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SETS WHICH ARE TAME IN ARCS IN E^3

RICHARD C. DETMER

Results of McMillan and Cannon may be combined to give an algebraic condition which is sufficient to show that an arc topologically embedded in E^3 is tame in E^3. The main theorem of this paper gives an essentially algebraic condition involving an arc embedded in E^3 and a compact subset of that arc which is sufficient to show that the arc may be approximated arbitrarily closely without moving the subset, to obtain a tame arc.

1. Preliminaries. The usual Euclidean distance function will be denoted by d. An open neighborhood having radius r about a set S will by denoted by $N(S, r)$. An r-set will be a set having diameter less than r.

1.1. DEFINITION. Suppose that X is a compact subset of a finite complex K which is topologically embedded in E^3. Then X is said to be tame in K iff given $r > 0$ there is a homeomorphism $h: K \to E^3$ such that

(1) $d(x, h(x)) < r$ for each x in K,
(2) $h(x) = x$ for each x in X, and
(3) $h(K)$ is tame.

1.2. DEFINITION. Suppose that X is a compact subset of an arc A which is topologically embedded in E^3. Then X is said to be untangled iff for each $r > 0$, there is an $s > 0$ such that if J is a loop in $E^3 - X$ which bounds (homologically) on an s-set in $E^3 - X$, then J shrinks (homotopically) on an r-set in $E^3 - X$.

McMillan [3] has noted that an arc is untangled iff it has free local fundamental groups (1-FLG) at each of its points. He also proved that an arc which has 1-FLG at each point is tame if each of its subarcs pierces a disk. Cannon [2, Theorem 3.16] has shown that an arc which has 1-FLG at each point does pierce a disk. Hence, an arc which is untangled is tame.

1.3. NOTATION. For the remainder of this paper A will denote an arc topologically embedded in E^3 and X will denote a compact subset of A which is untangled. The arc A will be assumed to have a fixed order, compatible with, and inducing, the given topology on A.

1.4. DEFINITION. Let Y be a subset of A. An indexed collection $C_1, \cdots, C_n$ of disjoint connected subsets of E^3 is said to be ordered with respect to Y iff, in the order on A, each point of $C_i \cap Y$ precedes each point of $C_{i+1} \cap Y$. $(i = 1, \cdots, n-1.)$

The lemma below gives a way to separate components of X by open sets in E^3 which are, roughly speaking, not much larger than the components.

1.5. SEPARATION LEMMA. Suppose that s is a positive number. Then there is a finite cover $C_1, \cdots, C_n$ of X by connected open subsets of E^3 with disjoint, polyhedral closures such that:

(1) $C_1, \cdots, C_n$ is ordered with respect to X, and

(2) For each i, there is a component X_i of X in C_i such that $C_i \subset N(X_i, s)$.

Proof. The lemma follows easily from the fact that X is a compact set.

2. **Cellularity lemmas.** In this section it is shown that if X is untangled, then each component K of X can be enclosed in a polyhedral ball which is "close" to K and which has boundary missing X. The proof falls naturally into two cases, depending on whether or not K is a nondegenerate component of X. The two cases are handled in 2.1 and 2.4 respectively. These results are referred to as cellularity lemmas.

2.1. CELLULARITY LEMMA FOR NONDEGENERATE COMPONENTS. Suppose that e is a positive number and that K is a nondegenerate component of X. Then there is a polyhedral ball B such that K is contained in B, $\mathrm{Bd}\, B$ does not intersect X, and $B \subset N(K, e)$.

Proof. By the results of McMillan [3] and Cannon [2], K is a tame arc. Therefore, there is a 3-cell $C \subset N(K, e)$ which contains $K - \mathrm{Bd}\, K$ in its interior, which does not intersect $A - K$, and which has boundary which is polyhedral modulo K.

In view of Dehn's lemma [5], it suffices to prove the fact stated below. Indeed, the fact may be used in conjunction with Dehn's lemma to alter $\mathrm{Bd}\, C$ slightly near $K \cap \mathrm{Bd}\, C$ and in $N(K, e)$ so as to miss X. The adjusted 2-sphere will then bound a ball B satisfying the requirements of the lemma.

Fact. Suppose that J is a simple closed curve in $\mathrm{Bd}\, C$ which separates the endpoints of K from each other in $\mathrm{Bd}\, C$, and suppose that J bounds a disk D in $\mathrm{Bd}\, C$ of diameter less than some given positive number q. Then J bounds a singular disk D' in $E^3 - X$ of diameter less than q.

The fact is proved as follows. Let $r = q - \operatorname{diam} D$. Choose $s > 0$ so small that loops which bound on s-sets in $E^3 - X$ shrink on r-sets in $E^3 - X$. Pick a 3-cell T of diameter less than s such that $\operatorname{Bd} T$ separates the endpoints of K in E^3 and $(\operatorname{Bd} T) \cap (\operatorname{Bd} C)$ is a simple closed curve in $\operatorname{Int} D$. Let E denote the disk $C \cap \operatorname{Bd} T$. Use the separation lemma 1.5 to cover components of X which intersect $(\operatorname{Bd} T) - E$ by a finite collection of disjoint open sets whose polyhedral closures miss $E \cup K$. Let W be the union of these sets and assume that $\operatorname{Bd} W$ is in general position with respect to $\operatorname{Bd} T$. Because $\operatorname{Cl} W$ does not intersect $K \cup E$, $\operatorname{Bd} E$ bounds homologically on the s-set $\operatorname{Bd}(T - W) - \operatorname{Int} E$ in $E^3 - X$ and therefore bounds a singular r-disk F in $E^3 - X$. The set $(D - K) \cup F$, which lies in $E^3 - X$, contains a singular disk D' of diameter less than q which is bounded by J. This establishes the fact and completes the proof of the lemma.

If, in the proof of 2.1, C is first partitioned by means of disjoint spanning disks D_1 and D_2 into three 3-cells — a central 3-cell C_3, whose intersection with K is an arc $K_3 \subset \operatorname{Int} K$, and end cells C_1 and C_2 whose intersections with K are the closed components K_1 and K_2 of $K - K_3$ — and $\operatorname{Bd} C$ is adjusted only very near $(\operatorname{Bd} C_1 - D_1) \cap K$ and $(\operatorname{Bd} C_2 - D_2) \cap K$ in constructing $\operatorname{Bd} B$, then the following is evident.

2.2. ADDENDUM. The ball B in 2.1 may be chosen in such a manner that it can be partitioned by disjoint spanning disks D_1 and D_2 into 3-cells B_1, B_2, and $B_3 (B_i \cap B_3 = D_i$ for $i = 1, 2)$ satisfying

(1) $B_3 \cap A$ is a subarc of $\operatorname{Int} K$ which spans the cell B_3 from D_1 to D_2,

(2) the diameter of B_i is less than $e (i = 1, 2)$, and

(3) $B_i \cap A$ lies in an e-arc on $A (i = 1, 2)$ with one endpoint of this e-arc missing X (unless B_i contains an endpoint of A).

The next lemma is well-known.

2.3. LEMMA. *Suppose that J is a simple closed curve in E^3 which bounds an orientable surface T of diameter less than r, and suppose that L is an arc of diameter less than s which misses J. Then J bounds a surface of diameter less than $r + s$ in $E^3 - L$ and this surface may be chosen in an arbitrarily small neighborhood of $T \cup L$.*

2.4. CELLULARITY LEMMA FOR DEGENERATE COMPONENTS. Suppose that s is a positive number and that $\{p\}$ is a degenerate component of X. Then there is a polyhedral 3-cell B such that p lies in B, $\operatorname{Bd} B \cap X = \varnothing$, and $B \subset N(p, s)$.

Proof. As a first approximation to the desired 3-cell, let $B' = \operatorname{Cl}(N(p, r))$ where $0 < r < s/2$ and r is so small that B' intersects no component of X with diameter as large as $s/4$. Choose $q > 0$ so small that loops which bound on q-sets in $E^3 - X$ shrink on $r/2$-sets in $E^3 - X$.

Choose a collection $D_1, \cdots, D_n$ of small disjoint polyhedral disks on Bd B' with boundaries missing X and a collection $S_1, \cdots, S_m$ of small polyhedral spheres in $E^3 - X$ such that $X \cap \text{Bd } B'$ is contained in $D_1 \cup \cdots \cup D_n \cup (\text{Int } S_1 \cap \text{Bd } B') \cup \cdots \cup (\text{Int } S_m \cap \text{Bd } B')$. Specifically, select for each degenerate component $\{x\}$ of X which lies on Bd B' a disk $D_x \subset \text{Bd } B'$ with $\text{Bd } D_x \cap X = \emptyset$ and so small that diam $D_x < q/2$ and $D_x \cap A$ lies on a subarc of A with diameter less than $q/2$ which has endpoints in $A - X$. Select a sphere for each nondegenerate component of X which intersects Bd B' using the cellularity lemma for nondegenerate components, 2.1, with $s/4$ as the value of e in that lemma. A finite collection of these disks and spheres satisfies the conditions above except for the requirement that the disks be disjoint. Since their boundaries miss X, the disks may be made disjoint by simply putting their boundaries in general position and cutting them apart.

Application of Lemma 2.3 shows that, for each i, Bd D_i bounds on a q-set in $E^3 - X$, and hence shrinks on an $r/2$-set in $E^3 - X$. Because of this, if Int D_i is thrown away and Bd D_i shrunk, a singular 2-sphere R_0 may be produced which intersects X in "fewer" places than did Bd B' and which is essential in $E^3 - \{p\}$ since it is homotopic to Bd B' in $E^3 - N(p, r/2)$. If any of the spheres S_i contains p, the proof is finished since each S_i has diameter less than s. Otherwise, for each i in turn, choose a point x_i in $\text{Int } S_i - R_{i-1}$ and project "radially" the part of R_{i-1} in Int S_i into S_i to form a singular sphere R_i. The final result is a singular sphere R_m in $E^3 - X$, essential in $E^3 - \{p\}$, and lying in $N(p, s)$. Using a strong form of the sphere theorem (which is implicit in the original proof in [5] and is stated explicitly in [6]), there is a nonsingular, polyhedral sphere with the same properties. Taking B to be the closure of the interior of this sphere completes the proof.

3. The following theorem is the main result of this paper. It says that if a compact subset of an arc is untangled, then it is tame in that arc.

THEOREM 3.1. *Suppose that X is a compact subset of an arc A which is topologically embedded in E^3, and that for each positive number r, there is a positive number s such that if J is a loop in $E^3 - X$ which bounds on an s-set missing X, then J shinks on an r-set missing X. Then for each positive number q there is a homeomorphism $f\colon A \to E^3$ such that*

(1) *$f(x) = x$ for each x in X,*

(2) $d(x, f(x)) < q$ *for each* x *in* A, *and*
(3) $f(A)$ *is tame.*

Proof. The idea of the proof is as follows. Construct a homeomorphism $h: E^3 \twoheadrightarrow E^3$ such that the restriction of h to X takes X in an order preserving fashion into the x-axis. Define $f(A) = h^{-1}(I)$ where I is a suitably chosen subinterval of the x-axis. Care must be taken in the construction of h in order that A and $f(A)$ be close homeomorphically. The details of this process are described below.

It may be assumed that A is locally polyhedral modulo X. Use the cellularity lemma for nondegenerate components, 2.1, to construct a collection $B'_1, \cdots, B'_n$ of disjoint polyhedral 3-cells with boundaries in $E^3 - X$, one for each component of X with diameter as large as $q/12$. Choose e in that lemma and its addendum, 2.2, less than $q/6$, and so small that the collection of cells is ordered with respect to X (although it need not cover X).

The set $X' = X - \bigcup B'_i$ is a compact subset of A with no component having diameter as large as $q/12$; X' is untangled. "Small" 3-cells covering X' are now constructed. By an appropriate choice of s in the separation lemma, 1.5, a cover $C_1, \cdots, C_m$ of X' may be obtained, the closures of whose elements miss $\bigcup B'_i$, have diameters less than $q/6$, and are ordered with respect to X. A suitable choice of s also guarantees that each C_i intersects at most one of the $q/6$-arcs associated (by the Addendum 2.2) with any B'_j, and that $C_i \cap A$ lies in a $q/6$-arc on A.

Each C_i may be changed to a ball, using the two cellularity lemmas to cover each component of X inside C_i by a polyhedral ball, picking a finite subcover of these, and following the methods of Bing [1] to cut apart and reconnect these balls to form a single ball containing $X \cap C_i$ in its interior. Retain the notation C_i for such a polyhedral ball.

Modify each $B'_j (j = 1, \cdots, n)$ as follows. The set $B'_j \cap \mathrm{Cl}(A - B'_j)$ is contained in (at most) two $q/6$-arcs A_1 and A_2 on A, and the intersection of each of these with B'_j lies in a $q/6$-cell (by the addendum). If A_i intersects $X - B_j$, then $A_i \cap (X - B_j)$ is contained in some of the C_i's. Connect these C_i's to the $q/6$-cells using disjoint $q/6$-cells, so that the result is two $5q/6$-cells, each of which intersects one of A_1 or A_2 and no other point of $A - B'_j$.

Let $B_1, \cdots, B_n, B_{n+1}, \cdots, B_k$ be the modified B_j's $(j = 1, \cdots, n)$ plus the C_j's not used in the modifications. Assume that the indices are arranged so that this collection is ordered with respect to X.

Figure 1 illustrates the situation at this point of the proof.

The purpose in constructing $B_1, \cdots, B_n$ so carefully is to make it possible to change A homeomorphically by moving only points of

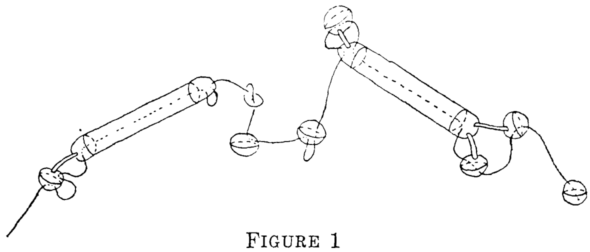

FIGURE 1

A which are near small components of X and near the endpoints of large components of X. Define a homeomorphism f' on A to move some subarcs of the $q/6$-arcs associated with $B_1, \cdots, B_n$ into the $5q/6$-cells at the "ends" of these "large" balls, and to move some subarcs of the $q/6$-arcs associated with $B_{n+1}, \cdots, B_k$ into these balls; do this so that $f'(A)$ intersects each $\mathrm{Bd}\, B_j$ in at most two points (and only one point if B_j (contains an endpoint of A.)

The first approximation h_1 to the homeomorphism h taking X into the x-axis is defined to take $f'(A) - \bigcup B_i$ into the x-axis, to take each B_i to a small neighborhood of an arc on the x-axis, and to take each component of X with diameter as large as $q/12$ into the x-axis, everything with order preserved. Subsequent approximations to h will be the identity outside the images under h_1 of the $5q/6$-cells associated with $B_1, \cdots, B_n$ and of $B_{n+1}, \cdots, B_k$, and this will ensure that conclusion (2) of the theorem is true.

Approximations to h are now obtained sequentially. At the second stage of the construction, a new collection of balls is chosen, inside the fiast, closer to X, and separating the components of X with diameters as large as $q/24$. A homeomorphism h_2 of E^3 onto itself is obtained which is the identity on each of the h_1-images of the components of X which have diameter as large as $q/12$. This homeomorphism maps h_1-images of the new balls to neighborhoods of arcs on the x-axis, and h_2h_1 also sends components of X with diameter as large as $q/24$ into the x-axis, preserving order of cells.

Continue in this manner, choosing the balls so small that the sequence $h_1, h_2h_1, h_3h_2h_1, \cdots$ of homeomorphisms converges to a homeomorphism h. Let I be the smallest subarc of the x-axis which contains $h(f'(A)) \cap$ (x-axis). The arc $h^{-1}(I)$ differs from $f'(A)$ only in the end cells of $B_1, \cdots, B_n$ and in $B_{n+1}, \cdots, B_k$ and so a homeomorphism $f\colon A \rightarrow E^3$ may be defined so that $f(A) = h^{-1}(I)$ and $d(x, f(x)) < q$ for each x in A. The homeomorphism f may also be chosen to fix points of X since the restriction of h to X is a homeomorphism of X into the x-axis. This completes the proof of the theorem.

There is also a relative version of 3.1:

THEOREM 3.2. *Suppose that X is a compact subset of an arc A*

which is topologically embedded in E^3 *and that* X *is untangled. Suppose that* $g\colon A \to [0, \infty)$ *is a continuous function so that* $g^{-1}(0) \cap X = \emptyset$. *Then there is a homeomorphism* $f\colon A \to E^3$ *such that*

(1) $f(x) = x$ *for each* x *in* X,

(2) $d(x, f(x)) < g(x)$ *for each* x *in* A, *and*

(3) $f(A)$ *is locally tame modulo the set* $g^{-1}(0)$.

Proof. The proof is almost exactly the same as for 3.1, except for beginning with an approximation to A which is locally polyhedral modulo $X \cup g^{-1}(0)$ and which is homeomorphically within g of A. This new arc is then modified on subarcs close to X in the same way as in the proof of 3.1.

4. Theorem 3.1 may be combined with a characterization of subsets of arcs due to R. L. Moore [4, Theorem 135] to yield a characterization of subsets of tame arcs in E^3. Moore's theorem is proved for a space satisfying his axioms 0–5 and E^3 does not satisfy axiom 4. However the proof is not difficult in this case.

THEOREM (R. L. Moore) 4.1. *In order that the compact point set* M *in* E^3 *be a subset of an arc it is necessary and sufficient that every closed and connected subset of* M *be either a degenerate point set or an arc* t *such that no point of* t, *except for its endpoints is a limit point of* $M - t$.

4.2. CHARACTERIZATION OF SUBSETS OF TAME ARCS IN E^3. Suppose that X is a compact subset of E^3. Then X is a subset of a tame arc in E^3 if and only if each component of X is a point or an arc t such that no point of t, except possibly for an endpoint, is a limit point of $X - t$, and for each positive number r, there is a positive number s such that each loop which bounds on a s-set in $E^3 - X$ shrinks on an r-set in $E^3 - X$.

Proof. Sufficiency follows from 3.1 and 4.1. Necessity is obvious.

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UNIVERSITY OF TENNESSEE AT CHATTANOOGA