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DUGUNDJI EXTENSION THEOREMS FOR LINEARLY ORDERED SPACES

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In this paper we prove a Dugundji Extension Theorem for a large class of monotonically normal spaces, the generalized ordered spaces. We show that if A is a closed subset of a generalized ordered space X and if $C^*(A)$ and $C^*(X)$ denote the vector spaces of continuous, bounded real-valued functions on A and X respectively, then there is a linear transformation $u: C^*(A) \to C^*(X)$ such that for each $g \in C^*(A)$, $u(g)$ extends g and the range of $u(g)$ is contained in the closed convex hull of the range of g. Furthermore, we give an example which shows that such linear transformations from $C(A)$, the vector space of all continuous, real-valued functions on A, to $C(X)$ cannot always be found, even when A is a closed, separable metrizable subspace of a hereditarily paracompact linearly ordered space.

1. Introduction. For any space S, let $C(S)$ be the vector space of all continuous, real-valued functions on S and let $C^*(S)$ be the space of all bounded members of $C(S)$. In [4], Dugundji proved that if A is a closed subset of a metrizable space X, then there is a linear transformation $u: C(A) \to C(X)$ such that for each $g \in C(A)$:

(a) $u(g)$ is an extension of the function g; and

(b) the range of $u(g)$ is a subset of the closed convex hull of the range of g.

We shall call u a *simultaneous extender from* $C(A)$ *to* $C(X)$; the notion of a *simultaneous extender from* $C^*(A)$ *to* $C^*(X)$ is analogously defined[1].

Subsequent generalizations of Dugundji's theorem have relaxed the requirement that X be metrizable and have considered functions having values in a locally convex topological vector space [1, 2, 3, 10, 12, 13]. The largest class of spaces for which a Dugundji Extension Theorem has been proved is the class of stratifiable spaces [3] (which includes all metric spaces). Recently, a theorem reminiscent of Dugundji's result has been obtained for the still larger class of monotonically normal spaces [6]. In this paper we present a Dugundji Extension Theorem for a large and important subclass of the monotonically normal spaces, vis., the generalized ordered spaces. Our theorem,

[1] It is interesting to note that if A is a closed subset of a normal space X, then there will always exist linear transformations $u: C(A) \to C(X)$ such that $u(g)$ extends g for each $g \in C(A)$: this may be deduced from the Tietze extension theorem by considering a Hamel base for the real vector space $C(A)$.

unlike some of the ones mentioned above [1, 2, 3, 10, 12], deals with simultaneous extenders from $C^*(A)$ to $C^*(X)$ and an example in §3 shows that it is not always possible to obtain simultaneous extenders from $C(A)$ to $C(X)$, even when X is hereditarily paracompact and such extenders from $C^*(A)$ to $C^*(X)$ are known to exist.

A *linearly ordered topological space* is a linearly ordered set endowed with the usual open-interval topology of the linear order. A *generalized ordered space* is a linearly ordered set X endowed with a topology having a base $\mathscr{B}$ every member of which is an interval (open, closed, half-open and degenerate intervals are allowed) and which contains all open intervals of X. The generalized ordered spaces may also be characterized as those spaces which can be embedded in linearly ordered spaces; spaces of this type were studied in [8] and it is known that any generalized ordered space is monotonically normal [6]. Perhaps the most familiar pathological generalized ordered spaces are the Sorgenfrey line (3.1) and the Michael line (3.3).

2. The extension theorem for bounded functions.

The key to our extension theorem is the notion of a Banach limit. Let us begin by recalling some definitions.

DEFINITION 2.1. Let $(D, \leqq)$ be a directed set [7] and let f be a bounded, real-valued function on D. Then

$$\operatorname{limsup}_D(f) = \inf\{\sup\{f(x): x \geqq y\}: y \in D\} \quad \text{and}$$
$$\operatorname{liminf}_D(f) = \sup\{\inf\{f(x): x \geqq y\}: y \in D\}\,.$$

If $\operatorname{limsup}_D(f) = \operatorname{liminf}_D(f)$, then we define $\lim_D(f) = \operatorname{limsup}_D(f)$.

LEMMA 2.2. *Let $(D, \leqq)$ be a directed set and let V be the vector space of all bounded, real-valued functions on the set D. Then there is a linear transformation $L: V \to R$, the space of real numbers, such that*

(a) *for any $f \in V$,* $\operatorname{liminf}_D(f) \leqq L(f) \leqq \operatorname{limsup}_D(f)$;

(b) *if* $\lim_D(f)$ *exists, then* $L(f) = \lim_D(f)$.

Proof. See [15; p. 104].

REMARK 2.3. In the classical case of this result, D is the set of positive integers and the functional L is called a Banach limit. We shall call the functional L in 2.2 a *Banach limit over D*. Observe that if the directed set $(D, \leqq)$ has a last element d', then $L(f) = f(d')$ for each f in V. In our applications, the set D will carry a linear ordering.

THEOREM 2.4. *Let A be a closed subset of a generalized ordered space X. Then there is a simultaneous extender $u: C^*(A) \to C^*(X)$.*

Proof. Let $\mathscr{I} = \{I_\gamma: \gamma \in \Gamma\}$ be the collection of all convex components of set $X \setminus A$.[2] For each $\gamma \in \Gamma$ let $A_\gamma^+ = \{a \in A: a \geqq x$ for each $x \in I_\gamma\}$ and let $A_\gamma^- = \{a \in A: a \leqq x$ for each $x \in I_\gamma\}$. Using the ordering inherited from X, A_γ^- is a directed set; using the *reverse* of the ordering inherited from X, A_γ^+ is also a directed set. Provided $A_\gamma^+ \neq \varnothing$, let L_γ^+ be a Banach limit over A_γ^+; provided $A_\gamma^- \neq \varnothing$, let L_γ^- be a Banach limit over A_γ^-. For each $\gamma \in \Gamma$, let $\psi_\gamma: X \to [0, 1]$ be a continuous function such that

(i) $\{x \in X: x < y$ for each $y \in I_\gamma\} \subset \psi_\gamma^{-1}(1)$

(ii) $\{x \in X: x > y$ for each $y \in I_\gamma\} \subset \psi_\gamma^{-1}(0)$

(iii) if I_γ has at least two points, then there are points r_γ and s_γ of I_γ having $r_\gamma < s_\gamma$ and

$$(\longleftarrow, r_\gamma] \subset \psi_\gamma^{-1}(1) \quad \text{and} \quad [s_\gamma, \longrightarrow) \subset \psi_\gamma^{-1}(0) .$$

For each $f \in C^*(A)$, define $\hat{f}$ on X by

$$\hat{f}(x) = \begin{cases} f(x) \text{ if } x \in A; \\ \psi_\gamma(x)L_\gamma^-(f) + (1 - \psi_\gamma(x))L_\gamma^+(f) \text{ if } x \in I_\gamma \text{ and } A_\gamma^+ \neq \varnothing \neq A_\gamma^- ; \\ L_\gamma^+(f) \text{ if } x \in I_\gamma \text{ and } A_\gamma^- = \varnothing ; \\ L_\gamma^-(f) \text{ if } x \in I_\gamma \text{ and } A_\gamma^+ = \varnothing . \end{cases}$$

Then $\hat{f}: X \to R$ is continuous and $\hat{f}(X) \subset \mathrm{cl}_R \{r \in R$: for some $a, b \in A$, $f(a) \leqq r \leqq f(b)\}$, the closed convex hull of the set $f(A)$. Hence $\hat{f} \in C^*(X)$. Therefore, if we define $u(f) = \hat{f}$, we obtain the required simultaneous extender.

REMARKS 2.5. It is clear that the simultaneous extender found in 2.4 preserves constant functions and that if $C^*(A)$ and $C^*(X)$ are each equipped with the sup-norm, then the simultaneous extender is a linear operator of norm 1.

3. Extending unbounded functions.

We begin this section with examples showing that simultaneous extension of all continuous real-valued functions is sometimes possible in spaces which are not stratifiable.

EXAMPLE 3.1. Let A be any closed subset of the Sorgenfrey line

[2] A subset S of a linearly ordered set X is *convex* provided $[a, b] \subset S$ whenever a and b are points of S having $a \leqq b$. A subset S of a set T in X is called a *convex component of* T provided S is a convex subset of X, $S \subset T$ and no strictly larger convex subset of X is a subset of T. Clearly any subset of X can be uniquely expressed as a union of its convex components.

X [14], i.e., the set of real numbers topologized in such a way that for each real number x, the collection $\{[x, x + t) : t > 0\}$ is an open neighborhood base at x. Then there is a simultaneous extender $u: C(A) \to C(X)$. For consider $\mathscr{I} = \{I_\gamma : \gamma \in \Gamma\}$, the family of all convex components of $X \backslash A$,[2] and choose $a_0 \in A$. Each I_γ will have one of the five forms described in the following definition: for any $g \in C(A)$, define $\hat{g}: X \to R$ by

$$\hat{g}(x) = \begin{cases} g(x) & \text{if } x \in A; \\ g(a_\gamma) & \text{if } x \in I_\gamma = (a_\gamma, b_\gamma) \text{ or if } x \in I_\gamma = (a_\gamma, +\infty); \\ g(b_\gamma) & \text{if } x \in I_\gamma = [a_\gamma, b_\gamma) \text{ or if } x \in I_\gamma = (-\infty, b_\gamma); \\ g(a_0) & \text{if } x \in I_\gamma = [a_\gamma, +\infty) \,. \end{cases}$$

Defining $u(g) = \hat{g}$, we obtain the required simultaneous extender from $C(A)$ to $C(X)$.

Because the Sorgenfrey line is not metrizable, it is not stratifiable [8; Theorem 5.3].

Our second example shows that a space may satisfy a Dugundji Extension Theorem without being paracompact or perfectly normal.

EXAMPLE 3.2. Let X be the usual space of countable ordinals [7]. For any closed subset $A \subset X$ there exist simultaneous extenders from $C(A)$ to $C(X)$. Theorem 2.4 provides a proof of this fact since $C^*(A) = C(A)$ and $C^*(X) = C(X)$.

Our final example shows that simultaneous extenders from $C(A)$ to $C(Y)$ may fail to exist even when A is a closed, separable metrizable subspace of a hereditarily paracompact linearly ordered space Y (whence simultaneous extenders from $C^*(A)$ to $C^*(X)$ do exist). In our example, P, Q, and R will denote the sets of irrational, rational and real numbers, respectively.

EXAMPLE 3.3. Let X be the Michael line [11], i.e., the set of real numbers endowed with the topology $\mathscr{M} = \{U \cup V : U \text{ is open in the usual topology of } R \text{ and } V \subset P\}$. Let $A = Q$. Then there is no simultaneous extender $u: C(A) \to C(X)$.

We argue indirectly. Enumerate A as $A = \{r_n : n \geqq 1\}$ and suppose there were a simultaneous extender $u: C(A) \to C(X)$. For any $g \in C(A)$, denote $u(g)$ by $\hat{g}$.

Let $a_0 = -\pi$ and $b_0 = \pi$. Define $f_1 \in C(A)$ by

$$f_1(x) = \begin{cases} 1 & \text{if } x \quad A \backslash [a_0, b_0] \\ 0 & \text{if } x \in A \cap [a_0, b_0] \,. \end{cases}$$

Then there are irrational numbers $a_1 < b_1$ such that $[a_1, b_1] \subset (a_0, b_0)\backslash\{r_1\}$ and such that $\hat{f}_1([a_1, b_1]) \subset [0, 1/4]$ because the set $\{x \in [a_0, b_0]: \hat{f}_1(x) \geqq 1/4\}$ is nowhere dense in $[a_0, b_0]$ with its usual topology. Inductively construct irrational numbers $a_n < b_n$ and functions $f_n \in C(A)$ satisfying:

(1) $[a_n, b_n] \subset (a_{n-1}, b_{n-1})\backslash\{r_n\}$;

(2) for $n \geqq 2$,

$$f_n(x) = \begin{cases} 1/2^{n-1} & \text{if} \quad x \in A \cap ([a_{n-2}, b_{n-2}]\backslash[a_{n-1}, b_{n-1}]) \\ 0 & \text{if} \quad x \in A\backslash([a_{n-2}, b_{n-2}]\backslash[a_{n-1}, b_{n-1}]) \end{cases}$$

(3) $(\sum_{i=1}^{n} \hat{f}_i)([a_n, b_n]) \subset [0, 1/2^{n+1}]$.

It follows from (1) that $\bigcap \{[a_n, b_n]: n \geqq 1\}$ consists of a single irrational number c. Let $g = \sum \{f_n: n \geqq 1\}$. Then $g \in C^*(A)$ because the series $\sum \{f_n: n \geqq 1\}$ converges uniformly. Furthermore, because the range of $g - \sum \{f_i: 1 \leqq i \leqq n\}$ is a subset of $[0, 1/2^n]$, so is the range of $u(g - \sum \{f_i: 1 \leqq i \leqq n\}) = \hat{g} - \sum \{\hat{f}_i: 1 \leqq i \leqq n\}$; hence $\hat{g} = \sum \{\hat{f}_n: n \geqq 1\}$ so that $\hat{g}(c) = 0$.

Define $h \in C(A)$ by $h(x) = 1 + 1/(|x - c|)$ and let k be the number $\hat{h}(c)$. There is an $\varepsilon > 0$ such that $h(x) > k + 1$ whenever $x \in A \cap [c - \varepsilon, c + \varepsilon]$. Let m be the minimum value of $g(x)$ on the set $A\backslash[c - \varepsilon, c + \varepsilon]$. Then $m > 0$ so that, for some positive integer N, $Nm > k + 1$. Therefore, the range of the function $h + Ng$ is a subset of $[k + 1, +\infty)$; hence the same is true of the function $\hat{h} + N\hat{g}$. But that is impossible since $(\hat{h} + N\hat{g})(c) = k$.

To obtain the hereditarily paracompact linearly ordered space in which simultaneous extenders for unbounded functions cannot be found, observe that there is a linearly ordered space Y which contains the Michael line X as a closed subspace and which is also hereditarily paracompact (in the notation of [8], $Y = X^*$).

REMARK 3.4. Example 3.3 shows that the assertion on page 806 of [10] that simultaneous extenders from $C(A)$ to $C(X)$ can be found provided A is a closed metrizable subspace of a paracompact space X is erroneous. The correct statement is that if A is a closed, metrizable, G_δ-subspace of the paracompact space X then simultaneous extenders from $C(A)$ to $C(X)$ exist [9].

Our final theorem contrasts with the situation in 3.3 and illustrates the special role of perfect normality in generalized ordered spaces; see also [8; Theorem 4.8].

THEOREM 3.5. *Suppose that A is a closed subset of a perfectly normal generalized ordered space X. If A is σ-compact, then there is a simultaneous extender from $C(A)$ to $C(X)$.*

Our proof requires two lemmas.

LEMMA 3.6. *Let A be a closed, σ-compact subset of a generalized ordered space X. Suppose that $X\backslash A$ is an F_σ-subset of X and that each convex component of $X\backslash A$ is closed in X. Then there is a simultaneous extender from $C(A)$ to $C(X)$.*

Proof. Let $\{I_\gamma: \gamma \in \Gamma\}$ be the family of all convex components of $X\backslash A$. Write $X\backslash A = \bigcup \{F_n: n \geqq 1\}$ where each F_n is closed in X and let $\Gamma'_n = \{\gamma \in \Gamma: I_\gamma \cap F_n \neq \emptyset\}$. Let $\Gamma_1 = \Gamma'_1$ and for $n \geqq 1$ let $\Gamma_{n+1} = \Gamma'_{n+1} \backslash \bigcup \{\Gamma_i: 1 \leqq i \leqq n\}$. Then each collection $\{I_\gamma: \gamma \in \Gamma_n\}$ is a discrete collection of closed and open subsets of X.

Write $A = \bigcup \{A_n: n \geqq 1\}$ where $A_1 \subset A_2 \subset \cdots$ are compact sets. For each $n \geqq 1$ let $u_n: C(A_n) \to C(X)$ be the simultaneous extender constructed in 2.4. For each $g \in C(A)$ define $\hat{g}$ on X by

$$\hat{g}(x) = \begin{cases} g(x) & \text{if} \quad x \in A; \\ u_n(g/A_n)(x) & \text{if} \quad x \in \bigcup \{I_\gamma: \gamma \in \Gamma_n\} . \end{cases}$$

Because the collections Γ_n are pairwise disjoint, $\hat{g}(x)$ is well-defined. Furthermore, $\hat{g}$ is continuous because each u_n has the following property, as may be seen from the proof of 2.4: if $a < b$ are points of A_n and if $f \in C(A_n)$, then $(u_n(f))([a, b])$ is contained in the closed convex hull of the set $f(A_n \cap [a, b])$. Then, defining $u(g) = \hat{g}$, we obtain the required simultaneous extender.

LEMMA 3.7. *Suppose that B is a closed subset of the generalized ordered space X and that no convex component of $X\backslash B$ is closed in X. Then there is a simultaneous extender from $C(B)$ to $C(X)$.*

Proof. Let $\{I_\gamma: \gamma \in \Gamma\}$ be the family of convex components of $X\backslash B$. Each set I_γ must have either one or two limit points in B and these limit points must be end-points of I_γ. If I_γ has a right end-point which belongs to B, denote it by b_γ^+; if I_γ has a left end-point which belongs to B, denote it by b_γ^-. If both b_γ^+ and b_γ^- exist, choose a continuous function $\psi_\gamma: X \to [0, 1]$ having $\psi_\gamma(\{x \in X: x \leqq b_\gamma^-\}) = 0$ and $\psi_\gamma(\{x \in X: x \geqq b_\gamma^+\}) = 1$. For each $g \in C(B)$ define $\hat{g}$ on X by

$$\hat{g}(x) = \begin{cases} g(x) \text{ if } x \in B; \\ g(b_\gamma^-) \text{ if } x \in I_\gamma \text{ and } b_\gamma^-, \text{ but not } b_\gamma^+, \text{ exists;} \\ g(b_\gamma^+) \text{ if } x \in I_\gamma \text{ and } b_\gamma^+, \text{ but not } b_\gamma^-, \text{ exists;} \\ \psi_\gamma(x)g(b_\gamma^+) + (1 - \psi_\gamma(x))g(b_\gamma^-) \text{ if } x \in I_\gamma \\ \text{and both } b_\gamma^+ \text{ and } b_\gamma^- \text{ exist} . \end{cases}$$

Then $\hat{g}(x)$ is well-defined and continuous, and the required simultaneous extender is obtained by defining $u(g) = \hat{g}$.

Proof of Theorem 3.5. Let $\{I_\gamma : \gamma \in \Gamma\}$ be the family of all convex components of $X \backslash A$ and let $\Gamma_1 = \{\gamma \in \Gamma : I_\gamma$ is closed in $X\}$. Let $X_1 = A \cup (\cup \{I_\gamma : \gamma \in \Gamma_1\})$ and apply 3.6 to the closed subset A of the generalized ordered space X_1 to obtain a simultaneous extender $u_1 : C(A) \rightarrow C(X_1)$. Observe that X_1 is a closed subset of X and that no convex component of $X \backslash X_1$ is closed in X. Apply 3.7 to find a simultaneous extender $u_2 : C(X_1) \rightarrow C(X)$. Then the composite function $u = u_2 \circ u_1$ is the required simultaneous extender from $C(A)$ to $C(X)$.

Let us conclude with some questions suggested by the results above; further questions related to the Dugundji Extension Theorem may be found in [5].

(1) If A is a closed subspace of a perfectly normal generalized ordered space X, must there be a simultaneous extender from $C(A)$ to $C(X)$? What if X is assumed to be linearly ordered?

(2) Is there an analogue of Theorem 2.4 for functions with values in a locally convex topological vector space?

REFERENCES

1. R. Arens, *Extension of functions on fully normal spaces*, Pacific J. Math., **2** (1952), 11-22.
2. C. R. Borges, *On stratifiable spaces*, Pacific J. Math., **17** (1966), 1-16.
3. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105-125.
4. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math., **1** (1951), 353-367.
5. R. W. Heath, *Remarks on Dugundji extension theorems*, Proceedings of the Second Pittsburgh Topology Conference. Springer-Verlag Lecture Notes in Mathematics, #378, New York, 1975.
6. R. W. Heath, D. Lutzer, and P. Zenor, *On monotonically normal spaces*, Trans. Amer. Math. Soc., **178** (1973), 481-493.
7. J. L. Kelley, *General Topology*, Van Nostrand, New York, 1965.
8. D. J. Lutzer, *On generalized ordered spaces*, Dissertations Mathematicae (Rozprawy Mat.), **89** (1971), 1-36.
9. D. J. Lutzer and H. Martin, *A note on the Dugundji extension theorem*, Notices Amer. Math. Soc. Proc. Amer. Math. Soc., **45** (1974), 137-139.
10. E. A. Michael, *Some extension theorems for continuous functions*, Pacific J. Math., **3** (1953), 789-806.
11. ———, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc., **69** (1963), 375-376.
12. E. A. Michael and A. Pelczynski, *A linear extension theorem*, Illinois J. Math., **11** (1967), 563-579.
13. A. Pelczynski, *On simultaneous extension of continuous functions*, Studia Math., **24** (1964), 285-304.
14. R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc., **53** (1947), 631-632.
15. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1965.

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