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ON GRAPHICAL REGULAR REPRESENTATIONS OF CYCLIC EXTENSIONS OF GROUPS

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A simple graph X is said to be a graphical regular representation (GRR) of an abstract group G if the automorphism group of X is a regular permutation group and is isomorphic to G. If a group G_1 is a cyclic extension of a group G which admits a GRR, the question is posed whether G_1 also admits a GRR. Nowitz and Watkins have given an affirmative answer if G_1 is non-abelian and finite and the index $[G_1: G] \geqq 5$. This paper applies some new graph theoretical techniques to investigate the problem if $[G_1: G] = 2$, 3 or 4, whether or not G_1 is finite. As long as G_1 is non-abelian, an affirmative answer can again be given except in only finitely many unresolved cases.

1. Introduction. A simple graph X with vertex set $V(X)$ and automorphism group $A(X)$ is a *graphical regular representation* (*GRR*) *of the group* G if (i) $G \cong A(X)$ and (ii) $A(X)$ acts on $V(X)$ as a regular permutation group; that is, given $u, v \in V(X)$, there exists a unique $\varphi \in A(X)$ such that $\varphi(u) = v$. The graph X is a *GRR* if it is a GRR of some group.

A number of authors have investigated the question as to which finite groups G admit a GRR. A complete and easily stated solution is known in case G is abelian (see [10], [1], and [4]). If G is non-abelian, the problem is more difficult and the results that have emerged to date seem to fall far less readily into a neat pattern, (see [6], [11], [7], [8], [12], and [13]). Nonetheless certain general principles appear to take form. For example, we are tempted to conjecture that if G admits a GRR and $G \lhd G_1$ where G is non-abelian, then G_1 admits a GRR. In particular, it has been shown [7, Theorem 1] that:

If G admits a GRR and if the non-abelian group G_1 is a cyclic extension of G such that the index $[G_1: G] \geqq 5$, then G_1 admits a GRR.

The techniques of [7] are inadequate, however, if $[G_1: G] = 2$, 3 or 4, which is the task of the present paper. Elements of orders 2 and 3 have generally posed extra difficulties in the work on this problem. Thus [8, Theorem 2]:

Every non-abelian finite group whose order is coprime to 6 *admits a GRR.*

Only affirmative answers are obtained for the existence of a GRR if $[G_1: G] = 2$, 3 or 4. While the results are not complete, as in [7] and [8], the number of unresolved cases is finite, the results being "complete" if $|G|$ is sufficiently large. This fits in with another emerging principle that the only "large" groups not admitting a GRR are either abelian or generalized dicyclic (see [6] and [11]).

In §2 notation will be introduced. The main result of the section (Theorem 1) is a graph theoretical result that bypasses the obstacles due to elements of orders 2 and 3 encountered in [7].

In §3 existence of a GRR is demonstrated when G (a group with a GRR) is extended to a non-abelian group by the cyclic groups C_2 and C_4, provided G satisfies any of a variety of conditions. One such sufficient condition is simply that $|G| > 32$.

In §4 the group G is extended by C_3. The existence of a GRR follows when $|G| > 36$ as well as when $|G|$ assumes certain smaller values and satisfies certain conditions.

2. **Preliminaries.** The letter X will always denote a finite simple graph (except as noted in Corollary 1A below, where X is infinite) with vertex set $V(X)$, edge set $E(X)$, and automorphism group $A(X)$. The complementary graph of X relative to the complete graph on $|V(X)|$ vertices will be denoted by X'. Thus $A(X)$ and $A(X')$ are identical, even as permutation groups. If $u \in V(X)$, the *stabilizer* of u is the subgroup

$$A_u(X) = \{\varphi \in A(X) \mid \varphi(u) = u\} .$$

The valence (degree) of the vertex u is denoted $\rho(u)$. The graph X is said to be *ρ-valent* if $\rho(u) \equiv \rho$ for all $u \in V(X)$, and X is *isovalent* if X is ρ-valent for some cardinal ρ. (We depart here from the more conventional term "regular", which serves in another capacity in this paper.)

The letter G (with or without a subscript) will always denote a finite group. Its automorphism group will be denoted by $\mathrm{Aut}(G)$, its center by $Z(G)$, and its identity element by e. The cyclic group of order n will be denoted by C_n and the dihedral group of order $2n$ by D_n. If K is a subset of G, then $\langle K\rangle$ represents the subgroup of G generated by K. If $g \in G$, the *order* of g is $o(g) = |\langle g\rangle|$. The *exponent* of G, abbreviated by $\exp(G)$, is the least common multiple of the orders of the elements of G.

The letter H (with or without a subscript) will always denote a subset of the group G (bearing the same subscript, if any) with the following three properties: (i) $e \notin H$, (ii) $h \in H \Rightarrow h^{-1} \in H$, (iii) $\langle H\rangle = G$.

The *Cayley graph* $X_{G,H}$ *of* G *with respect to* H is the graph with $V(X_{G,H}) = G$, and $[x, y] \in E(X_{G,H})$ if and only if $y = xh$ for some $h \in$

H. A Cayley graph $X = X_{G,H}$ is always connected, and since $A(X)$ contains all left-multiplications by elements of G, it is transitive on $V(X)$. If X is a GRR of G and $|G| > 2$, then X must be a Cayley graph $X_{G,H}$ for some H, (see [9, Theorem 2]). Moreover, if X is a GRR of G, then so is X', and X is ρ-valent with $3 \leqq \rho \leqq |V(X)| - 4$, (see [13, Lemma 1]).

The group G is in *Class I* if G admits a GRR. It is in *Class II* if for each H there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi[H] = H$ but φ is not the identity. These two classes are disjoint [11, Theorem 1], and it has been conjectured that every finite group is in one of these two classes.

To prove that a given Cayley graph $X = X_{G,H}$ is a GRR of G, it suffices to show (see [7, Corollary 2.4]) that for some set $K \subset H$ such that $\langle K \rangle = G$,

$$\varphi \in A_e(X),\ h \in K \Longrightarrow \varphi(h) = h .$$

The only abelian groups in Class I are the elementary abelian 2-groups $(C_2)^n$ for $n \neq 2, 3, 4$. (See [4] and [10].)

For integers $n \geqq 2$, $k \geqq 1$, a graph X will belong to the class $\mathscr{P}_{n,k}$ if it is isovalent and if there exists a partition $[V_1, V_2, \cdots, V_p]$ of $V(X)$ with $2 \leqq p \leqq n$ such that every $x \in V_i$ is adjacent to $\leqq k$ elements of V_j, $(i \neq j)$. Such a partition $[V_1, \cdots, V_p]$ is called a $\mathscr{P}_{n,k}$-*partition*. We also allow $[V_1, \cdots, V_p]$ to denote a $\mathscr{P}_{n,k}$-partition when as many as p-2 cells are empty.

It is an immediate consequence of these definitions that if $[V_1, \cdots, V_p]$ is a $\mathscr{P}_{n,k}$-partition of X and if a vertex u of X is adjacent to $> k$ elements of V_i, then $u \in V_i$. Also, if $n \leqq n'$ and $k \leqq k'$, then $\mathscr{P}_{n,k} \subset \mathscr{P}_{n',k'}$.

THEOREM 1. *Let $X \in \mathscr{P}_{n,k}$ and suppose $|V(X)| > n(mh + 2k)$, where m, h are integers such that $m \geqq 2$, $h \geqq 1$. Then $X' \notin \mathscr{P}_{m,h}$.*

Proof. Suppose X satisfies the hypothesis of the theorem but not the conclusion. Let $[V_1, \cdots, V_p]$ be a $\mathscr{P}_{n,k}$-partition of $V(X)$ where the cells have been indexed so that

$$|V_i| \geqq |V_{i+1}| \qquad (i = 1, \cdots, p-1) .$$

Then $n|V_1| \geqq p|V_1| \geqq |V(X)| > n(mh + 2k)$, and so

$$|V_1| > mh + 2k . \tag{2.1}$$

Arbitrarily choose and fix $a \in V(X) \backslash V_1$, and let

$$N_1 = \{x \mid x \in V_1,\ [a, x] \in E(X)\} .$$

Then $|N_1| \leqq k$, since $X \in \mathscr{P}_{n,k}$.

By assumption there exists a $\mathscr{P}_{m,h}$-partition $[W_1, \cdots, W_q]$ of X'. Define $U_i = W_i \cap (V_1 \backslash N_1)$. Without loss of generality we may assume that $|U_1| \geqq |U_i| (i = 2, \cdots, q)$.

By (2.1), $mh + k < |V_1| = |N_1| + \sum_{i=1}^{q} |U_i| \leqq k + m|U_1|$, and so $h < |U_1|$. Since the vertex a is adjacent in X' to all vertices of U_1, we have $a \in W_1$. Moreover, a is adjacent to $\leqq h$ vertices in each cell $W_i (i = 2, \cdots, q)$. Hence $|U_i| \leqq h$, and so $|N_1| + \sum_{i=1}^{q} |U_i| \leqq k + |U_1| + (m-1)h$. By (2.1), $|U_1| > h + k$. Since each vertex in $V(X) \backslash V_1$, is adjacent to $\leqq k$ and nonadjacent to $> h$ vertices in U_1,

$$V(X) \backslash V_1 \subset W_1 . \tag{2.2}$$

Arbitrarily choose and fix $c \in V_1 \backslash U_1$. It will be shown that $c \in W_1$, which together with (2.2) implies that $W_1 = V(X)$, contrary to the assumption that $[W_1, \cdots, W_q]$ is nontrivial, and the theorem will follow.

Suppose $c \notin W_1$. By (2.2), $c \in V_1$ and so c is adjacent in X to $\leqq (p-1)k$ vertices of $V(X) \backslash V_1$. Hence c is adjacent in X' to $\geqq |V(X) \backslash V_1| - (p-1)k$ vertices of $V(X) \backslash V_1$. But c is adjacent in X' to $\leqq h$ vertices of W_1. Hence $h \geqq |V(X) \backslash V_1| - (p-1)k$. This implies $h \geqq (p-1)(|V_p| - k)$, whence $h + k \geqq |V_p|$, since $p - 1 \geqq 1$.

Suppose X is ρ-valent. Then by consideration of a vertex in V_p it is easily seen that

$$\rho \leqq (p-1)k + |V_p| - 1 \leqq pk + h - 1 . \tag{2.3}$$

Also, in the light of (2.2),

$$|V(X) \backslash W_1| \leqq |N_1| + \sum_{i=2}^{q} |U_i| \leqq k + (m-1)h . \tag{2.4}$$

Now c is adjacent in X to $\leqq \rho$ vertices in W_1. Hence by (2.1), (2.3), and (2.4), the number of vertices of W_1 to which c is adjacent in X' is at least

$$\begin{aligned} |W_1| - \rho &= |V(X)| - |V(X) \backslash W_1| - \rho \\ &\geqq n(mh + 2k) - [k + (m-1)h] - (pk + h - 1) \\ &= (n-1)mh + (2n - p - 1)k + 1 > h , \end{aligned}$$

which implies that $c \in W_1$.

COROLLARY 1A. *Let $X \in \mathscr{P}_{n,k}$ be a infinite graph. Then X' belongs to no class $\mathscr{P}_{m,h}$.*

Proof. Let X be infinite and let $[V_1, \cdots, V_p]$ be a $\mathscr{P}_{n,k}$-partition of X. Suppose $X' \in \mathscr{P}_{m,h}$ and let $[W_1, \cdots, W_q]$ be a $\mathscr{P}_{m,h}$-partition of X'. Without loss of generality, we can assume that $V_1 \cap W_1$ is infinite. Since each vertex in $V(X)\backslash V_1$ is adjacent in X to $\leqq k$ vertices of $V_1 \cap W_1$, it is adjacent in X' to infinitely many vertices of $V_1 \cap W_1$ and is therefore in W_1. Thus $V(X)\backslash V_1 \subset W_1$.

If V_i were infinite for some $i \neq 1$, then each vertex of V_1—and hence each vertex of $V(X)\backslash W_1$—would be adjacent in X' to infinitely many vertices of V_i. That is to say, each vertex in each $W_j (j \neq 1)$ would be adjacent in X' to infinitely many vertices of W_1, contrary to assumption. Hence $V_2, \cdots, V_p$ are each finite sets. Consideration of a vertex, say in V_p, yields that X is ρ-valent for some finite cardinal ρ. But then the valence ρ' of X' is infinite. A vertex in, say, W_q is therefore adjacent in X' to infinitely many vertices of W_q which is absurd, for W_q is finite since each vertex of W_1 is adjacent to $\leqq \rho$ vertices of W_q and nonadjacent to $\leqq h$ vertices of W_q.

For integers $n \geqq 2$, $k \geqq 1$, define $\nu(n, k)$ to be the smallest integer for which it holds that whenever $X \in \mathscr{P}_{n,k}$, X is a GRR, and $|V(X)| > \nu(n, k)$, then $X' \notin \mathscr{P}_{n,k}$. The following is immediate.

COROLLARY 1B. *If $n \geqq 2$ and $k \geqq 1$, then*

$$\nu(n, k) \leqq kn(n + 2) .$$

The inequality in Corollary 1B is not necessarily as strong as possible. While the authors have no uniform procedure for determining $\nu(n, k)$ in general, improvements over the corollary exist in special cases. For example, Corollary 1B gives only $\nu(3, 1) \leqq 15$ and $\nu(2, 2) \leqq 16$. The following stronger results will be used in §§3 and 4.

LEMMA 1. $\nu(3, 1) \leqq 11$.

Proof. Let $[V_1, V_2, V_3]$ be a $\mathscr{P}_{3,1}$-partition of a GRR X where $|V(X)| \geqq 12$, and suppose $|V_1| \geqq |V_2| \geqq |V_3|$. Clearly $|V_1| \geqq 4$. Note that $\rho \geqq 3$, since X is a GRR. Supposing the corollary false, let $[W_1, W_2, W_3]$ be a $\mathscr{P}_{3,1}$-partition of X'.

Suppose $|V_1| = 4$ or 5. Clearly then $|V_2| \geqq 4$ and $|V_3| \geqq 2$. Also V_1 must intersect some cell of $[W_1, W_2, W_3]$ in at least two vertices; say $a_1, a_2 \in V_1 \cap W_1$. With $k = 1$, at least two vertices $b_1, b_2 \in V_2$ are adjacent in X to neither a_1 nor a_2 and so they are adjacent in X' to both a_1 and a_2. Hence $b_1, b_2 \in W_1$. If $c \in V_3$, then c is adjacent in X to at most one of a_1, a_2 and at most one of b_1, b_2. So c is adjacent in X' to at least two of these four vertices. Hence $c \in W_1$. We now have that

$$|V_i \cap W_1| \geqq 2 \qquad (i = 1, 2, 3) .$$

Moreover, any vertex $d \in V_i$ is adjacent in X' to at least one vertex in $V_j \cap W_1$ for both values of $j \neq i$, whence $d \in W_1$. We infer the contradiction that $W_1 = V(X)$.

Now suppose $|V_1| \geqq 6$. We assert that the inequality

$$|V_1 \cap W_j| \geqq 2 \tag{2.5}$$

can hold for at most two values of $j = 1, 2, 3$. For if (2.5) held for $j = 1, 2, 3$, then any vertex $b \in V_2$, being adjacent in X to at most one vertex of V_1, is adjacent in X' to at least two vertices in each of two of the sets $V_1 \cap W_j$, implying that b is in two different cells of $[W_1, W_2, W_3]$. Since $|V_1| \geqq 6$, we may say for definiteness that $|V_1 \cap W_1| \geqq 3$.

If $c \in V_2 \cup V_3$, then c is adjacent in X' to at least two vertices in $V_1 \cap W_1$, and so $c \in W_1$. Thus $V_2 \cup V_3 \subset W_1$.

Pick $a \in W_2 \cup W_3$. Then a is adjacent in X to all but at most one vertex in W_1. But since $a \in V_1$, a is adjacent to at most one vertex in each of V_2 and V_3. Hence $|V_2| \leqq 2$ and $|V_3| \leqq 1$. But $|V_3| \neq 1$, or else the lone vertex in V_3 would be adjacent in X to two vertices in V_1 or in V_2 since $\rho \geqq 3$. Hence $V_3 = \emptyset$. But then a vertex in V_2 would be adjacent (in X) to at least two vertices in V_1, giving a contradiction. It follows that $W_2 \cup W_3 = \emptyset$, and no $\mathscr{P}_{3,1}$-partition of X' exists.

LEMMA 2. $\nu(2, 2) \leqq 15$.

Proof. Since $\nu(2, 2) \leqq 16$ by Corollary 1B, it may be assumed that $X = X_{G,H}$ is a GRR where $|G| = 16$. Let $[V_1, V_2]$ be a $\mathscr{P}_{2,2}$-partition of X, and suppose $[W_1, W_2]$ is a $\mathscr{P}_{2,2}$-partition of X'. We show first that

$$|V_i \cap W_j| = 4 , \qquad i, j = 1, 2 . \tag{2.6}$$

Supposing (2.6) false, we may assume without loss of generality that $|V_1 \cap W_1| \geqq 5$. Then each vertex in $V(X)\backslash V_1$ is adjacent in X' to at least three vertices of $V_1 \cap W_1$ and is therefore in W_1. Thus $V_2 \subset W_1$.

If $|V_2| \geqq 5$ we conclude by similar argument that V_1 is also contained in W_1, which is not possible. If $|V_2| \leqq 4$, then X is ρ-valent for $\rho \leqq 5$, and $|V_1| \geqq 12$. Now $|V_1 \cap W_2| \leqq 4$ or else $V_2 \subset W_2$, which is absurd. But then $|V_1 \cap W_1| \geqq 8$ and, as every vertex in W_2 can be adjacent in X to at most 5 vertices of $V_1 \cap W_1$, it is adjacent in X' to at least three vertices in $V_1 \cap W_1$ and hence is in W_1. This proves (2.6).

Each vertex $a \in V_1 \cap W_1$ must be adjacent in X to exactly two vertices in $V_2 \cap W_2$; a cannot have more than two neighbors in $V_2 \cap W_2$ since $X \in \mathscr{P}_{2,2}$, while if there were fewer than two, a would be adjacent in X' to at least three vertices of $V_2 \cap W_2$, in contradiction to the assumption that $a \in W_1$. Furthermore, a must be adjacent in X to all vertices in $V_1 \cap W_2$. Otherwise a would be adjacent in X' to at least one vertex in $V_1 \cap W_2$ and to two vertices in $V_2 \cap W_2$, making a adjacent in X' to at least three vertices in W_2, which is not possible. In summary, with $i \neq i'$ and $j \neq j'$, each vertex in $V_i \cap W_j$ is adjacent to all vertices in $V_i \cap W_{j'}$, to exactly two vertices in $V_{i'} \cap W_{j'}$, and to no vertices of $V_{i'} \cap W_j$, $(i, i', j, j' = 1, 2)$.

Thus $\rho \geqq 6$. It may be assumed that $\rho \leqq 7$; otherwise X and X' can be interchanged.

The eight edges joining $V_1 \cap W_1$ with $V_2 \cap W_2$ span a 2-valent subgraph on eight vertices which must be either the union of two disjoint 4-gons or a single 8-gon. In the first case let $[v_1, \cdots, v_4]$ and $[v_5, \cdots, v_8]$ denote the two 4-gons. In the second case let $[v_1, \cdots, v_8]$ denote the 8-gon. In either case, one may suppose that the vertices with odd index lie in $V_1 \cap W_1$.

Let T_i denote the subgraph of X induced by the vertices in $V_i \cap W_i (i = 1, 2)$. If $\rho = 6$, then T_i is totally disconnected. If $\rho = 7$, then each T_i consists of two edges without common end-point.

Consider the permutation β on $V(X)$ whose cyclic decomposition is given by

$$\beta = (v_1 v_5)(v_2 v_6)(v_3 v_7)(v_4 v_8) .$$

Since β has fixed-points without being the identity, $\beta \notin A(X)$, for $A(X)$ is a regular permutation group. But clearly $\beta \in A(X)$ when $\rho = 6$, and the verification is straightforward when $\rho = 7$.

The result in Theorem 1 is far more general than is required for the applications in the next two sections. Actually, a rather narrow formulation is involved repeatedly. In the interest ultimately of economy, we here formulate that application in the following way:

LEMMA 3. *Let G be a group in Class I. Let $n \geqq 2$ and $k \geqq 1$ be given and suppose $|G| > \nu(n, k)$. Then G has a GRR $X = X_{G,H} \notin \mathscr{P}_{n,k}$. Moreover, let $G \lhd G_1$ and suppose that for some $b \in G_1 \backslash G$, G_1 admits the coset decomposition $G_1 = G \cup bG \cup \cdots \cup b^{n-1}G$. Let H_1 be a set of generators of G_1 such that*

(i) $e \notin H_1 = H_1^{-1}$,

(ii) $H_1 \cap G = H$,

(iii) $k \geqq |H_1 \cap b^i G|$, $(i = 1, \cdots, n-1)$.

Form the Cayley graph $Y = X_{G_1, H_1}$, and let X_i be the subgraph of Y induced by vertices in the coset $b^i G$. Then $X_i \cong X$ and the sets

$\{V(X_i) \mid i = 0, \cdots, n-1\}$ form a complete system of imprimitivity for the permutation group $A(Y)$. Finally, if $\varphi \in A(Y)$ satisfies $\varphi x_0 = x_0$ for some $x_0 \in V(X_i)$, then $\varphi x = x$ for all $x \in V(X_i)$.

Proof. If $X = X_{G,H}$ is a GRR of G, then so is X'. Since $|V(X)| = |G| > \nu(n, k)$, at most one of X and X' belongs to $\mathscr{P}_{n,k}$ by definition of $\nu(n, k)$. For definiteness, suppose $X \notin \mathscr{P}_{n,k}$.

Assume the remainder of the hypothesis. That $X_i \cong X$ is immediate from the definition of a Cayley graph. Let $\varphi \in A(Y)$. Choose one of the subgraphs X_j and let $V_i = V(X_i) \cap \varphi[V(X_j)]$. If $i \neq j$, each vertex of X_i is adjacent to at most k vertices of X_j. If more than one of the sets V_i are nonempty, the sets V_i form a $\mathscr{P}_{n,k}$-partition of $\varphi[X_j]$. But $\varphi[X_j] \cong X_j \cong X \notin \mathscr{P}_{n,k}$. Hence $\varphi[X_j] = X_i$ for some i. Since Y is a Cayley graph, $A(Y)$ is transitive, and the assertion concerning imprimitivity is proved.

Now suppose $\varphi x_0 = x_0$ for some $x_0 \in V(X_i)$. Since $V(X_i)$ is a block of imprimitivity, $\varphi[X_i] = X_i$. Hence the restriction φ_i of φ to X_i belongs to $A(X_i)$. But $X_i \cong X$ and so $A(X_i)$ is a regular permutation group. Since φ_i has a fixed-point x_0, φ_i is the identity of $A(X_i)$, which proves the lemma.

LEMMA 4. *Every non-abelian group G of square-free order $|G| > 10$ is Class I.*

Proof. Since $|G|$ is square-free, G is generated by two elements a and b satisfying relations of the form

$$\langle a, b \mid a^r = b^s = e, b^{-1}ab = a^m \rangle \tag{2.7}$$

where $|G| = rs$ and $m^s \equiv 1 (\mathrm{mod}\ r)$. (See [5, p. 261].) All groups (2.7) have been classified [7, Theorem 2]. Except for the groups D_3 and D_5, the only groups (2.7) not in Class I are either abelian or have order divisible by 4.

3. Classification of Extensions by C_2 and C_4.

The main result of this section is:

THEOREM 2. *Let the non-abelian group G_1 contain a subgroup G of index 2, and suppose that G is in Class I. Then each of the following conditions is sufficient for G_1 to be in Class I:*

(a) *$|G| > 2$ and G_1 is a semi-direct product of G by C_2.*

(b) *$|G| > 12$ but $|G| \neq 16$ or 32.*

(c) *$|G| = 16$ or 32, and either:*

(i) *there exists an element $b \in G_1 \backslash G$ such that $o(b) \neq 4$, or*

(ii) *$Z(G)$ does not contain $C_2 \times C_2$ as a subgroup.*

REMARK. The reader should not infer that condition (c) is necessary if $|G| = 16$ or 32, but merely that the profusion of groups of orders 16 and 32 not satisfying (ii) and of their cyclic extensions not satisfying (i) renders a case-by-case study inordinately tedious and without anticipation of surprising outcome. Theorem 2 is stated in the given form merely to convey as much information as possible.

Groups of order 16 have been classified (see [12]) and exactly two such groups both are in Class I and fail to satisfy (ii). There are six groups G_1 of order 32 failing to satisfy (i) which are extensions of these two exceptional groups of order 16. There are more than 100 groups G_1 of order 64 failing to satisfy (i) which are extensions of those groups G of order 32 failing to satisfy (ii). For a complete list of such groups, see [3].

Proof. First consider the case where there exists an element $b \in G_1 \backslash G$ of order 2. Since G is in Class I and $G \not\cong C_2$, we have $|G| \geqq 12$ (see [7, 1.10]). By Corollary 1B and Lemma 3, G has a GRR $X = X_{G,H} \notin \mathscr{P}_{2,1}$. If $H_1 = H \cup \{b\}$, one may form the Cayley graph $Y_0 = X_{G_1,H_1}$. Let $\varphi \in A_e(Y_0)$. With X_0 and X_1 defined as in Lemma 3, it is immediate that φ fixes each vertex of X_0. Hence $\varphi(b) = b$, and φ is the identity on all of $V(Y_0)$, whence Y_0 is a GRR of G_1. It has been shown that (a) is a sufficient condition.

Suppose now that $b^2 \neq e$ for all $b \in G_1 \backslash G$. If $|G| = 14$, then $G \cong D_7$. Of the two non-abelian groups of order 28 [2, p. 135], only one contains D_7 as a subgroup, namely D_{14} which is known to be in Class I, [11, Theorem 2]. Since no group in Class I has order 13 or 15 we now let $|G| \geqq 16$. By Lemmas 2 and 3, G has a GRR $X = X_{G,H} \notin \mathscr{P}_{2,2}$, as before. Arbitrarily choose then fix $b \in G_1 \backslash G$. Let $H_1 = H \cup \{b, b^{-1}\}$, and form $Y_1 = X_{G_1,H_1}$. Let $\varphi \in A_e(Y_1)$. As before, φ fixes each vertex of X_0, and the theorem is proved unless φ has no fixed-point in X_1, which we now assume to be the case. In particular, φ interchanges b and b^{-1}.

The vertices in X_0 adjacent to b are e and b^2 while those in X_0 adjacent to b^{-1} are e and b^{-2}. Thus $b^2 = b^{-2}$, and $b^4 = e$. Since the choice of $b \in G_1 \backslash G$ was arbitrary, it follows that $o(b) = 4$ for all $b \in G_1 \backslash G$.

We next show that $b^2 \in Z(G_1)$ for all $b \in G_1 \backslash G$. Let $h \in H$. Since b^{-1} is adjacent to $b^{-1}h$, $b = \varphi(b^{-1})$ must be adjacent to $\varphi(b^{-1}h)$. The neighbors in X_1 of $b^{-1}hb \in V(X_0)$ are $b^{-1}h$ and $b^{-1}hb^2$. Thus $\varphi(b^{-1}h) = b^{-1}hb^2$. There exists some $h' \in H$ such that $bh' = b^{-1}hb^2$, i.e., $h' = b^{-2}hb^2$. This implies that the set H is fixed under conjugation by b^2, but since X is a GRR, the automorphism $x \mapsto b^{-2}xb^2$ in Aut(G) must be the identity. Thus $b^2 \in Z(G)$. But for any $x \in G$, $(bx)b^2 = b(b^2x) = b^2(bx)$, so $b^2 \in Z(G_1)$, as required.

Recalling that the choice of $b \in G_1 \backslash G$ was arbitrary, we have that the set

$$K = \{c^2 \mid c \in G_1 \backslash G\}$$

generates an elementary abelian 2-group contained in $Z(G_1) \cap G$. If K consists of a single element, then $b^2 = (bx)^2$ for all $x \in G$, $b \in G_1 \backslash G$. This implies that

$$(3.1) \qquad b^{-1}xb = x^{-1} \; ; \qquad x \in G,\ b \in G_1 \backslash G \; .$$

Thus G is abelian. Since the only abelian groups in Class I are elementary abelian 2-groups, (3.1) implies that G_1 is abelian, contrary to the hypothesis of this theorem. Hence

$$(3.2) \qquad |K| \geqq 2 \; .$$

At this point condition (c) has been shown sufficient.

Having obtained the foregoing information about G_1, we continue to prove the sufficiency of (b) by first supposing that $|G| > 32$. By Corollary 1B and Lemma 3 we may assume that $X = X_{G,H} \notin \mathscr{P}_{2,4}$. By (3.2), there exist $b, c \in G_1 \backslash G$ such that $b^2 \neq c^2$. Redefine $H_1 = H \cup \{b, b^{-1}, c, c^{-1}\}$, and form $Y_2 = X_{G_1, H_1}$. As before, let $\varphi \in A_e(Y_2)$. Again, φ fixes each vertex of X_0, and we may assume that φ fixes no vertex of X_1. For each $x \in V(X_0)$, its neighbors in X_1 are xb, xb^{-1}, xc, xc^{-1} while the neighbors in X_1 of xb^2 are $xb^{-1}, xb, xb^2c, xb^2c^{-1}$. Since $b^2 \neq c^2$ (and so $b \neq c^{-1}$), the vertices x and xb^2 have precisely two common neighbors in X_1, namely xb and xb^{-1}. They must be interchanged by φ, as are xc with xc^{-1}. This is so for all $x \in V(X_0)$. In particular, for $x = e$ or $b^{-1}c$ we obtain

$$b = \varphi(b^{-1}) = \varphi((b^{-1}c)c^{-1}) = b^{-1}c^2 \; ,$$

implying that $b^2 = c^2$, a contradiction. Hence Y_2 is a GRR of G_1.

It remains only to consider the cases where $16 < |G| < 32$. Because of (3.2), G contains the central subgroup $M = \langle K \rangle$ of order divisible by 4, leaving only 20, 24, and 28 as possible orders of G. If $|G| = 20$ or 28, then G/M is cyclic. But then G is abelian but not an elementary abelian 2-group, contrary to the hypothesis that G is in Class I.

Suppose finally that $|G| = 24$. It suffices to prove that G is abelian, for then it cannot be in Class I. There exists an element $a \in G$ of order 3. We show first that

$$(3.3) \qquad a^{-1} = d^{-1}ad \equiv a^d \qquad \text{for all } d \in bG \; .$$

Since $d^2, (da)^2 \in K$, one has immediately that

$$d^2aa^d = ad^2a^d = (ad)^2 = d^{-1}(da)^2d = (da)^2 = d^2d^{-1}ada = d^2a^da .$$

It follows that a and a^d commute and that $(aa^d)^2 = [d^{-2}(da)^2]^2 = d^{-4}(da)^4 = e$. Since $o(a^d) = 3$,

$$aa^d = (aa^d)^2aa^d = a^3(a^d)^3 = e ,$$

and hence (3.3) holds. Now write $d = xb$ for arbitrary $x \in G$. By (3.3), for any $x \in G$,

$$axb = (xb)a^{-1} = x(ba^{-1}) = xab .$$

This shows that $a \in Z(G)$. Therefore, the subgroup $\langle M \cup \{a\}\rangle$ is central and has order 12, which is possible only if G is abelian. This concludes the proof of the theorem.

REMARKS. 1. Except in the case where G is abelian, Theorem 2 generalizes the result of Watkins and Nowitz [13] that the direct product of C_2 with any group in Class I other than C_2 is also in Class I.

2. In addition to the special cases of groups of orders 16 and 32 previously remarked upon, the question remains open concerning the classification of extensions by C_2 (other than semi-direct products) of the one group in Class I of order 12, namely D_6.

COROLLARY 2A. *Let the non-abelian group G_1 contain a normal subgroup G such that $G_1/G \cong C_4$, and suppose that G is in Class I. If $|G| > 32$, then G_1 is in Class I.*

Proof. Let π be the projection morphism of G_1 onto $G_1/G \cong C_4 = \langle a\rangle$. If $\pi(b) = a$, we have the coset decomposition

$$G_1 = G \cup bG \cup b^2G \cup b^3G .$$

Furthermore, $G_0 = G \cup b^2G$ is a subgroup of G_1 and $[G_0: G] = [G_1: G_0] = 2$. If G_0 is non-abelian, it is in Class I by Theorem 2, and therefore so is G_1.

We have only to consider the case where G_0 is abelian. Then G is abelian, and since G is in Class I, we have $G \cong (C_2)^{k+1}$ with $k \geqq 5$ because $|G| > 32$. If $b^4 = e$, the group G_0 is an elementary abelian 2-group and so is in Class I. Then G_1 is also in Class I by Theorem 2. We can therefore assume that $b^4 \neq e$. As $b^4 \in G$ and $\exp(G) = 2$, we have $o(b) = 8$.

Denote by K the subgroup of G such that $G = K \times \langle b^4\rangle$. Note that $K \cong (C_2)^k$ is in Class I since $k \geqq 5$. If $K \lhd G_1$, then G_1 is an extension of K by C_8 and is in Class I by [7, Theorem 1]. Otherwise $bK \neq Kb$, and there exists an element $c \in K$ such that $bc \neq cb$. Using this c we shall construct a GRR of G_1.

Since $|G| \geqq 64 > 48 \geqq \nu(4, 2)$, G admits a GRR $X = X_{G,H} \notin \mathscr{P}_{4,2}$, by Lemma 3. The set

$$H_1 = H \cup \{b, b^{-1}, b^2, b^{-2}, bc, cb^{-1}\}$$

generates G_1. We shall demonstrate that the Cayley graph $Y_1 = X_{G_1,H_1}$ is a GRR.

First observe the distribution of elements of $H_1 \backslash H$ amongst the cosets of G:

$$b,\ bc \in bG;\ b^2,\ b^{-2} \in b^2G;\ b^{-1},\ cb^{-1} \in b^3G\ .$$

Let $\varphi \in A_e(Y_1)$. By Lemma 3 (and in the same notation as in the lemma), φ fixes each vertex of X_0. In particular, φ fixes the four distinct vertices b^4, bcb^{-1}, c, and e. The only neighbors of b and bc in X_0 are e and bcb^{-1}; the vertices b^2 and b^{-2} have the neighbors e and $b^4 \in V(X_0)$; and b^{-1} and cb^{-1} are each adjacent to e and $c \in V(X_0)$. Therefore, φ fixes each $V(X_i)$ setwise. As b is adjacent to b^2 but not to b^{-2}, while bc is adjacent to neither b^2 nor b^{-2}, it is clear that φ cannot interchange b with bc. Hence $\varphi(b) = b$. As φ fixes every element of the generating set $\{b\} \cup H$ of G_1, it fixes every element of G_1. This proves the corollary.

4. Classification of extensions by C_3.

THEOREM 3. *Let the non-abelian group G_1 contain a normal subgroup G of index 3, and suppose $G \not\cong C_2$ is in Class I. Then each of the following conditions is sufficient for G_1 to be in Class I:*

(a) *There exists an element $b \in G_1 \backslash G$ such that $b^6 \neq e$.*

(b) $|G| > 30$ *and either*:

(i) $G \not\cong (C_2)^5$,

or

(ii) *if* $|G| = 36$, *then* $\exp(G) \neq 6$.

(c) $G_1 = G \times C_3$, *and if* $|G| = 24$ *or* 36, *then* $\exp(G) \neq 6$.

Proof. (a) By hypothesis, $|G| \geqq 12$. Lemmas 1 and 3 imply that G has a GRR $X = X_{G,H} \notin \mathscr{P}_{3,1}$. Assuming condition (a), let $H_1 = H \cup \{b, b^{-1}\}$ and form $Y_0 = X_{G_1,H_1}$. Let $\varphi \in A_e(Y_0)$. By Lemma 3 (and in the same notation as in the lemma), φ fixes each vertex of X_0. It suffices to prove that φ fixes b (and hence b^{-1}). We suppose not, and so φ interchanges b with b^{-1}. Lemma 3 further implies that φ maps the subgraphs X_1 and X_{-1} isomorphically onto each other.

Since the only neighbor of b^{-1} lying in X_1 is b^{-2}, $\varphi(b^{-2})$ must be the unique neighbor of $b = \varphi(b^{-1})$ lying in X_{-1}, namely b^2. But $b^3 \in$

$V(X_0)$ is fixed by φ, and so φ interchanges its neighbors $b^4 \in V(X_1)$ with $b^2 \in V(X_{-1})$. Thus $\varphi(b^4) = b^2 = \varphi(b^{-2})$, whence $b^4 = b^{-2}$, or $b^6 = e$, contrary to assumption.

(b) In the light of (a) above, it may be assumed that

$$o(u) = 3 \text{ or } 6 \qquad \text{for all } u \in G_1 \backslash G . \tag{4.1}$$

If $o(u) = 6$, then $o(u^2) = 3$. So we may always select an element $d \in G_1 \backslash G$ of order 3. Write the coset decomposition $G_1 = G \cup dG \cup d^{-1}G$.

We shall now prove that:

$$\exists b, c \in dG \quad \text{such that} \quad b^3 = e \quad \text{and} \quad bc \neq cb . \tag{4.2}$$

If G contains an element g such that $g^6 \neq e$, let $c = gb$, where $b \in dG$ has order 3. If b commutes with c, then b commutes with g, and

$$c^6 = (gb)^6 = g^6 b^6 = g^6 \neq e ,$$

contrary to (4.1).

Suppose then that $g^6 = e$ for all $g \in G$ and that (4.2) fails. Since $d^3 = e$, d commutes with every element of $G_1 \backslash G$. Thus, for all $g \in G$ $(dg)d = d(dg)$, and $d \in Z(G_1)$. Define

$$K = \{x \in G \mid o(dx) = 3\} .$$

Then $x \in G$ implies $x^3 = (dx)^3$, while $(dx)^3 = e$ if and only if $x \in K$. Thus for $x \in G$,

$$x \in K \Longleftrightarrow o(x) = 1 \text{ or } 3 ; \tag{4.3}$$

$$x \in G \backslash K \Longleftrightarrow o(x) = 2 \text{ or } 6 . \tag{4.4}$$

If $x \in K$ and $g \in G$, then since (4.2) is assumed false, $(dx)(dg) = (dg)(dx)$. Since d is central we have $xg = gx$, and so $K \subset Z(G)$. If $x_1, x_2 \in K$, then $(dx_1x_2)^3 = d^3x_1^3x_2^3 = e$, so $x_1x_2 \in K$, and K is a normal subgroup of G.

Consider the cosets of G with respect to K. Since by (4.3) and (4.4) $x^2 \in K$ for all $x \in G$, the quotient group G/K is an elementary abelian 2-group. Moreover, if $x_1 = x_2k$ for some $x_1, x_2 \in G$ and $k \in K$, then

$$x_1x_2 = x_2kx_2 = x_2(x_2k) = x_2x_1 .$$

That is to say, any two elements of the same coset with respect to K commute.

Let $L = \{x \in G \mid x^2 = e\}$, and let $y, z \in L$. If $o(yz) = 3$ or 6, then $o((yz)^2) = 3$ whence $(yz)^2 \in K$ and is therefore central. Hence

$$e = (yzyz)y^2(zyzy) = (yz)^2y(yz)^2y = (yz)^4y^2 = (yz)^4 .$$

As no element of G has order 4, $(yz)^2 = e$ and $yz \in L$. Hence L is a subgroup of G. In particular, L is an elementary abelian 2-group.

Let $g \in G$. If $o(g) = 1$, 2 or 3, clearly $g \in LK$. If $o(g) = 6$, then $g = g^3 g^{-2} \in LK$, and so $G = LK$. Since K is central and L is abelian, G is abelian. But $G_1 = \langle G \cup \{d\} \rangle$. We have a contradiction, since G_1 is non-abelian by hypothesis. This proves (4.2).

We now assume the hypothesis (b). By Corollary 1B and Lemma 3, G has a GRR $X = X_{G,H} \notin \mathscr{P}_{3,2}$ Select b, c fulfilling (4.2). Let $H_1 = H \cup \{b, b^{-1}, c, c^{-1}\}$ and form the Cayley graph $Y_1 = X_{G_1,H_1}$. Let $\varphi \in A_e(Y_1)$. As in all the previous cases, φ fixes each vertex of X_0, in particular bc^{-1}.

The neighbors in X_{-1} of e are of course b^{-1} and c^{-1}; those of bc^{-1} are $bc^{-1}b^{-1}$ and bc^{-2}. Since $b \neq c$, certainly $b^{-1} \neq bc^{-1}b^{-1}$ and $c^{-1} \neq bc^{-2}$. If $c^{-1} = bc^{-1}b^{-1}$, then $bc = cb$, contrary to assumption. Also the equality $b^{-1} = bc^{-2}$ implies, since $b^3 = e$, that $b = c^{-2}$ which in turn implies that b and c commute. Hence e and bc^{-1} have no common neighbors in X_{-1}.

The vertex $b \in V(X_1)$ is a common neighbor of e and bc^{-1}. Thus $\varphi(b) \in V(X_1)$, and by Lemma 3, $\varphi[X_i] = X_i$ $(i = \pm 1)$. If b is the only such common neighbor, then $\varphi(b) = b$, φ is necessarily the identity, and (b) is proved. Suppose that b is not the sole common neighbor. This is possible only if $c = (bc^{-1})b$, which implies

$$(cb^{-1})^2 = e\ , \tag{4.5}$$

which we assume to hold.

If G contains an element g such that $g^6 \neq e$, then, as previously noted, we may choose $c = gb$. Substitution into (4.5), however, implies that $g^2 = e$. Assume that $g^6 = e$ for all $g \in G$. Inasmuch as the group $(C_2)^5$ and all groups of order 36 with exponent 6 are exempted from consideration, no cases are lost if one assumes that $|G| > 45$. Thus it may be assumed that $X \notin \mathscr{P}_{3,3}$. In addition to $b, c \in dG$ chosen according to (4.2), we select an additional element $f \in dG$ such that

$$f \notin \{b, bcb^{-1}, bc^{-1}b, b^{-1}c^{-1}, c, cbc^{-1}, cb^{-1}c, c^2b^{-1}\}\ .$$

Redefine $H_1 = H \cup \{b, b^{-1}, c, c^{-1}, f, f^{-1}\}$ and form $Y_2 = X_{G_1,H_1}$. One can verify straightforwardly that despite the additional edges, bc^{-1} and e still have no common neighbors in X_{-1} and that b and c remain their only common neighbors in X_1. Since f is the only neighbor in X_1 of e which is not a neighbor of bc^{-1}, $\varphi(f) = f$. Thus φ fixes pointwise the set $H \cup \{f\}$ which generates G_1. Hence φ is the identity and Y_2 is a GRR of G.

(c) Let $G_1 = G \times C_3$. We first suppose that G contains an element g such that $g^6 \neq e$. There exists an element $c \in G_1 \backslash G$ such that $c^3 = e$.

If $b = gc$,

$$b^6 = (gc)^6 = g^6c^6 = g^6 \neq e ,$$

and the result follows from part (a) above.

Suppose therefore, that $\exp(G) = 2$, 3 or 6. By part (b), we need only assume $12 \leqq |G| \leqq 36$ and ignore the orders 24 and 36. If $\exp(G) = 2$, then G is abelian. Since G_1 is non-abelian, there are no cases to consider. If $\exp(G) = 3$, then $|G| = 27$. But the (unique) group G of order 27 with exponent 3 is in Class II, [8, Theorem 3].

If $\exp(G) = 6$, then $|G| = 12$ or 18.

Since G is in Class I, $G \cong C_3 \times D_3$ if $|G| = 18$, and $G \cong D_6$ if $|G| = 12$. (See [12].)

Following Coxeter and Moser [2, p. 134], $C_3 \times D_3$ can be represented by

$$s^3 = t^6 = e, tst = s^{-1} .$$

G_1 is formed by adjoining a generator u with

$$u^3 = e, su = us, tu = ut .$$

Let

$$H_1 = \{s, s^{-1}, t, t^{-1}, ts, t^{-1}s, t^2s, t^{-2}s^{-1}, u, u^{-1}, tu, t^{-1}u^{-1}\} .$$

We assert that the Cayley graph $Y = X_{G_1,H_1}$ is a GRR of G_1. Let Y_e denote the subgraph of Y induced by the neighbors of e, namely the set H_1, and let $\varphi \in A_e(Y)$. The restriction of φ to Y_e belongs to $A(Y_e)$. One straightforwardly verifies that Y_e has the form of Figure 1.

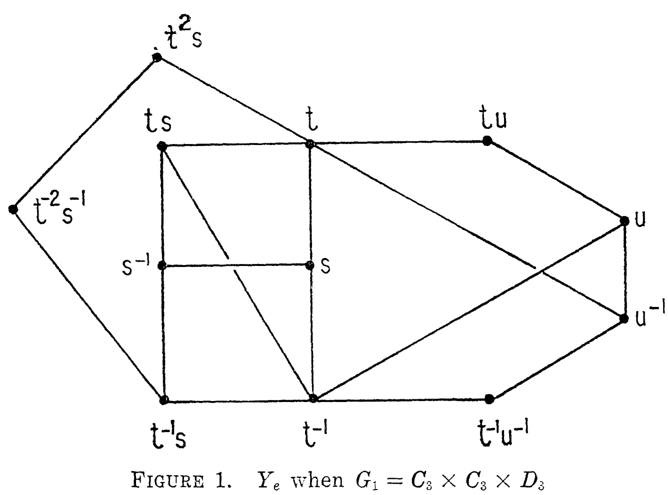

FIGURE 1. Y_e when $G_1 = C_3 \times C_3 \times D_3$

The two vertices whose valence in Y_e is 5 are t and t^{-1}, but only t^{-1} has exactly one neighbor $t^{-1}u^{-1}$ of valence 2. Therefore, φ fixes both of these vertices and hence all vertices in the subgroup $\langle t, u\rangle$ generated by them. Then φ fixes the second neighbor t^2s of t whose valence in Y_e is 2. But $G_1 = \langle t^2s, t, u\rangle$, and so φ is the identity and Y is a GRR of G_1.

If $G = D_6$, we represent $G_1 = C_3 \times D_6$ by

$$s^6 = t^2 = u^3 = e,\ tst = s^{-1},\ su = us,\ tu = ut\ .$$

Then G_1 is generated by

$$H_1 = \{s, s^{-1}, t, ts^{-1}, u, u^{-1}, su, s^{-1}u^{-1}, tu, tu^{-1}\}\ .$$

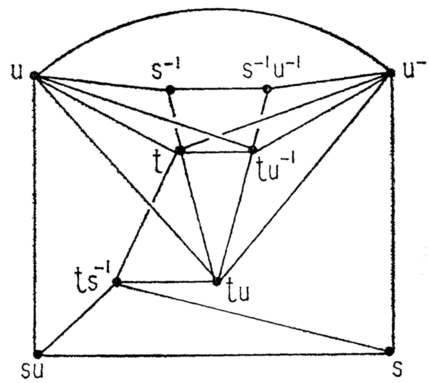

FIGURE 2. Y_e when $G_1 = C_3 \times D_6$

Define Y, Y_e, and φ as in the preceding paragraph. In this case (see Figure 2) φ fixes ts^{-1} since it is the only vertex whose valence in Y_e is 4. Its only neighbor with valence 5 (in Y_e) is tu and with valence 6 is t. But $\langle ts^{-1}, t, tu\rangle = G_1$, and so, as before Y is a GRR of G_1.

REMARKS 1. When $|G| \leqq 30$, the only cyclic extensions by C_3 which we have considered here are the direct products of C_3 with the various groups G in Class I. Some of the other cases are readily dispensed with. For example, if $|G| = 14$, 22 or 26, then G_1 is square-free and is in Class I by Lemma 4.

There is a semi-direct product G_1 of D_6 by C_3 given by

$$s^6 = t^2 = u^3 = e,\ tst = s^{-1},\ su = us,\ u^{-1}tu = ts^2\ .$$

If one lets $H_1 = \{s, s^{-1}, t, ts^{-1}, u, u^{-1}, tu^{-1}, ts^2u\}$, then the Cayley graph X_{G_1,H_1} is a GRR of G_1.

2. All cyclic extensions by C_3 of the two Class I groups G of order 16 with exponent 8 are in Class I. Indeed in the light of (a), we need consider only the semi-direct product. First G is generated by s and t where

$$s^8 = t^2 = e,\ tst = t^m,\ m \equiv 3 \quad \text{or} \quad 7 \pmod 8) .$$

Pick $c \in G_1 \backslash G$ such that $c^3 = e$. The subgroup generated by s is the only cyclic subgroup of order 8 and so is invariant. We have $c^{-1}sc = s^j$ where $o(s^j) = 8$. Thus $j^3 \equiv 1 \pmod 8$, and the only solution is $j \equiv 1 \pmod 8$. But $G_1 = G \times C_3$ is known to be in Class I by part (c).

3. Infinite graphs satisfy condition (b) in both Theorem 2 and Theorem 3. By virtue of Corollary 1A, these theorems and Corollary 2A hold for infinite graphs.

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