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DEFORMING P. L. HOMEOMORPHISMS ON A CONVEX POLYGONAL 2-DISK

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It is shown that for each *p. l.* homeomorphism f on a convex polygonal disk which is pointwise fixed on the boundary of the disk, there exists a triangulation K of the disk such that f may be obtained by successively moving the vertices of K (with the motion being extended linearly to each triangle of K) in a finite number of steps such that no triangles will be collapsed in the process of motion. An algebraic interpretation of this result is also given.

1. **Introduction.** By a *convex polygonal disk* in R^2, we mean a closed region D in R^2 enclosed by a convex polygon P. The vertices of P will also be called the vertices of the disk D. We shall always assume that no three consecutive vertices of D lie on a straight line. By a *triangulation* of D, we mean a (rectilinear) simplicial complex with D as its underlying space. If K is a triangulation of D, we shall let $L(K)$ denote the set of all homeomorphisms from D onto D which are linear on each simplex of K and are pointwise fixed on $\mathrm{Bd}(D)$. Elements of $L(K)$ will be called *linear homeomorphisms* of D with respect to K. We shall consider $L(K)$ as a topological space with the compact open topology. A map $f: D \to D$ will be called a *p. l. homeomorphism* of D if $f \in L(K)$ for some triangulation K of D. A triangulation K of D will be called a *proper triangulation* if the only 0-simplices of K lying on $\mathrm{Bd}(D)$ are those which are the vertices of D. Finally, a *p.l.* homeomorphism f of D is called *proper* if $f \in L(K)$ for some proper triangulation K of D. In the next section, we shall establish the following:

THEOREM A. *Let D be a convex polygonal disk in R^2. For any proper triangulation K of D, the space $L(K)$ is pathwise connected.*

Note that a path in the space $L(K)$ with the initial point f and terminal point g corresponds to a deformation from f to g through a family of homeomorphisms of D onto D which are linear with respect to K and are pointwise fixed on $\mathrm{Bd}(D)$. When D is a triangle in R^2, Theorem A is a consequence of S. S. Cairns' result on the deformation of isomorphically imbedded triangulations in the plane [2], [3] also [4, Proposition 2.19]. However, if D is not a triangle, Cairns' technique, unlike ours, will in general carry the deformation outside the disk D. Using Theorem A, we shall show in §3 that a general (i.e., not necessarily proper) $p.l.$ homeomorphism f of D may still be deformed into the identity map of D (or vice versa) in *some* space $L(K)$. Finally in §4, we shall improve our result of §3 to the following statement, which together with Theorem A form the main results of this paper.

THEOREM B. *For each $p.l.$ homeomorphism f of D, there exists a triangulation K of D such that f may be obtained in a finite number of steps by successively moving the vertices of K (with the motion being extended linearly to each simplex of K) such that none of the simplices is collapsed in the process.*

The problem of deforming a prescribed map of a space into the identity map, or vice versa, in a specific manner has been studied by many mathematicians. We shall mention some results which are more directly related to the problems considered in this paper. H. Tietze [9] and H. L. Smith [8] showed in 1914 and 1917 respectively that a homeomorphism of a 2-dimensional disk which leaves the boundary pointwise fixed is deformable into the identity map through a family of homeomorphisms which leave the boundary pointwise fixed. O. Veblen in 1917 [10] and J. W. Alexander in 1923 [1] extended the result for homeomorphisms of n-cells. The technique used by Alexander ("*The Alexander Trick*") can in fact be used to show that each $p.l.$ homeomorphism f on a polyhedral n-cell in R^n which leaves the boundary fixed can be deformed into the identity map through a family of such $p.l.$ homeomorphisms f_t. However, each of the homeomorphisms f_t in this process requires a different triangulation of the domain space. In fact, as t approaches 1, the triangulation for f_t requires triangles which are arbitrarily small. It is therefore natural to ask whether the given $p.l.$ homeomorphism on the polyhedral n-cell may be deformed into the identity map through a family of $p.l.$ homeomorphisms which are all linear with respect to a fixed triangulation of the n-cell. Our Theorem B clearly answers this question in affirmative for a convex 2-dimensional polyhedral disk. In fact, when the disk is a triangle, we have shown that an entire loop of $p.l.$ homeomorphisms of the disk can be deformed into the constant loop at the identity with respect to a fixed triangulation of the disk [5].

2. Deforming proper linear homeomorphisms. In this section, we shall first collect some of the basic properties of the spaces $L(K)$ and then carry out a proof of Theorem A. We shall use a process similar to that used by the author in [4] to resolve a special case which will appear in our proof of Theorem A. For this reason, we found it necessary to restate some of the definitions and theorems of [4] in the present framework. The proofs of the theorems will be omitted. In the following, if K is a triangulation of a convex polygonal disk D in R^2 and if v is a vertex of K, we shall let $\mathrm{St}(v, K)$ and $\mathrm{Lk}(v, K)$ to denote respectively the *star* and *link* of v with respect to K. A vertex v of K will be called an *inner vertex* of K if $v \notin \mathrm{Bd}(D)$. We shall start with the following observations.

REMARK 2.1. Let D be a convex polygonal disk in R^2 and K be any triangulation of D. For each $f \in L(K)$, the set $\{f(\sigma) \mid \sigma \in K\}$ also forms a triangualtion of D. This triangulation will be denoted by $f(K)$. Observe that for each $f \in L(K)$, the spaces $L(K)$ and $L(f(K))$ are homeomorphic; the map: $L(f(K)) \to L(K)$ carrying each $g \in L(f(K))$ onto gf is clearly a homeomorphism.

DEFINITION 2.2. A triangulation K of a polygonal disk D is said to be *decomposable* if there are vertices v_a, v_b, v_c of K such that the 1-simplices $\langle v_a, v_b\rangle$, $\langle v_b, v_c\rangle$, and $\langle v_c, v_a\rangle$ all belong to K but the 2-simplex $\langle v_a, v_b, v_c\rangle$ does not belong to K. If D is a triangle, we shall require in addition that at least one of the open simplices (v_a, v_b), (v_b, v_c), and (v_c, v_a) lies in the interior of D.

A triangulation K of D is called *indecomposable* if it is not decomposable.

The following two propositions may be proved essentially the same way as [4, Propositions 1.4 and 1.5].

PROPOSITION 2.3. *If K is a decomposable triangulation of a convex polygonal disk D in R^2, the space $L(K)$ is homeomorphic to a Cartesian product $L(K_1) \times L(K_2)$ where K_1 is a triangulation of D and K_2 is a triangulation of a triangle in R^2 such that each of K_1 and K_2 has a fewer number of vertices than K.*

PROPOSITION 2.4. *Let K be a proper, indecomposable triangulation of a convex polygonal disk D in R^2. For each inner vertex v_0 of K, there is an $f \in L(K)$ such that $St(f(v_0), f(K))$ is strictly convex (i.e., $St(f(v_0), f(k))$ is convex and no three consecutive vertices of $Lk(f(v_0), f(K))$ lie on a straight line).*

Let K be a triangulation of a convex polygonal disk D in R^2. To study the topological problems of the space $L(K)$, it turns out to be convenient to study some nice subspaces of it. We shall now describe these nice subspaces of $L(K)$.

DEFINITION 2.5. Let P be a polygonal circle in R^2 (i.e., P is a simplicial complex in R^2 with $|P|$ homeomorphic to the 1-sphere S^1). Let $[P]$ be the union of $|P|$ with the bounded component of $R^2 - |P|$. For each 1-simplex $\langle v_1, v_2\rangle$ of P, we shall let $H_P\langle v_1, v_2\rangle$ to denote the open half-plane of R^2 such that:

1. $\langle v_1, v_2\rangle$ lies on $\mathrm{Bd}\,(H_P\langle v_1, v_2\rangle)$.
2. $\bar{H}_P\langle v_1, v_2\rangle$ (the closed half-plane) contains a neighborhood of $\mathrm{Int}\,\langle v_1, v_2\rangle$ in $[P]$.

Now, for each polygonal circle P in R^2, we define the *core* of P, $\mathrm{cor}\,(P)$, to be the set

$$\mathrm{cor}\,(P) = \bigcap \{H_P\langle v_i, v_j\rangle \mid \langle v_i, v_j\rangle \text{ is a 1-simplex of } P\} .$$

REMARK 2.6. Let K be a triangulation of a convex polygonal disk D in R^2. Let v_0 be an inner vertex of K. Observe that for each $f \in L(K)$, the vertex $f(v_0)$ must lie in $\mathrm{cor}\,(Lk(f(v_0), f(K)))$. Conversely, for each point $x \in \mathrm{cor}\,(Lk(f(v_0), f(K)))$, there is a unique element $g \in L(K)$ such that $g(v) = f(v)$ for each vertex $v \neq v_0$ of K and $g(v_0) = x$.

DEFINITION 2.7. Let K be a triangulation of a convex polygonal disk D. For each inner vertex v_i of K, we shall let $L_{v_i}(K)$ or simply $L_i(K)$ denote the subspace of $L(K)$ consisting of all elements $f \in L(K)$ such that $f(v_i)$ is located at the centroid of the set $\mathrm{cor}\,(Lk\,(f(v_i), f(K)))$.

The space $L_i(K)$ is clearly of the same homotopy type as $L(K)$ for each inner vertex v_i of K (cf. [4, Proposition 2.6]). The reason for introducing these subspaces $L_i(K)$ is that under suitable conditions, they may be decomposed into a finite union of the spaces $L(K_i)$ where each K_i is a triangulation of the disk D with one fewer inner vertex than K. This makes it possible to carry out an induction argument on the number of vertices of K to prove the pathwise connectedness of $L(K)$. We shall now describe conditions which make such a decomposition of $L_i(K)$ possible.

DEFINITION 2.8. Let K be a triangulation of a convex polygonal disk D. For each vertex v_i of K, the incidence number of v_i in K, denoted by m_{v_i} or simply m_i, is defined as the number of vertices of K lying on $Lk(v_i, K)$ (hence, the number of 1-simplices of K incident on v_i).

PROPOSITION 2.9. *Let K be a triangulation of a convex polygonal disk D with n inner vertices. Let v_0 be an inner vertex of K such that*

1. *The incidence number* m_0 *of* v_0 *is* $\leqq 5$.
2. $St(v_0, K)$ *is strictly convex.*

Then the space $L_0(K)$ *may be written as the union of at most five subspaces* $\{L(K_i)\}$ *where each* K_i *is a triangulation of* D *with exactly* $n-1$ *inner vertices. Furthermore, if* K *is a proper triangulation of* D, *each of the* K_i*'s is also a proper triangulation.*

The proof of this proposition, though needing more work to establish, is still similar to that given for [4, Proposition 2.16], hence, will be omitted. Finally, we shall quote a combinatorial lemma [4, Proposition 2.17] which will be used to establish the existence of an inner vertex v_0 with the incidence number $\leqq 5$ when the triangulation is sufficiently nice.

PROPOSITION 2.10. *Let* K *be a triangulation of a polygonal disk* D *in* R^2. *Then*

$$\sum_{v_i=\text{inner vertex of } D} (6\text{-}m_i) = 6 + \sum_{v_i \in \mathrm{Bd}(D)} (m_i - 4) .$$

REMARK 2.11. Let K be a triangulation of a polygonal disk D in R^2 with at least one inner vertex. It follows immediately from Proposition 2.10 that if D is a triangle or if D is a general polygonal disk with $m_i \geqq 4$ for each vertex $v_i \in \mathrm{Bd}\,(D)$, then there is at least one inner vertex v_0 of K with $m_0 \leqq 5$.

We can now prove Theorem A *by induction on the number of vertices of the triangulation* K. The theorem is clearly true if the number of vertices of K is equal to three, since in that case, D must be a triangle and K has no inner vertex. Hence, $L(K)$ consists only of a single element. Assuming the theorem to be true for any convex polygonal disk in R^2 with a proper triangulation consisting of less than n vertices, we now consider a convex polygonal disk in D in R^2 and a proper triangulation K of D of n vertices. We assume that K has at least one inner vertex, for otherwise the theorem is trivially true. By Proposition 2.3, we may also assume that K is an indecomposable triangulation.

We now consider the following special case: *For each vertex* v_i *of* K *lying on* Bd (D), *the incidence number* $m_i \geqq 4$. The theorem may be proved very easily for this case as follows: By Remark 2.11, there is at least one inner vertex v_0 of K with $m_0 \leqq 5$. Using Proposition 2.4 and then Remark 2.1 if necessary, we may assume that St (v_0, K) is strictly convex. Applying Proposition 2.9, we see that the space $L_0(K)$, which is of the same homotopy type as $L(K)$, may be written as a finite union of the spaces $L(K_i)$, where each K_i is a triangulation of D with one fewer vertex than K. We may threfore apply the induction hypothesis to conclude that each $L(K_i)$ is pathwise connected. Since the identity map of the disk D clearly belongs to all the $L(K_i)$'s, these pathwise connected spaces $L(K_i)$ have a nonempty intersection. Therefore, $L(K)$ is pathwise connected. With the above special case being taken care of, we need only consider the following case in our inductive step:

Prove that space $L(K)$ is pathwise connected provided that K has a vertex $v_0 \in \mathrm{Bd}\,(D)$ with $m_0 \leqq 3$. Here, as before, K is a proper, indecomposable triangulation of a convex polygonal disk D in R^2, and has $n(>3)$ vertices.

For any vertex on $\mathrm{Bd}\,(D)$, the incidence number is clearly at least 2. We first show that the case for $m_0 = 2$ is trivial. Let v_1, v_2 be the two vertices adjacent to v_0 in either sense along $\mathrm{Bd}\,(D)$. The fact that $m_0 = 2$ implies that the triangle $\langle v_0, v_1, v_2\rangle$ is a 2-simplex of K. Since v_0, v_1, v_2 are all on $\mathrm{Bd}\,(D)$, each $f \in L(K)$ must be pointwise fixed on this triangle $\langle v_0, v_1, v_2\rangle$. Hence, the space $L(K)$ is homeomorphic to a space $L(K')$ where K' is the triangulation inherited from K on a smaller convex disk D' obtained from D by cutting off the triangle $\langle v_0, v_1, v_2\rangle$. Our induction hypothesis then guarantees the pathwise connectedness of $L(K')$, and hence, of $L(K)$.

Henceforth, we may assume that $m_0 = 3$. Again let v_1, v_2 be the two vertices adjacent to v_0 in either sense along $\mathrm{Bd}\,(D)$. Let v_3 be the third vertex such that $\langle v_0, v_3\rangle$ is a 1-simplex of K. Note that we may assume v_3 to be an inner vertex. For otherwise, the 1-simplex $\langle v_0, v_3\rangle$ would cut the disk into two convex disks D_1, D_2, hence, the space $L(K)$ would be homeomorphic to the Cartesian product $L(K_1) \times L(K_2)$ where K_i is the triangulation inherited from K on the disk $D_i (i = 1, 2)$. Pathwise connectedness of $L(K)$ would then be immediate from the induction hypothesis applied to $L(K_1)$ and $L(K_2)$.

Now by Proposition 2.4 and then by Remark 2.1 if necessary, we may assume that $\mathrm{St}\,(v_3, K)$ is strictly convex. Let an arbitrary $f \in L(K)$ be given. We wish to show that f may be connected to the identity map by a path in the space $L(K)$. We shall consider three cases.

Case 1. *The vertex $f(v_3)$ lies in the open triangular region $\langle v_0, v_1, v_2\rangle$.* In this case, we shall first consider a map g defined by $g(v) = v$ for each vertex $v \neq v_3$ of K and $g(v_3) = f(v_3)$. Since $\mathrm{St}\,(v_3, K)$ is convex and $f(v_3)$ lies in $\mathrm{St}\,(v_3, K)$, g is a well-defined element in $L(K)$. Note that g agrees with the identity map outside the region $\mathrm{St}\,(v_3, K)$ in D and g agrees with f on the triangles $\langle v_0, v_1, v_3\rangle$ and $\langle v_0, v_2, v_3\rangle$. Also note that the identity map may be connected to g by a straight line homotopy which moves the vertex v_3 to the point $f(v_3)$ along a straight line and keeps all the other vertices of K fixed. Let γ be the path in $L(K)$ corresponding to this homotopy. γ is a path from the identity map to g. Observe that $f \circ g^{-1}$ is an element in the space $L(g(K))$ which is pointwise fixed on the triangles $\langle g(v_0), g(v_1), g(v_3)\rangle$ and $\langle g(v_0), g(v_2), g(v_3)\rangle$. We may therefore view $f \circ g^{-1}$ as a map on the smaller disk D' obtained from D by cutting off the triangles $\langle g(v_0), g(v_1), g(v_3)\rangle$ and $\langle g(v_0), g(v_2), g(v_3)\rangle$. The map $f \circ g^{-1}$ is clearly linear with respect to the triangulation K' on D' inherited from the triangulation $g(K)$ of D. The fact that $g(v_3) = f(v_3)$ lies in the interior of the triangular region $\langle v_0, v_1, v_2\rangle$ implies that the disk D' is still strictly convex. Since the triangulation K' has a fewer number of vertices than K (v_0 belongs to K but not K'), we may apply the induction hypothesis on the space $L(K')$ to get a path τ' from the identity map of D' to the map $f \circ g^{-1} | D'$. This path τ' then gives rise to a path τ in the space $L(g(K))$ from the identity map to the map $f \circ g^{-1}$. Note that $\tau \circ g$ is then a path in the space $L(K)$ from the map g to the map f. We may then connect the identity map to the map f in the space $L(K)$ by the path γ followed by $\tau \circ g$.

Case 2. *The vertex $f(v_3)$ lies on the line segment $\langle v_1, v_2\rangle$.* In §4 (Corollary 4.3), we shall prove that the vertex $f(v_3)$ may be moved slightly off the line segment $\langle v_1, v_2\rangle$ to produce a path γ_1 in $L(K)$ connecting the element f with an element $f' \in L(K)$ where $f'(v) = f(v)$ for each $v \neq v_3$ in K and $f'(v_3)$ lies inside the triangular region $\langle v_0, v_1, v_2\rangle$. Then by Case 1, f' may be connected to the identity element of $L(K)$ by a path γ_2. Then γ_1 followed by γ_2 will connect f to the identity element.

Case 3. *The vertex $f(v_3)$ lies outside the closed triangular region* $\langle v_0, v_1, v_2\rangle$. By assumption, $\mathrm{St}\,(v_3, K)$ is a convex open set in the plane. It therefore contains a neighborhood of the open line segment (v_1, v_2), and hence, contains points outside the triangle $\langle v_0, v_1, v_2\rangle$. Moving the vertex v_3 to any such point if necessary, we may assume that the vertex v_3 is outside the triangular region $\langle v_0, v_1, v_2\rangle$.

Now, let M denote the set of all maps in $L(K)$ which carry v_3 outside the triangular region $\langle v_0, v_1, v_2\rangle$. By the above assumptions, both f and the identity element of $L(K)$ are in M. Therefore, it suffices to prove that M is pathwise connected. We shall do this by showing that M is the homeomorphic image of a map j from some pathwise connected space into the space $L(K)$. Such a pathwise connected space and map j are defined as follows: Let D' be the polygonal disk obtained from D by cutting off the triangle $\langle v_0, v_1, v_2\rangle$. Note that D' is still a convex polygonal disk. We now describe a triangulation K' of D'. Outside the triangular region $\langle v_1, v_2, v_3\rangle$, we let K' be the same as the triangulation K. While on that triangular region, we let $\langle v_1, v_2, v_3\rangle$ be a 2-simplex of K'. Note that this determines a proper triangulation K' of D' which has one fewer vertex K (for v_0 belongs to K but not to K'). By the induction hypothesis, $L(K')$ is pathwise connected. We now describe a map $j: L(K') \to L(K)$. Consider D' as a subset of D. Since the disk D is convex, for each element $g \in L(K')$, observe that the open line segment $(g(v_0), g(v_3))$ lies completely in the quadrilateral $[v_0, v_1, g(v_3), v_2]$. We may therefore define an element $f \in L(K)$ by $f(v) = g(v)$ for all vertices $v \in K'$ and $f(v_0) = v_0$ (i.e., f, as a map on the whole disk D, is obtained from g by deleting the 1-simplex $\langle v_1, v_2\rangle$ from both K' and the image $g(K')$, and then inserting the 1-simplex $\langle v_0, v_3\rangle$ to the domain space and the 1-simplex $\langle v_0, g(v_3)\rangle$ to the image). The map f is a well-defined element in $L(K)$. We let $f = j(g)$. Since for each vertex v of K' and for any two elements $g_1, g_2 \in L(K')$, $\mathrm{dist}\,(g_1(v), g_2(v)) = \mathrm{dist}\,(j(g_1)(v), j(g_2)(v))$, the map j is continuous (in fact, it is an isometry). Also note that for each $g \in L(K')$, the vertex $j(g)(v_3)$ is outside the triangular region $\langle v_0, v_1, v_2\rangle$. Hence, $j(g)$ belongs to the subset M. Conversely, we may easily show that each element $f \in M$ is the image $j(g)$ of some $g \in L(K')$. This shows that M is the homeomorphic image of the pathwise connected space $L(K')$ under the map j. M must be pathwise connected. This finishes the proof of Theorem A.

3. Deforming arbitrary p.l. homeomorphisms. In this section, we shall show that each arbitrary *p.l.* homeomorphism on a convex polygonal disk D is deformable into the identity map in the space $L(K)$ for some triangulation K of D. We do this by first showing that each *p.l.* homeomorphism on D is deformable into a proper one. The deformation process may then be completed by using Theorem A. We start with a trivial observation which will be needed in the deformation process.

LEMMA 3.1. *Let D be a convex polygonal disk in R^2 and K be an arbitrary triangulation of D. For each subdivision K' of K, the space $L(K)$ may be considered in a natural way as a subspace of the space $L(K')$.*

Proof. Observe that each element f in $L(K)$ is also linear with respect to the triangulation K', and hence, may be considered as an element of $L(K')$.

PROPOSITION 3.2. *Let K be an arbitrary triangulation of a convex polygonal disk D in R^2. For each element $f \in L(K)$, there exist triangulations K_1, K_2 of D such that*

1. *K_1 is a proper triangulation of D. K_2 is a common subdivision of K and K_1. (Hence, $L(K_1)$ and $L(K)$ may both be considered as subspaces of $L(K_2)$.)*

2. *The element f may be deformed in $L(K_2)$ into an element f' contained in $L(K_1)$.*

Outline of the proof. When D is a triangle, this theorem is proved in detail in [5, Theorem 1.3]. In fact, we showed there how to deform, instead of a single element f in $L(K)$, a compact set of elements in $L(K)$. The same proof, with obvious modifications, also works here. We shall therefore only give a brief outline of the proof.

For a sufficiently small positive number δ, we let D_δ be the disk lying inside D and concentric to D (for some point in $\operatorname{Int}(D)$) such that the perpendicular distances between the corresponding parallel sides of D and D_δ are δ. We also let B_δ be the annular strip $\operatorname{Cl}(D - D_\delta)$ where $\operatorname{Cl}(X)$ means the closure of X with respect to D. Each D_δ gives rise to a decomposition of D into a rectilinear cell complex (in the sense of [7, p. 74] or [6, p. 5]) which is obtained by letting the disk D_δ be a 2-cell and by cutting the region B_δ into 2-cells by connecting each vertex of D to the closest vertex of D_δ by a 1-cell. We denote this cell complex by $R(\delta)$.

Let K be an arbitrary triangulation of the disk D and consider an arbitrary element $f \in L(K)$. If K is proper, we simply let K_1 be K and f' be f and there is nothing to prove. Hence, we may assume that K is not proper. We first fix a number $\alpha > 0$ such that

1. All the inner vertices of K are contained in $\operatorname{Int}(D_\alpha)$.

2. If q is an inner 1-simplex of K with both vertices on $\operatorname{Bd}(D)$, then $q \cap \operatorname{Int}(D_\alpha) \neq \varnothing$.

We then fix a sufficiently small number $\delta (0 < \delta < \alpha)$ and let $J(\delta)$ be the rectilinear cell complex obtained by imposing the two cell complexes $R(\alpha)$ and $R(\delta)$ on K (i.e., each cell in $J(\delta)$ is an intersection of cells of $R(\alpha)$, $R(\delta)$, and K). Finally, we let $K(\delta)$ be a simplicial subdivision of $J(\delta)$ without adding any more vertices (i.e., we get $K(\delta)$ by adding a number of diagonals to each 2-cell of $J(\delta)$ which is not a triangle). $K(\delta)$ is then a simplicial subdivision of K, hence, $f \in L(K(\delta))$.

The idea of the proof is this. We first deform the element f in the space $L(K(\delta))$ into an element f' such that $f' | B_\delta$ is the identity map of B_δ. We then observe that any such f' may be considered as a linear homeomorphism with respect to a *proper* triangulation K_1 of D. If we let K_2 be a common simplicial subdivision of K_1 and $K(\delta)$, the triangulations K_1 and K_2 will clearly satisfy all the conditions of the proposition.

A deformation F carrying f into an f' described above can be defined as follows: For each vertex v of $K(\delta)$ which does not lie on the polygon $\mathrm{Bd}(D_\delta)$, we let $F(v, t) = f(v)$ for all t (i.e., we fix the image of f on the set $f(D_\alpha)$ and on $\mathrm{Bd}(D)$). For all the vertices v's of $K(\delta)$ which lie on $\mathrm{Bd}(D_\delta)$, we pull all the $f(v)$'s simultaneously back to the v's. It can be shown that if δ is sufficiently small (e.g., δ satisfies the conditions given in [5, Definition 3.5]) this will give us a deformation in the space $K(\delta)$. The resulting map f' then agrees with f on the set $f(D_\alpha)$ and $f'|B_\delta =$ identity map. The reader is referred to [5] for details.

We then observe that f' may be considered as an element in $L(K_1)$ for some proper triangulation K_1 of D. We now describe such a proper triangulation. We let K_1 be identical with $K(\delta)$ on the closed region D_δ of D. While on the annular strip B_δ, we firrst cut the strip into a number of 2-cells by the cell decomposition $R(\delta)$ on this strip (each of these 2-cells will then be a quadrilateral). We then triangulate the strip by cutting each of these 2-cells into triangles in the following manner: Pick a vertex v of the 2-cell which is also a vertex of D. Note that exactly one of the sides of this 2-cell lies in the polygon $\mathrm{Bd}(D_\delta)$. Let $v_1, v_2, \cdots, v_k$ be vertices of $K(\delta)$ lying on this side of the 2-cell. We shall then cut the 2-cell into triangles by inserting a diagonal from v to each vertex v_i for $i = 1, 2, \cdots, k$. This gives us a desired proper triangulation K_1.

We contend that the map f' belongs to K_1. Observe that f' is linear on each simplex σ of K_1 lying in D_δ, for σ is also a simplex of $K(\delta)$. f' is also linear on each simplex δ of K_1 lying on B_δ, for f' is the identity map there. Hence $f' \in L(K_1)$. The proposition then follows by letting K_2 be a common simplicial subdivision of K_1 and $K(\delta)$.

THEOREM 3.3. *Let K be an arbitrary triangulation of a convex polygonal disk D in R^2. For each linear homeomorphism $f \in L(K)$, there exists a simplicial subdivision K' of K and path $\gamma: I \to L(K')$ such that $\gamma(0) = f$ and $\gamma(1) =$ the identity map of D.*

Proof. Consider any triangulation K and any $f \in L(K)$. By Proposition 3.2, we may find a proper triangulation K_1 of D and a triangulation K_2 which is a common subdivision of K and K_1 such that there is a path γ_1 in $L(K_2)$ connecting f to some element f' of $L(K_1)$. By Theorem A, we may also get a path γ_2 in the space $L(K_1)$ from the element f' to the identity map of D. Since K_2 is a subdivision of K_1, the path γ_2 is also a path in the space $L(K_2)$. We may therefore set $K' = K_2$. Using the path γ_1 followed by γ_2, we get a path in the space $L(K')$ connecting f to the identity map of D.

COROLLARY 3.4. *Each p.l. homeomorphism of a convex polygonal disk D in R^2 is deformable to the identity map of D in the space $L(K)$ for some triangulation K of D.*

4. Decomposing a deformation into a finite sequence of single moves. In this section, we shall prove our Theorem B of §1. Let D be a convex polygonal disk D in R^2. We first make some observations on the topology of the space $L(K)$ for an arbitrary triangulation K of D.

REMARK 4.1. For each triangulation K of D, observe that the space $L(K)$ is metrizable, say with a metric $\gamma(f, g) = \max_{v \in S} \{d(f(v), g(v))\}$, where S is the set of inner vertices of K and d is the sup-metric of the Euclidean plane i.e., $d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} |x_i - y_i|$.

PROPOSITION 4.2. *Let K be any triangulation of D with k inner vertices. The space $L(K)$ may be identified with an open subset of $R^2 \times R^2 \times \cdots \times R^2$ (k copies).*

Proof. In the following, all the maps from D into R^2 are assumed to be pointwise fixed on Bd(D). By a linear map $f: K \to R^2$, we mean a map $f: D \to R^2$ (not necessarily a homeomorphism into) which is linear with respect to each simplex of K. First observe that each linear map from K into R^2 is completely determined by its images of the inner vertices of K. Suppose that an ordering is assigned to the inner vertices, say $v_1, v_2, \cdots, v_k$ of K. The correspondence between a linear map $f: K \to D$ and the point $(f(v_1), f(v_2), \cdots, f(v_k)) \in R^2 \times R^2 \times \cdots \times R^2$ (k copies) gives rise to a one-to-one correspondence between the set of all such linear maps and R^{2k}. Under this correspondence, the space $L(K)$ is identified with a subset of R^{2k}. This identification is indeed a homeomorphism of $L(K)$ into R^{2k}. In fact, it is an isometry into R^{2k} if $L(K)$ is given the metric γ and R^{2k} is given the sup-metric of the Euclidean space.

To see that the image of $L(K)$ in R^{2k} is an open subset, we need only show that any linear map g from K to R^2 belongs to $L(K)$ if it is sufficiently close to some element of $L(K)$ under the metric γ. This can be done by means of a result of J. H. C. Whitehead, that a sufficiently close C^1 approximation to an immersion from a complex into a manifold is itself an immersion [11] (also see [7, Theorem 8.8]). However, we shall sketch a more elementary argument here. Observe that a linear map g from K into R^2 belongs to $L(K)$ if and only if the image under g of each 2-simplex has positive area (i.e., nonzero, nonnegative area) (cf. 5, Lemma 2.1]). If g has this property, any linear map: $K \to K^2$ sufficiently close to g must also have this property.

This shows that $L(K)$ must correspond to an open subset of R^{2k}. This finishes the proof.

The following corollary was needed in §1. It follows immediately from the Proposition 4.2 and the fact that Euclidean spaces are locally pathwise connected.

COROLLARY 4.3. *For each triangulation K of D, the space $L(K)$ is locally pathwise connected. In fact, for each $f \in L(K)$, there exists a number $\delta > 0$ such that for each linear map $g: K \rightarrow R^2$ with $g|\mathrm{Bd}(D) =$ identity and $d(g(v), f(v)) < \delta$ for all inner vertices v of K, the map g belongs to $L(K)$ and may be connected to f by a path in $L(K)$.*

Proof of Theorem B. Consider any *p.l.* homeomorphism f of D. By Corollary 3.4, there exists a triangulation K of D and a path $\gamma: I \rightarrow L(K)$ such that γ connects f to the identity map of D. Let k be the number of inner vertices of K. Using Proposition 4.2, we may find, for each point $x \in \gamma(I)$ an open rectangular box $U_x = V_1 \times V_2 \times \cdots \times V_k$ of R^{2k} containing x such that $U_x \subset L(K)$ and each V_i is an open rectangle in R^2. Let ε be the Lebesegue number of the open covering $\{\gamma^{-1}(U_x) | x \in \gamma(I)\}$ of the unit interval I and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of the unit interval such that the length of each closed interval $[t_{i-1}, t_i]$ is less than ε.

We now show that for each $i = 1, \cdots, n$, the element $\gamma(t_{i-1})$ may be deformed into the element $\gamma(t_i)$ in the space $L(K)$ in a finite number of steps by succesively moving the vertices of $\gamma(t_{i-1})$. Consider any particular $i(1 \leqq i \leqq n)$. Let $\gamma(t_{i-1}) = (a_1, a_2, \cdots, a_k) \in R^{2k}$ and $\gamma(t_i) = (b_1, b_2, \cdots, b_k) \in R^{2k}$. We may choose an open rectangular box $U = V_1 \times V_2 \times \cdots \times V_k$ in $L(K)(\subset R^{2k})$ containing both $\gamma(t_{i-1})$ and $\gamma(t_i)$. Since the whole rectangular box U is contained in $L(K)$, we may move $\gamma(t_{i-1})$ to $\gamma(t_i)$ (considered as points of R^{2k}) within the set $L(K)$ by a sequence of k moves such that for each $j = 1, 2, \cdots, k$, the jth move carries the point $(b_1, b_2, \cdots, b_{j-1}, a_j, \cdots, a_k)$ to the point $(b_1, \cdots, b_j, a_{j+1}, \cdots, a_k)$. Under the identification of $L(K)$ as an open subset of R^{2k}, each of these k-moves clearly corresponds to a deformation from an element g to some other element in $L(K)$ by moving a single vertex of $g(K)$. Hence, $\gamma(t_i)$ may be reached from $\gamma(t_{i-1})$ by finitely many such single moves. This finishes the proof of Theorem B.

Theorem B also allows an algebraic interpretation as follows: Let D be a convex polygonal disk in R^2 and PL(D) be the set of all *p.l.* homeomorphisms of D onto D. Observe that PL(D) forms a group with respect to the composition of functions. For each triangulation K of D, we let $S(K) = \{f \in L(K) |$ there is a vertex

$v \in K$ such that $f|(D\text{-St}(v, K)) = \text{identity}\}$. It can be shown with the help of Remark 2.6 that for each triangulation K of D, the set $S(K)$ consists of all elements of $L(K)$ which may be obtained by moving a single vertex of K. Now let $S(D) = \bigcup \{S(K) | K$ a triangulation of $D\}$. The set $S(D)$ may be called the set of single moves. Theorem B says that for each *p.l.* homeomorphism f of D, there exists a triangulation K of D and a finite sequence of elements $f_1, f_2, \cdots, f_m \in S(D)$ such that

1. $f_1 \in S(K)$ and $f_{i+1} \in S(f_i(\cdots(f_1(K))\cdots))$ for each $i = 1, \cdots, m-1$.
2. $f = f_m \circ f_{m-1} \circ \cdots \circ f_1$.

In particular, each element of PL (D) is a finite product of elements of $S(D)$. Hence, we have the following.

THEOREM 4.4. *For any convex polygonal disk D in R^2, the group* PL (D) *is generated by the subset $S(D)$ of single moves.*

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