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STABILITY OF MEASURE DIFFERENTIAL EQUATIONS

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The preservation of stability properties of perturbed differential equations when the perturbations are impulsive are discussed relative to asymptotically self invariant sets. The fact that the solutions of perturbed systems are discontinuous offers many difficulties in applying the usual methods of perturbation theory and thus makes the study interesting.

1. **Introduction.** Generally, in perturbation theory, we consider the perturbed system given by

$$\frac{dx}{dt} = f(t, x) + R(t, x) ,$$

where the perturbation term $R(t, x)$ is continuous and small in some sense. But it is of much importance to consider the case when the perturbation term is rather wildly impulsive in character and it is also natural to expect such a situation in pulse frequency modulation systems, models for biological neural nets and some automatic control problems. Thus the study of equations of the type

$$DX = f(t, x) + G(t, x)Du , \tag{1.1}$$

where Du denotes the distributional derivative of the function u is important in itself. Equations of the form (1.1) are called measure differential equations. The existence of solutions of such equations has been studied by Schmaedeke, W. W. [6] and Das, P. C. and Sharma, R. R., (see [2], [3]).

Our interest here is to treat Eq. (1.1) as a perturbation of the ordinary differential system

$$\frac{dx}{dt} = f(t, x) \tag{1.2}$$

and to investigate the preservation of stability properties of solutions of (1.2) under the effect of impulsive perturbations. Some simple situations of the stability of such problems have been considered by Das, P. C. and Sharma, R. R. [3], Barbashin, E. A.[1] and Zabalishchin, S. T. [7].

The fact that the solutions of (1.1) are discontinuous (i.e., functions of bounded variation) offers many difficulties in applying the usual techniques of perturbation theory. It is known (see [4], [5]) that even in the case of ordinary differential systems, it is more general and natural to consider the stability of asymptotically self invariant (ASI) sets rather than that of usual invariant sets. Thus, it seems more appropriate to consider ASI sets and their stability properties with respect to (1.1). In this paper, we study the effect of impulsive perturbations on the stability in variation of ASI sets, making use of the methods developed in our earlier work [5].

2. Notation and basic theorems. Let R^n and M denote the n-Euclidean space and the set of all $n \times m$ matrices of real numbers, with the norms $||x|| = \sum_{i=1}^{n} |x_i|$, $x \in R^n$ and $||G|| = \sum_{i=1}^{n} \sum_{j=1}^{m} |g_{ij}|$, $(g_{ij}) = G \in M$, respectively. Let $Bv(R^+, R^m)$ denote the set of all vector functions defined on R^+ with values in R^m, whose components are scalar functions of bounded variation on R^+.

Let us consider the measure differential equation

$$Dx = f(t, x) + G(t, x)Du \ , \quad x(t_0) = x_0,\ t_0 \geqq 0 \ , \tag{2.1}$$

where (i) the functions f and G are defined on $R^+ \times S(\rho)$, $S(\rho) = [x \in R^n\colon ||x|| < \rho]$, with values in R^n and M respectively; (ii) u is a right continuous function belonging to the set $Bv(R^+, R^m)$; and (iii) Dx, Du denote the distributional derivatives of functions x and u (identified with Stieltjes measure) respectively.

DEFINITION 1. A function $y(\cdot) = y(\cdot, t_0, x_0)$ is said to be a solution of Eq. (2.1) on R^+ if $y(\cdot)$ is a right continuous function $\in Bv(R^+, S(\rho))$ and the distributional derivative of $y(\cdot)$ on (t_0, T), $T \in R^+$, satisfies the Eq. (2.1).

For the existence and uniqueness of solutions of Eq. (2.1) and for more details about equations of the type (2.1), refer [2], [3]. In the sequel, we shall assume that the solutions of Eq. (2.1) exist and are unique for $t \geqq t_0$.

The following classes of functions will be used often in our discussion. So, we define

$$L = [\sigma\colon \sigma \in C[R^+, R^+] \ , \quad \sigma(t) \text{ is decreasing in } t \text{ and } \sigma(t) \to 0 \text{ as } t \to \infty] \ ;$$

$$A = [a\colon a \in C[R^+ x[0, \rho), R^+] \ , \quad a(t, r) \text{ is decreasing in } t \text{ for each } r \text{ and increasing in } r \text{ for each } t \text{ such that } \lim_{\substack{t \to \infty \\ r \to 0}} a(t, r) = 0] \ ;$$

$$B = \left[H\colon H \in C[R^+ x R^+, R^+] \text{ and for some } \tau > 0, \ \lim_{t \to \infty} \begin{Bmatrix} \sup H(t, t_0) \\ t_0 \geqq \tau \end{Bmatrix} = 0 \right] ,$$

where $C[Y, Z]$ denote the set of continuous functions on Y taking values in Z.

We shall now suppose that $f \in C[R^+ x S(\rho), R^n]$, $f_x(t, x)$ exists and is continuous on $R^+xS(\rho)$ and $x(t, t_0, x_0)$ is the solution of the unperturbed system

$$\frac{dx}{dt} = f(t, x) , \quad x(t_0) = x_0 . \tag{2.2}$$

Let $\Phi(t, t_0, x_0)$ denote the fundamental matrix solution of the variational system

$$\frac{dz}{dt} = f_x(t, x(t, t_0, x_0))z , \quad z(t_0) = 1 \text{ (identity matrix)} .$$

We use the following definitions for the characterization of the asymptotically self invariant (ASI) set with respect to the system (2.2) and the stability criteria of such sets.

DEFINITION 2. The set $x = 0$ is said to be ASI relative to (2.2) if every solution $x(t, t_0, 0)$ of (2.2) satisfies

$$\|x(t, t_0, 0)\| \leqq \lambda(t_0),\ t \geqq t_0,\ \lambda \in L .$$

DEFINITION 3. The ASI set $x = 0$ is said to be

(i) uniformly stable in variation, if for each α, $0 < \alpha \leqq \rho$, there exists a constant $M(\alpha) > 0$ such that

$$\|\Phi(t, t_0, x_0)\| \leqq M(\alpha) , \quad t \geqq t_0 ,$$

provided $\|x_0\| \leqq \alpha$;

(ii) uniformly asymptotically stable in variation, if for each α, $0 < \alpha \leqq \rho$, there exists a function $\sigma_\alpha \in L$ such that whenever $\|x_0\| \leqq \alpha$,

$$\|\Phi(t, t_0, x_0)\| \leqq \sigma_\alpha(t - t_0) , \quad t \geqq t_0 ;$$

(iii) uniformly stable, if there exists a function $a \in A$ such that

$$\|x(t, t_0, x_0)\| \leqq a(t_0, \|x_0\|) , \quad t \geqq t_0 ;$$

(iv) uniformly asymptotically stable, if there exist functions $a \in A$, $\eta \in L$, and $H \in B$ such that

$$\|x(t, t_0, x_0)\| \leqq a(t_0, \|x_0\|)\eta(t - t_0) + H(t, t_0) , \quad t \geqq t_0 ;$$

(v) quasi-equi asymptotically stable if the estimate in (iv) is replaced by

$$\|x(t, t_0, x_0)\| \leqq \{a(t_0, \|x_0\|) + \beta\}\eta(t - t_0) + H(t, t_0) , \quad t \geqq t_0 ,$$

where β is a constant > 0 and the functions a, η, and H are as in (iv);

(vi) exponentially asymptotically stable if, for $t \geqq t_0$,

$$\| x(t, t_0, x_0) \| \leqq \{a(t_0, \| x_0 \|) + \beta\} e^{-\delta(t-t_0)} + H(t, t_0) ,$$

where the constant $\beta \geqq 0$, $\delta > 0$, $a \in A$ and $H \in B$.

REMARK. If $f(t, 0) \equiv 0$ so that $x \equiv 0$ is the unique solution of (2.2), then the ASI set reduces to the usual invariant set.

We next state two known results [5] which are very useful in our investigation of stability properties of Eq. (2.1).

THEOREM 2.1. *Assume that the ASI set* $x = 0$ *with respect to* (2.2) *is uniformly stable in variation. Then, there exists a Lyapunov function* $V(t, x)$ *with the following properties*:

(1) $V(t, x)$ *is defined and continuous on* $R^+ \times S(\alpha)$;

(2) $\| x \| \leqq V(t, x) \leqq a(t, \| x \|)$, $(t, x) \in R^+ \times S(\alpha)$, $a \in A$;

(3) $| V(t, x) - V(t, y) | \leqq M(\alpha) \| x - y \|$, $(t, x), (t, y) \in R^+ \times S(\alpha)$;

(4) $D^+ V_{(2.2)}(t, x) = \lim_{h \to 0^+} \sup 1/h [V(t + h, x + hf(t, x)) - V(t, x)] \leqq 0$, *for* $(t, x) \in R^+ \times S(\alpha)$.

THEOREM 2.2. *Suppose that the ASI set* $x = 0$ *relative to* (2.2) *is uniformly asymptotically stable in variation. Then, there exists a Lyapunov function* $W(t, x)$ *verifying the following properties*:

(1′) $W(t, x)$ *is defined and continuous on* $R^+ \times S(\alpha)$;

(2′) $\| x \| \leqq W(t, x) \leqq b(t, \| x \|)$, $(t, x) \in R^+ \times S(\alpha)$, $b \in A$;

(3′) $| W(t, x) - W(t, y) | \leqq L(\alpha) \| x - y \|$, $(t, x), (t, y) \in R^+ \times S(\alpha)$;

(4′) $D^+ W_{(2.2)}(t, x) \leqq -\delta W(t, x) + \lambda(t)$, $(t, x) \in R^+ \times S(\alpha)$, *where* $\delta > 0$ *and* $\lambda \in L$.

REMARK. It is important to note that the Lyapunov functions obtained in the foregoing theorems are Lipschitzian in x for a constant and this fact plays a crucial role in studying the effect of perturbations. The usefulness of the stability in variation notions lies in the above fact.

3. Main results. In this section, we shall assume that the set $x = 0$ is ASI relative to the unperturbed system (2.2) and give sufficient conditions for the stability criteria of the ASI set $x = 0$ with respect to the measure differential system (2.1). First we state the following hypotheses:

(H_1) $f(t, x)$ and $f_x(t, x)$ are defined and continuous on $R^+ \times S(\rho)$ and $f_x(t, x)$ is bounded on $R^+ \times S(\rho)$;

(H_2) the ASI set $x = 0$ relative to (2.2) is uniformly stable in variation;

(H_3) the ASI set $x = 0$ relative to (2.2) is uniformly asymptotically stable in variation;

(H_4) $\|G(t, x)\| \leqq g(t)\|x\|$ on $R^+ \times S(\rho)$ where $G(t, x)$ is measurable in t for each x, continuous in x for each t and $g(t)$ is dv_u integrable function v_u being the total variation function of u;

(H_5) $\int_0^\infty w(s)ds < \infty$ where

$$w(t) = \limsup_{h\to 0+} \frac{1}{h}\int_t^{t+h} g(s)dv_u(s)$$

is the upper right Dini derivative of the indefinite integral $\int^t g(s)dv_u(s)$;

(H_6) the function $w(t)$ in H_5 is such that

$$\int_t^{t+1} w(s)ds \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \ ;$$

(H_7) the discontinuities of u occur at isolated points $\{t_k\}$, $t_1 < t_2 < \cdots < t_k < \cdots$ and are such that

$$\|u(t_k) - u(t_k^-)\| \leqq \lambda_k \exp\left(\int_{t_0}^{t_k} w(s)ds\right),\ k = 1, 2, \cdots, \tag{3.1}$$

and λ_k are constants ;

(H_8) $\sum_{k=1}^\infty g(t_k)\lambda_k$ converges.

We are now in a position to state our first main result which deals with the sufficient conditions for the uniform stability of the ASI set $x = 0$ with respect to the perturbed system (2.1).

THEOREM 3.1. *Assume that the hypotheses* (H_1), (H_2), (H_4), (H_5), (H_7), *and* (H_8) *are satisfied. Then, the set* $x = 0$ *is ASI with respect to* (2.1) *and is uniformly stable.*

Naturally, the proof depends on the construction of a Lyapunov function for the unperturbed system (2.2), obtaining the estimate of solutions of the perturbed system (2.1) in terms of that Lyapunov function and estimating the jumps of the solutions of (2.1) at the points of discontinuity. For convenience, we first state and prove the following lemmas.

LEMMA 3.1. *Let the hypotheses* (H_1), (H_2), (H_4), *and* (H_5) *hold. Then,*

$$\begin{cases} D^+m(t) = \limsup_{h\to 0+} \frac{1}{h}[m(t+h) - m(t)] \\ \qquad \leqq M(\alpha)\,\|y(t, t_0, x_0)\|\,w(t) + \underset{(2.2)}{D^+}V(t, x)\ , \end{cases} \tag{3.2}$$

where $m(t) = V(t, y(t, t_0, x_0))$, $y(t, t_0, x_0)$ *is the solution of* (2.1), $V(t, x)$

is the Lyapunov function obtained in Theorem 2.1 *and* $M(\alpha)$ *is the Lipschitz constant for the function* $V(t, x)$.

Proof. Set $x = y(t, t_0, x_0)$ and note that $y(t+h, t_0, x_0) \equiv y(t+h, t, x)$, $h > 0$, in view of the assumed uniqueness of solutions of (2.1). Also, let $x(s, t, x)$ denote the solution of the system (2.2). Then, using the definition of $m(t)$, the Lipschitzian property of V, we get,

$$\begin{aligned}
D^+m(t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, y(t+h, t_0, x_0)) - V(t, y(t, t_0, x_0))] \\
&= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, y(t+h, t, x)) - V(t, x)] \\
&\leqq \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, y(t+h, t, x)) - V(t+h, x(t+h, t, x))] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x(t+h, t, x)) - V(t+h, x+hf(t, x))] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x+hf(t, x)) - V(t, x)] \\
&\leqq M(\alpha) \limsup_{h\to 0^+} \left[\frac{1}{h}\, ||\, y(t+h, t, x) - x(t+h, t, x)\, ||\right] + D^+_{(2.2)} V(t, x) \\
&\leqq M(\alpha) \limsup_{h\to 0^+} \frac{1}{h}\left\{\int_t^{t+h} ||\, f(s, y(s, t, x)) - f(s, x(s, t, x))\, ||\, ds \right. \\
&\quad \left. + \int_t^{t+h} ||\, G(s, y(s, t, x))\, ||\, dv_u(s)\right\} + D^+_{(2.2)} V(t, x) ,
\end{aligned}$$

where v_u is the total variation function of u.

Since f is Lipschitzian in x,

$$\begin{aligned}
\lim_{h\to 0^+} \sup \frac{1}{h}\left[\int_t^{t+h} ||\, f(s, y(s, t, x)) - f(s, x(s, t, x))\, ||\, ds\right] \\
\leqq L \limsup_{h\to 0^+} \left[\sup_{t\leqq s\leqq t+h} (||\, y(s, t, x) - x(s, t, x)\, ||)\right] = 0 ,
\end{aligned}$$

and we obtain, using (H_4), $D^+m(t) \leqq M(\alpha)w(t)\, ||\, y(t, t_0, x_0)\, || + D^+ V_{(2.2)}(t, x)$.

LEMMA 3.2. *Let the hypotheses* (H_1), (H_2), (H_4), *and* (H_5) *hold. If* $y(t, t_0, x_0)$ *is a solution of* (2.1), *then, at the points of discontinuity* $\{t_k\}$, *we have*

$$\text{(3.3)} \qquad \left\{\begin{aligned} &|\, V(t_k, y(t_k, t_0, x_0)) - V(t_k, y(t_k^-, t_0, x_0))\, | \\ &\quad \leqq M(\alpha)\rho g(t_k)\, ||\, u(t_k) - u(t_k^-)\, || \, . \end{aligned}\right.$$

Proof. Following the arguments in [3], we get

$$\| y(t_k, t_0, x_0) - y(t_k^-, t_0, x_0) \| = \| G(t_k, y(t_k, t_0, x_0))[u(t_k) - u(t_k^-)] \| .$$

Hence, using the assumption (H_4),

$$\begin{aligned} &| V(t_k, y(t_k, t_0, x_0)) - V(t_k, y(t_k^-, t_0, x_0)) | \\ &\quad \leqq M(\alpha) \| y(t_k, t_0, x_0) - y(t_k^-, t_0, x_0) \| \\ &\quad \leqq M(\alpha) g(t_k) \| y(t_k, t_0, x_0) \| \| u(t_k) - u(t_k^-) \| , \end{aligned}$$

which yields the estimate (3.3).

Proof of Theorem 3.1. Recalling that $D^+ V_{(2.2)}(t, x) \leqq 0$ and $\| x \| \leqq V(t, x)$, we have from (3.2), the following differential inequality

$$D^+ m(t) \leqq M(\alpha) m(t) w(t) . \tag{3.4}$$

Since $y(t, t_0, x_0)$ is continuous on $[t_{k-1}, t_k)$, $k = 1, 2, \cdots$, the inequality (3.4) yields, for each $t \in [t_{k-1}, t_k)$,

$$V(t, y(t, t_0, x_0)) \leqq V(t_{k-1}, y(t_{k-1}, t_0, x_0)) \exp\left(M(\alpha) \int_{t_{k-1}}^{t} w(s)ds\right) . \tag{3.5}$$

In view of (3.3), (3.5), and (3.1), we now obtain

$$\begin{aligned} &V(t_k, y(t_k, t_0, x_0)) \\ &\quad \leqq V(t_k, y(t_k^-, t_0, x_0)) + M(\alpha)\rho g(t_k)\lambda_k \exp\left(M(\alpha) \int_{t_0}^{t_k} w(s)ds\right) \\ &\quad \leqq V(t_{k-1}, y(t_{k-1}, t_0, x_0)) \exp\left(M(\alpha) \int_{t_{k-1}}^{t_k} w(s)ds\right) \\ &\qquad + M(\alpha)\rho g(t_k)\lambda_k \exp\left(M(\alpha) \int_{t_0}^{t_k} w(s)ds\right) , \end{aligned}$$

where $M(\alpha) \geqq 1$ (without any loss of generality).

Also, we have

$$V(t, y(t, t_0, x_0)) \leqq V(t_0, x_0) \exp\left(M(\alpha) \int_{t_0}^{t} w(s)ds\right), \; t \in [t_0, t_1) ;$$

$$V(t, y(t, t_0, x_0)) \leqq V(t_1, y(t_1, t_0, x_0)) \exp\left(M(\alpha) \int_{t_1}^{t} w(s)ds\right), \; t \in [t_1, t_2) .$$

Hence, for $t \in [t_0, t_2)$ we get

$$\begin{aligned} V(t, y(t, t_0, x_0)) &\leqq \Big[V(t_0, x_0) \exp\left(M(\alpha) \int_{t_0}^{t_1} w(s)ds\right) \\ &\qquad + M(\alpha)\rho g(t_1)\lambda_1 \exp\left((M(\alpha)\int_{t_0}^{t_1} w(s)ds\right)\Big] \exp\left(M(\alpha)\int_{t1}^{t} w(s)ds\right) \\ &= [V(t_0, x_0) + M(\alpha)\rho g(t_1)\lambda_1] \exp\left(M(\alpha) \int_{t_0}^{t} w(s)ds\right) . \end{aligned}$$

Thus, in general, for $t \geqq t_0$,

$$V(t, y(t, t_0, x_0)) \leqq \left[V(t_0, x_0) + M(\alpha)\rho \sum_{k=1}^{\infty} g(t_k)\lambda_k \right] \exp\left(M(\alpha) \int_{t_0}^{t} w(s)ds \right) .$$

Now, using the upper and lower estimates of V(see Theorem 2.1), we get

$$\| y(t, t_0, x_0) \| \leqq \left[a(t_0, \| x_0 \|) + M(\alpha)\rho \sum_{k=1}^{\infty} g(t_k)\lambda_k \right] N(t_0) \equiv \hat{a}(t_0, \| x_0 \|) ,$$

where $N(t_0) = \exp\left(M(\alpha) \int_{t_0}^{\infty} w(s)ds \right)$. Clearly $\hat{a} \in A$ since $\sum_{k=1}^{\infty} g(t_k)\lambda_k$ converges and the proof is complete.

THEOREM 3.2. *Let the hypotheses* (H_1), (H_3), (H_4), (H_6), *and* (H_8) *hold. Further, let*

$$(3.6) \qquad \| u(t_k) - u(t_k^-) \| \leqq \lambda_k \exp(-\delta(t_k - t_0)) , \quad \delta > 0 ,$$

be satisfied in place of (3.1) *in* (H_7). *Then, the ASI set* $x = 0$ *is exponentially asymptotically stable with respect to* (2.1).

Proof. We now employ the Theorem 2.2 and set $m(t) = W(t, y(t, t_0, x_0))$, where $W(t, x)$ is the Lyapunov function obtained in Theorem 2.2. Using the same arguments as in proof of Lemma 3.1 for this W, we get, in place of (3.2), the following inequality:

$$(3.7) \qquad \begin{cases} D^+m(t) \leqq L(\alpha)\rho w(t) + D^+_{(2\cdot 2)} W(t, x) \\ \qquad\qquad \leqq L(\alpha)\rho w(t) + \lambda(t) - \delta m(t) \\ \qquad\qquad = \lambda_0(t) - \delta m(t) , \quad \delta > 0 \end{cases}$$

with $\lambda_0(t) = L(\alpha)\rho w(t) + \lambda(t)$. Obviously, $\int_t^{t+1} \lambda_0(s)ds \to 0$ as $t \to \infty$. Since $y(t, t_0, x_0)$ is continuous on $[t_{k-1}, t_k)$, $k = 1, 2, \cdots$, the differential inequality (3.7) implies, for each $t \in [t_{k-1}, t_k)$,

$$(3.8) \qquad \begin{cases} W(t, y(t, t_0, x_0)) \leqq W(t_{k-1}, y(t_{k-1}, t_0, x_0)) \exp(-\delta(t - t_{k-1})) \\ \qquad\qquad + H(t, t_{k-1}) , \end{cases}$$

where $v(t, t_0, v_0) = v_0 \exp(-\delta(t - t_0)) + H(t, t_0)$ is the solution of the scalar differential equation

$$v' = -\delta v + \lambda_0(t) , \qquad v(t_0) = v_0 ,$$

and $H(t, t_0) \equiv \int_{t_0}^{t} \exp(-\delta(t - s))\lambda_0(s)ds$. For the properties $H(t, t_0)$, see [4] and [5].

In view of (3.3) (which is true with W in place of V and $L(\alpha)$ instead of $M(\alpha)$), (3.8) and the estimate (3.6) for the jumps of u, we get

$$\begin{aligned} W(t_k, y(t_k, t_0, x_0)) &\leqq W(t_k, y(t_k^-, t_0, x_0)) + L(\alpha)\rho g(t_k) \| u(t_k) - u(t_k^-) \| \\ &\leqq W(t_{k-1}, y(t_{k-1}, t_0, x_0)) \exp(-\delta(t_k - t_{k-1})) \\ &\quad + H(t_k, t_{k-1}) + L(\alpha)\rho g(t_k)\lambda_k \exp(-\delta(t_k - t_0)) . \end{aligned}$$

We also have

$$W(t, y(t, t_0, x_0)) \leqq W(t_0, x_0) \exp(-\delta(t - t_0)) + H(t, t_0),\ t \in [t_0, t_1) ;$$
$$W(t, y(t, t_0, x_0)) \leqq W(t_1, y(t_1, t_0, x_0)) \exp(-\delta(t - t_1)) + H(t, t_1),\ t \in [t_1, t_2) .$$

Thus, for $t \in [t_0, t_2)$, we obtain

$$\begin{aligned} W(t, y(t, t_0, x_0)) &\leqq \{[W(t_0, x_0) + L(\alpha)\rho g(t_1)\lambda_1] \exp(-\delta(t_1 - t_0)) \\ &\quad + H(t_1, t_0)\} \exp(-\delta(t - t_1)) + H(t, t_1) \\ &= [L(\alpha)\rho g(t_1)\lambda_1 + W(t_0, x_0)] \exp(-\delta(t - t_0)) + H(t, t_0) , \end{aligned}$$

since, by the definition of $H(t, t_0)$, it is obvious that

$$H(t_1, t_0) \exp(-\delta(t - t_1)) + H(t, t_1) \equiv H(t, t_0) .$$

Hence, in general, it can be shown that, for $t \geqq t_0$,

$$\begin{aligned} &W(t, y(t, t_0, x_0)) \\ &\qquad \leqq [W(t_0, x_0) + L(\alpha)\rho \sum_{k=1}^{\infty} g(t_k)\lambda_k] \exp(-\delta(t - t_0)) + H(t, t_0) . \end{aligned}$$

Now, by using the upper and lower estimates of $W(t, x)$ (see Theorem 2.2), we have

$$\| y(t, t_0, x_0) \| \leqq [b(t_0, \| x_0 \|) + \beta] \exp(-\delta(t - t_0)) + H(t, t_0) ,$$

where $\beta = L(\alpha)\rho \sum_{k=1}^{\infty} g(t_k)\lambda_k$. Further, using the facts that

(i) $\lim_{t\to\infty} [\sup_{t_0 \geqq 1} H(t, t_0)] = 0$ (see [4], p. 113),

(ii) $H(t, t_0) \leqq \sigma(t_0)$, $\sigma \in L$ (see [4]) and

(iii) β can be made small by choosing k sufficiently large, we see that the set $x = 0$ is ASI relative to the perturbed system (2.1) and that the ASI set $x = 0$ is exponentially asymptotically stable with respect to (2.1). The proof is complete.

REMARK. If in the above proof, we do not take $t_0 \geqq T_k$ (k sufficiently large enough to make β as small as we wish), we could still conclude that the solutions of (2.1) tend to zero as $t \to \infty$, without the ASI set $x = 0$ being uniformly stable, i.e., we can conclude that the ASI set is quasi-equi asymptotically stable.

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