, make the convention that homomorphisms of left R-modules are written on the right, and composed accordingly. Then $\Lambda \cong \mathrm{End}_\Lambda(\Lambda) \cong \mathrm{End}_\Lambda(Q^m) \cong M_m(\Delta)$, where $\Delta = \mathrm{End}_\Lambda(Q)$. (If we write homomorphisms on the left, we should write $\Delta = \mathrm{End}_\Lambda(Q)^{opp}$.)

Now consider the artin ring $\Sigma = S^{-1}\Lambda$. Note that the number of prime ideals of Σ = the number of prime ideals of $\Sigma/J(\Sigma)$ = the rank of $K_0(\Sigma/J(\Sigma))$ (as an additive group; by the structure theory for modules over a semi-simple ring) = the rank of $K_0(\Sigma)$ (by the lifting of idempotents). By Lemma 2 this is $\leqq$ the rank of $K_0(\Lambda)$, which by Lemma 3, part (i), is $\leqq$ the rank of $K_0(\Lambda/J(\Lambda))$. But this last term is 1 since $\Lambda/J(\Lambda)$ is simple artinian. Hence Σ has precisely one prime ideal and is thus quasi-local.

Since Σ is left Λ-flat, gl. dim. $\Sigma \leqq$ gl. dim. Λ. By Lemma 1, Σ will now be simple artin, i.e., it will have the form $M_{m'}(L)$ for some division ring L and some integer m'. We shall show that $m' = m$. Let V be the minimal projective Σ-module. Then the free Σ-module of rank 1 is isomorphic to $V^{m'}$. We have seen above that the free Λ-module of rank 1 is isomorphic to Q^m. For some r, $\Sigma \otimes Q \approx V^r$. Now $V^{m'} \approx \Sigma \cong \Sigma \otimes \Lambda \cong \Sigma \otimes Q^m \approx (\Sigma \otimes Q)^m \approx V^{rm}$. Hence $m' = rm$.

On the other hand, by Lemma 2, there exist integers a and b so that $V \oplus (\Sigma \otimes Q^a) \approx \Sigma \otimes Q^b$. Hence $V \oplus V^{ra} \approx V^{rb}$, and so $1 + ra = r$ b. Thus $r = 1$ and $m' = m$. Therefore, we can take for the matrix units of Σ the images of those of Λ, and this gives a map $\Delta \rightarrow L$ inducing the map $\Lambda \rightarrow \Sigma$. Since Λ is an order in Σ, Δ is an order in L [4, Theorem. 3.3].

Taking $n = 1$ in the above theorem we get:

COROLLARY 5. *Suppose that Λ is a local left noetherian ring of finite* (*left*) *global dimension and S is a left Ore set for Λ such that $\Sigma = S^{-1}\Lambda$ is artinian. Then Σ is a division ring.*

Another immediate consequence of Theorem 4 is:

COROLLARY 6. *Suppose F is a field, and Λ a left noetherian order in $M_p(F)$. If Λ is quasi-local and* gl. dim. $\Lambda < \infty$, *then* $\Lambda \approx M_p(R)$, *where $R \subset F$ is the center of Λ.*

Proof. Since p is the matrix-rank of $M_p(F)$, p will be the m of Theorem 4 and F will be the L.

3. **A conjecture.** It would be nice to be able to remove the hypothesis, in Theorem 4, that Λ is an order in an artin ring. Thus we raise the

Question. Is every left noetherian quasi-local ring of finite global dimension a left order in an artin ring?

The evidence is meager. It is known that any left hereditary ring with maximum condition on left annihilators has an artinian left quotient ring (see for example [1, p. 243]). On the other hand, Small [2] has produced a two-sided noetherian ring of global dimension two which is neither a left nor a right order in an artin ring. However, this ring is *not* quasi-local. We conclude this paper with an affirmative answer for global dimension 2.

PROPOSITION 7. *Suppose Λ is a left noetherian ring such that Λ is quasi-local and left and right* gl. dim. $\Lambda = 2$. *Then Λ is a left order in an artin ring.*

Proof. By Small [3, Theorem 2.11] it suffices to establish the regularity condition: If $x + N(\Lambda)$ is regular in $\Lambda/N(\Lambda)$ (where $N(\Lambda)$ is the maximum nilpotent ideal of Λ) then x if regular in Λ.

So assume that $x + N(\Lambda)$ is regular in $\Lambda/N(\Lambda)$, and let $l(x)$ be the left annihilator of x in Λ. Since the sequence

$$0 \longrightarrow l(x) \longrightarrow \Lambda \xrightarrow{x} \Lambda \longrightarrow \Lambda/\Lambda x \longrightarrow 0$$

is exact and gl. dim. $\Lambda = 2$, $l(x)$ is Λ-projective.

Now by the regularity of $x + N(\Lambda)$ in $\Lambda/N(\Lambda)$, we have $l(x) \subseteqq N(\Lambda)$. So, as $N(\Lambda)$ is nilpotent, $l(x)$ has nonzero left annihilator. But, as we observed in the proof of Lemma 1, a nonzero projective module over a quasi-local ring is faithful. Hence $l(x) = 0$, that is, x is left regular in Λ. We have not used the one-sided Noetherian hypothesis in establishing left regularity (it is wanted only for applying Small's result), so by the same argument x is right regular, as required.

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