PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

# CONDITIONS UNDER WHICH A CONNECTED REPRESENTABLE SPACE IS LOCALLY CONNECTED

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**In this paper it is shown that every strongly locally homogeneous Hausdorff continuum is locally connected. It is known that every connected representable space is homogeneous and that every locally connected representable space is strongly locally homogeneous; this paper investigates the problem of whether or not every connected representable space is locally connected.**

The discovery that homogeneity did not characterize the circle among planar continua motivated other classifications of continua in terms of the actions of their homeomorphism groups, and the pseudo-arc remains the most tortuous proving grounds for testing such topological properties. Those homogeneity-like properties that are not possessed by the pseudo-arc appear to be closely related to local connectedness. Such properties include strong local homogeneity, representability, (strong) 2-homogeneity and countable dense homogeneity.

In this paper it is shown that every strongly locally homogeneous Hausdorff continuum is locally connected. It is known that every connected representable space is homogeneous and that every locally connected representable space is strongly locally homogeneous; this paper investigates the problem of whether or not every connected representable space is locally connected. It follows from results of Ben Fitzpatrick, Jr. and Ralph Bennett that every connected locally compact separable metric representable space is locally connected. Arguments of Fitzpatrick and de Groot are used to show that if $(X, \tau)$ is a representable connected complete metric space such that $\tau$ does not have a base of totally disconnected sets, then $(X, \tau)$ is locally connected (and hence arcwise connected and strongly locally homogeneous). The authors do not know of even a homogeneous connected complete metric space with a base of totally disconnected sets; however, there exists a separable connected metric topology that has such a base.

Let $(X, \tau)$ be a topological space. We let $H(X)$ denote the group of all homeomorphisms from the space $(X, \tau)$ onto itself and let $i$ denote the identity of $H(X)$. If $A \subset X$, then $A' = \{h \in H(X): h \mid A = i \mid A\}$ and if $G$ is a subgroup of $H(X)$, then $G' = \{x \in X: g(x) = x$ for each $g \in G\}$. If $G \subset H(X)$ and $A \subset X$, then $G(A) = \{g(a): g \in G$ and $a \in A\}$. We write $G(x)$ in room of $G(\{x\})$. Throughout this paper all spaces are assumed to be Hausdorff.

DEFINITION [5]. A topological space $(X, \tau)$ is *representable* provided that for every neighborhood $U$ of any point $x \in X$, there exists a subneighborhood $V$ such that if $y \in V$, then there is $h \in (X - U)'$ such that $h(x) = y$.

DEFINITION [7]. A topological space $(X, \tau)$ is *strongly locally homogeneous* provided that for every neighborhood $U$ of any point $x \in X$, there exists a subneighborhood $V$ such that if $y \in V$, then there is $h \in (X - V)'$ such that $h(x) = y$.

It is known that a topological space $(X, \tau)$ is representable if and only if for each closed set $F$ and each $x \in X - F$, $F'(x) \in \tau$ [5, Corollary to Theorem 1].

LEMMA [6, Lemma 4.3]. *Let $(X, \tau)$ be a representable continuum, let $F$ be a closed subset of $X$ and let $x \in X - F$. Then $\overline{F'(x)}$ is a subcontinuum of $X$ with interior points.*

PROPOSITION 1. *Every representable continuum is locally connected.*

*Proof.* Let $(X, \tau)$ be a representable continuum and let $x \in U \in \tau$. Since $(X, \tau)$ is homogeneous it suffices to show that $(X, \tau)$ is connected im kleinen at $x$. Since $(X, \tau)$ is compact and Hausdorff, there is an open set $V$ such that $x \in V \subset \bar{V} \subset U$. Let $F = X - V$. By the preceding lemma, $\overline{F'(x)}$ is connected and since $F'(x) \subset V$, $\overline{F'(x)} \subset \bar{V} \subset U$. Since $(X, \tau)$ is representable $F'(x) \in \tau$ and hence $(X, \tau)$ is connected im kleinen at $x$.

COROLLARY. *Every representable continuum is strongly locally homogeneous.*

*Proof.* [6, Theorem 2.4].

DEFINITION [2]. A separable topological space $(X, \tau)$ is *countable dense homogeneous* provided that for any two countable dense subsets $M$ and $N$ of $X$ there exists $h \in H(X)$ such that $h(M) = N$.

We have need of an amalgam of the following two theorems proved by R. Bennett and J. de Groot respectively.

THEOREM 2 [2, Theorem 3]. *Every locally compact separable representable metric space is countable dense homogeneous.*

THEOREM 3 [8, Theorem 1]. *Every separable strongly locally homogeneous complete metric space is countable dense homogeneous.*

We have been unable to show that every representable space is strongly locally homogeneous, however strong local homogeneity can be replaced by representability in the above theorem; and for the sake of completeness we give a proof of this perhaps more general result, based on J. de Groot's arguments.

LEMMA [1, page 777 and 8, page 3]. *Let $(X, \rho)$ be a complete metric space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of homeomorphisms from $X$ onto itself. If, for every positive integer $n$ and every $x \in X$, $\rho(\varphi_n(x), \varphi_{n+1}(x)) < 1/2^n$ and $\rho(\varphi_n^{-1}(x), \varphi_{n+1}^{-1}(x)) < 1/2^n$, then $\{\varphi_n\}_{n=1}^{\infty}$ converges to a homeomorphism from $X$ onto itself.*

THEOREM 4. *Every separable representable complete metric space is countable dense homogeneous.*

*Proof.* Let $(X, \rho)$ be a separable complete metric space that is representable and let $A = \{a_n\}_{n=1}^{\infty}$ and $B = \{b_n\}_{n=1}^{\infty}$ be countable dense subsets of $X$. Let $V_1$ be an open set containing $b_1$ and contained in $B(b_1, 1/4)$ such that for every $x \in V_1$ there exists $f \in (X - B(b_1, 1/4))'$ with $f(x) = b_1$. Let $k(1)$ be the least positive integer such that $a_{k(1)} \in V_1$. Let $f_1 \in (X - B(b_1, 1/4))'$ such that $f_1(a_{k(1)}) = b_1$. For each $j$ such that $0 < j < k(1)$, if any such $j$ exists, define $U_1^j$, $V_1^j$, $p(1, j)$, and $g_{1,j}$ as follows. Let $U_1^j$ be pairwise disjoint open sets such that $a_j \in U_1^j \subset B(a_j, 1/4) - \{b_1\}$. Let $V_1^j$ be open sets such that $a_j \in V_1^j \subset U_1^j$ and for every $x \in V_1^j$ there exists $f \in (X - U_1^j)'$ with $f(a_j) = x$. Since $B$ is dense in $X$, there exist distinct integers $p(1, j) > 1$ such that $b_{p(1,j)} \in V_1^j$. Then there are $g_{1,j} \in (X - U_1^j)'$ with $g_{1,j}(a_j) = b_{p(1,j)}$. Define $h_1$ to be $g_{1,k(1)-1} \circ g_{1,k(1)-2} \cdots \circ g_{1,1} \circ f_1$(or $h_1 = f_1$ if $k(1) = 1$), and set $\varphi_1 = h_1$, $m(1) = 1$ and $k(0) = 0$.

Define $\varphi_{n+1}$ be induction as follows. Let $m(n+1)$ be the least integer that is greater than $m(n)$ and not equal to $p(i, j)$ for any $1 \leqq i \leqq n$ and $k(i-1) < j < k(i)$. Set $b = b_{m(n+1)}$ and $Y = \{b_{m(i)} : 1 \leqq i \leqq n\} \cup \{a_{k(i)} : 1 \leqq i \leqq n\} \cup \{a_j : 1 \leqq i \leqq n, k(i-1) < j < k(i)\} \cup \{b_{p(i,j)} : 1 \leqq i \leqq n, k(i-1) < j < k(i)\}$. Let $\varepsilon > 0$ be such that $B(b, \varepsilon) \subset (B(b, 1/2^{n+2}) \cap \varphi_n[B(\varphi_n^{-1}(b), 1/2^{n+1})]) - Y$. Let $V_{n+1}$ be an open set containing $b$ and contained in $B(b, \varepsilon/3)$ such that for every $x \in V_{n+1}$ there exists $f \in (X - B(b, \varepsilon/3))'$ with $f(x) = b$. Let $k(n+1)$ be the least integer greater than $k(n)$ such that $a_{k(n+1)} \in V_{n+1}$. Let $f_{n+1} \in (X - B(b, \varepsilon/3))'$ such that $f_{n+1}(a_{k(n+1)}) = b$. For every $j$ such that $k(n) < j < k(n+1)$, if any such $j$ exists, define $U_{n+1}^j$, $V_{n+1}^j$, $p(n+1, j)$, and $g_{n+1,j}$ as follows. Let $U_{n+1}^j$ be pairwise disjoint open sets such that $a_j \in U_{n+1}^j \subset (B(a_j, \varepsilon/3)) \cap \varphi_n[B(\varphi_n^{-1}(a_j, 1/2^{n+1})]) - (Y \cup \{b\})$. Let $V_{n+1}^j$ be open sets such that $a_j \in V_{n+1}^j \subset U_{n+1}^j$ and such that for every $x \in V_{n+1}^j$, there is $f \in (X - U_{n+1}^j)'$ with $f(a_j) = x$. Since $B$ is dense in $X$, there exist distinct integers

$p(n+1, j)$ which are greater than $m(n+1)$ and not equal to $p(i, j)$ for $1 \leqq i \leqq n$ and $k(i-1) < j < k(i)$ such that $b_{p(n+1,j)} \in V^j_{n+1}$. Then there are $g_{n+1,j} \in (X - U^j_{n+1})'$ with $g_{n+1,j}(a_j) = b_{p(n+1,j)}$. Define $h_{n+1} = g_{n+1,k(n+1)-1} \circ g_{n+1,k(n+1)-2} \circ \cdots g_{n+1,k(n)+1} \circ f_{n+1}$ (or $h_{n+1} = f_{n+1}$ if $k(n+1) = k(n)+1$). Finally let $\varphi_{n+1} = h_{n+1} \circ \varphi_n$.

The sequence $\{\varphi_n\}_{n=1}^{\infty}$ defined above satisfies the hypothesis of the lemma. Then if $\varphi = \lim \varphi_n$, $\varphi \in H(X)$ such that $\varphi(A) = B$. Therefore, $X$ is countable dense homogeneous.

It is known that every locally compact connected metric countable dense homogeneous space is locally connected [4, Theorem 1]. Consequently every metric locally compact separable connected representable space is locally connected. As in the proof of Theorem 1 of [4], we note that if $(X, \tau)$ is a separable countable dense homogeneous complete metric space such that every nonempty open set contains a nondegenerate connected set, then $(X, \tau)$ is locally connected. Moreover, if a representable space $(X, \tau)$ has a totally disconnected open subset, then $\tau$ has a base of totally disconnected sets. Thus we have the following proposition:

PROPOSITION 5. *Let $(X, \tau)$ be a connected representable separable complete metric space. If $\tau$ does not have a base of totally disconnected sets, then $(X, \tau)$ is locally connected.*

With respect to the above proposition it is noteworthy that a connected locally compact Hausdorff space cannot have a base of totally disconnected sets. It is possible to modify the Cantor's teepee [9, Example 129] in order to obtain a separable connected metric space $(X, \tau)$ such that $\tau$ has a base of totally disconnected sets; however, so far as we know, the following question is unanswered.

Does there exist a homogeneous connected complete metric space $(X, \tau)$ such that $\tau$ has a base of totally disconnected sets? In particular can such a space be representable?

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Received February 1, 1973.

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