

A "GOING DOWN" THEOREM FOR CERTAIN REFLECTED RADICALS

B. J. GARDNER AND PATRICK N. STEWART

In a category \mathcal{K} suitable for radical theory, a functor $\Phi: \mathcal{K} \rightarrow \mathcal{K}$ is studied which is associated with a natural transformation $1_{\mathcal{K}} \rightarrow \Phi$ in a way which bears a formal resemblance to the behavior of certain "extension" functors of rings, such as that which assigns to each A the polynomial ring $A[x]$: every normal subobject $N \rightarrow \Phi(A)$ has a "contraction" $N^c \rightarrow A$. For a radical class \mathcal{R} in \mathcal{K} such that $\mathcal{R}^* = \{A | \Phi(A) \in \mathcal{R}\}$ is also radical, some conditions are obtained which imply that $\mathcal{R}^*(A) = \mathcal{R}(\Phi(A))^c$.

1. Preliminaries. We shall work in a category \mathcal{K} for which the general theory of radicals can be developed (for a set of conditions on \mathcal{K} which ensure this and for some other remarks on radicals in categories, see [9]) and shall consider a left-exact functor $\Phi: \mathcal{K} \rightarrow \mathcal{K}$ which has associated with it a natural transformation $1_{\mathcal{K}} \rightarrow \Phi$, which will be fixed throughout the discussion. We shall further assume that for each normal subobject $N \rightarrow \Phi(A)$ there is a normal subobject $N^{c_A} \rightarrow A$ and a pullback

$$\begin{array}{ccc} N^{c_A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ N & \longrightarrow & \Phi(A) \end{array}$$

where the right-hand vertical map is defined by the natural transformation mentioned above. When no confusion can result, N^{c_A} will be abbreviated to N^c . We shall frequently find it convenient to write A^e for $\Phi(A)$. A prototypical example of such a functor is that which assigns to each ring A its polynomial ring $A[x]$, in which case $A^e = A[x]$ ("extension") and $N^c = N \cap A$ ("contraction"). The symbol $A \rightarrow A^e$ will always denote a map defined by the given natural transformation.

Our category-theoretic terminology is essentially that of [2]. We shall not distinguish notationally between a subobject and a representative map. In particular if $A \in \mathcal{K}$ and \mathcal{R} is a radical class, $\mathcal{R}(A) \rightarrow A$ will denote the \mathcal{R} -radical of A .

PROPOSITION 1.1.

- (a) If $N \rightarrow A$ is a normal subobject, then $N \rightarrow A \subseteq N^{cc} \rightarrow A$.
- (b) If $N_1 \rightarrow A^e \subseteq N_2 \rightarrow A^e$ are normal subobjects then $N_1^c \rightarrow A \subseteq N_2^c \rightarrow A$.

(c) $A^{e^e} = A$.

(d) If $I \rightarrow A$ and $J \rightarrow A^e$ are normal subobjects, with $J \rightarrow A^e \cong I^e \rightarrow A^e$, then there is a map $J^{eI} \rightarrow J^{eA}$ such that

$$\begin{array}{ccc} J^{eI} & \longrightarrow & I \\ \downarrow & & \downarrow \\ J^{eA} & \longrightarrow & A \end{array}$$

commutes.

(e) If

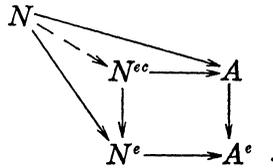
$$\begin{array}{ccc} P & \longrightarrow & N_1 \\ \downarrow & & \downarrow \\ N_2 & \longrightarrow & A^e \end{array}$$

is a pullback and $N_1 \rightarrow A^e, N_2 \rightarrow A^e, P \rightarrow A^e$ are normal subobjects, then

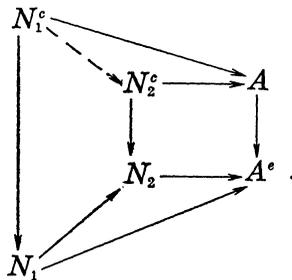
$$\begin{array}{ccc} P^e & \longrightarrow & N_1^e \\ \downarrow & & \downarrow \\ N_2^e & \longrightarrow & A \end{array}$$

is also a pullback.

Proof. (a) follows from consideration of the diagram

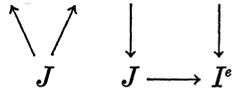


(b) follows from consideration of

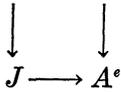


(c) is obtained from (a) by taking $N = A$.

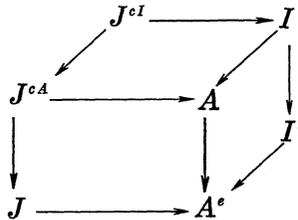
(d) Since $I^e \longrightarrow A^e, J^{eI} \longrightarrow I$



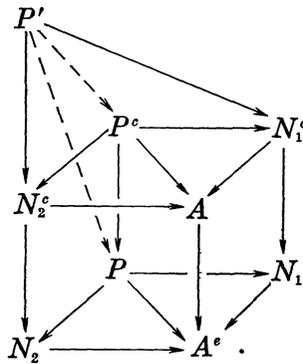
and $J^{eA} \longrightarrow A$ commute and the last square is a pullback, consideration



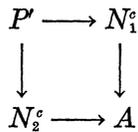
of the following diagram establishes the result.



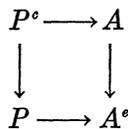
(e) Consider the diagram



By (b) the two triangles on the top of the cube commute and so the square on the top of the cube commutes. Suppose



commutes. Since the base of the cube is a pullback we obtain an appropriate map $P' \longrightarrow P$ and since



is a pullback, we obtain the required map $P' \dashrightarrow P^c$ which is unique since $P^c \rightarrow A$ is monic.

2. The results. If \mathcal{R} is a radical class in \mathcal{K} , we denote by \mathcal{R}^* the class $\{A \in \mathcal{K} \mid \Phi(A) \in \mathcal{R}\}$. Henceforth we shall only consider radical classes \mathcal{R} for which \mathcal{R}^* is also a radical class. Some conditions on Φ which imply that \mathcal{R}^* is radical for some or all radical classes \mathcal{R} are given in [5].

PROPOSITION 2.1. *For every $A \in \mathcal{K}$, we have $\mathcal{R}^*(A) \rightarrow A \cong \mathcal{R}(A^c) \rightarrow A$.*

Proof. Since $\mathcal{R}^*(A)^c \rightarrow A^c$ is a normal \mathcal{R} -subobject we have $\mathcal{R}^*(A)^c \rightarrow A^c \cong \mathcal{R}(A^c) \rightarrow A^c$. The result now follows from (a) and (b) of Proposition 1.1.

A radical class \mathcal{R} is *hereditary* if $N \in \mathcal{R}$ whenever $M \in \mathcal{R}$ and $N \rightarrow M$ is a normal subobject. \mathcal{R} is *normally-hereditary* if for every normal subobject $N \rightarrow A$ we have

$$\mathcal{R}(N) \longrightarrow N \longrightarrow A = (N \rightarrow A) \cap (\mathcal{R}(A) \longrightarrow A).$$

Clearly normally-hereditary radical classes are hereditary, but the converse need not be true.

In what follows we shall be concerned with the following conditions involving Φ and \mathcal{R} .

- (α) If $\mathcal{R}(A^c)^c = A$, then $A^c \in \mathcal{R}$.
- (α') $\mathcal{R}(A^c)^{cc} \rightarrow A^c \cong \mathcal{R}(A^c) \rightarrow A^c$ for each $A \in \mathcal{K}$.
- (α'') $\mathcal{R}(A^c)^{cc} \in \mathcal{R}$ for each $A \in \mathcal{K}$.

It is easy to see that $(\alpha'') \Rightarrow (\alpha') \Rightarrow (\alpha)$.

PROPOSITION 2.2. *If every pair of normal subobjects of each object in \mathcal{K} has a normal intersection, then for normally-hereditary radical classes \mathcal{R} , (α) and (α') are equivalent.*

Proof. We need only show that (α) implies (α'). Assume (α) is satisfied. The pullback

$$(*) \quad \begin{array}{ccc} P & \longrightarrow & \mathcal{R}(A^c)^{cc} \\ \downarrow & & \downarrow \\ \mathcal{R}(A^c) & \longrightarrow & A^c \end{array}$$

exists. By Proposition 1.1(e)

$$\begin{array}{ccc} P^c & \longrightarrow & \mathcal{R}(A^e)^{cec} \\ \downarrow & & \downarrow \\ \mathcal{R}(A^e)^c & \longrightarrow & A \end{array}$$

is a pullback and by Proposition 1.1(a), $\mathcal{R}(A^e)^c \rightarrow A \subseteq \mathcal{R}(A^e)^{cec} \rightarrow A$ and so $P^c = \mathcal{R}(A^e)^c$. Since \mathcal{R} is normally-hereditary, it follows from (*) that $P = \mathcal{R}(\mathcal{R}(A^e)^{ce})$ and so $P^c = \mathcal{R}(\mathcal{R}(A^e)^{ce})^c$. We conclude that $\mathcal{R}(A^e)^c = \mathcal{R}(\mathcal{R}(A^e)^{ce})^c$, i.e., $\mathcal{R}(I^e)^{cA} = I$, where $I = \mathcal{R}(A^e)^c$.

Consider the diagram

$$\begin{array}{ccccc} \mathcal{R}(I^e)^{cA} & \xrightarrow{\quad} & A & & \\ & \searrow & \downarrow & & \\ & & \mathcal{R}(I^e)^{cI} & \xrightarrow{\quad} & I \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{R}(I^e) & \xrightarrow{\quad} & I^e \\ & & & & \downarrow \\ & & & & A^e \end{array}$$

Since $I^e \rightarrow A^e$ is monic and

$$\begin{array}{ccc} \mathcal{R}(I^e)^{cI} & \longrightarrow & I \\ \downarrow & & \downarrow \\ \mathcal{R}(I^e) & \longrightarrow & I^e \end{array}$$

is a pullback we obtain a map $\mathcal{R}(I^e)^{cA} \rightarrow \mathcal{R}(I^e)^{cI}$ such that

$$\begin{array}{ccc} \mathcal{R}(I^e)^{cA} & \xlongequal{\quad} & I \\ \searrow & & \nearrow \\ & \mathcal{R}(I^e)^{cI} & \end{array}$$

commutes. Thus $\mathcal{R}(I^e)^{cI} = I$ and so by (α) we have $\mathcal{R}(A^e)^{ce} = I^e \in \mathcal{R}$, which establishes (α') .

PROPOSITION 2.3. *If \mathcal{R} is hereditary, then (α') and (α'') are equivalent.*

Proof. Obvious.

PROPOSITION 2.4. *If \mathcal{R} satisfies (α) then*

$$\mathcal{R}^* = \{A \mid \mathcal{R}(A^e)^c = A\} .$$

Proof. If $\mathcal{R}(A^e)^c = A$, then $A^e \in \mathcal{R}$, i.e., $A \in \mathcal{R}^*$, by (α) . Conversely, if $A \in \mathcal{R}^*$, then $A^e \in \mathcal{R}$ and so $\mathcal{R}(A^e) = A^e$. Hence $\mathcal{R}(A^e)^c = A^e = A$ by Proposition 1.1(c).

Proposition 2.4 gives a “global” description of \mathcal{R}^* . We can also give a “local” description of \mathcal{R}^* under more restrictive conditions. We shall need

LEMMA 2.5. *If \mathcal{R} satisfies (α'') then for each $A \in \mathcal{K}$ we have*

$$\mathcal{R}(A^\circ)^{\circ\circ\circ} = \mathcal{R}(A^\circ)^\circ.$$

Proof. By (α'') , $\mathcal{R}(A^\circ)^{\circ\circ} \in \mathcal{R}$, so

$$\mathcal{R}(\mathcal{R}(A^\circ)^{\circ\circ})^\circ = \mathcal{R}(A^\circ)^{\circ\circ\circ} \quad \text{and} \quad \mathcal{R}(A^\circ)^{\circ\circ} \longrightarrow A^\circ \subseteq \mathcal{R}(A^\circ) \longrightarrow A^\circ.$$

The latter implies, by Proposition 1.1(b), that $\mathcal{R}(A^\circ)^{\circ\circ\circ} \rightarrow A \subseteq \mathcal{R}(A^\circ)^\circ \rightarrow A$ and from Proposition 1.1(a) we see that $\mathcal{R}(A^\circ)^\circ \rightarrow A \subseteq \mathcal{R}(A^\circ)^{\circ\circ\circ} \rightarrow A$.

THEOREM 2.6. *If \mathcal{R} satisfies (α) and $\mathcal{R}(\mathcal{R}(A^\circ)^{\circ\circ})^\circ = \mathcal{R}(A^\circ)^\circ$, then $\mathcal{R}^*(A) = \mathcal{R}(A^\circ)^\circ$.*

Proof. Applying Proposition 2.4 to $\mathcal{R}(A^\circ)^\circ$, we see that $\mathcal{R}(A^\circ)^\circ \in \mathcal{R}^*$. The equality now follows from Proposition 2.1.

COROLLARY 2.7. *If every pair of normal subobjects of each object of \mathcal{K} has a normal intersection and if \mathcal{R} is normally-hereditary and satisfies (α) , then $\mathcal{R}^*(A) = \mathcal{R}(A^\circ)^\circ$ for each $A \in \mathcal{K}$.*

Proof. By Proposition 2.2, \mathcal{R} satisfies (α') and hence, by Proposition 2.3, (α'') . By Lemma 2.5, \mathcal{R} satisfies the requirements of Theorem 2.6 for all $A \in \mathcal{K}$.

3. EXAMPLES. In the category of associative rings, the functor Φ which associates the semigroup ring $A[S]$ with a ring A (S is a fixed semigroup with identity) and acts on maps in the obvious way satisfies the requirements listed in §1, the natural transformation being defined by the standard embedding $A \rightarrow A[S]$. Moreover, $\mathcal{R}^* = \{A \mid A[S] \in \mathcal{R}\}$ is radical for every radical class \mathcal{R} (cf. [5]) and $\mathcal{R}^* \subseteq \mathcal{R}$.

The following result is essentially due to Krempa [7] who proved it in the special case where S is the free semigroup with identity on one generator, i.e., $A[S]$ is the polynomial ring $A[x]$.

PROPOSITION 3.1. *Every radical class of associative rings satisfies (α') for the functor defined by the correspondence $A \mapsto A[S]$, for any semigroup S .*

From Proposition 2.4 we see that $\mathcal{R}^* = \{A \mid \mathcal{R}(A[S]) \cap A = A\}$ for every radical class \mathcal{R} . Thus in the case $A^\circ = A[x]$, \mathcal{R}^* coincides with the radical class discussed by Ortiz [8].

Proposition 2.1 and Theorem 2.6 therefore generalize Theorem 1 of [8]. By Corollary 2.7, $\mathcal{R}^*(A) = \mathcal{R}(A[S]) \cap A$ whenever \mathcal{R} is hereditary. For $A[S] = A[x]$, this was proved by the first author in [4].

Another example of a functor defined on the category of associative rings which meets our requirements is that which assigns to each ring A the ring A_n of $n \times n$ matrices for some (fixed) n . Again action on maps is defined in the obvious way. The natural transformation is defined by the embedding of A in A_n as the subring of scalar matrices. In this case too, $\mathcal{R}^* = \{A | A_n \in \mathcal{R}\}$ is radical for all radical classes \mathcal{R} [5].

The proof of the following result closely resembles that of Proposition 3.1.

PROPOSITION 3.2. *Every radical class \mathcal{R} of associative rings satisfies (α') for the functor defined by the correspondence $A \mapsto A_n$.*

By Proposition 2.4, $\mathcal{R}^* = \{A | A \subseteq \mathcal{R}(A_n)\}$ in this case and by Corollary 2.7, $\mathcal{R}^*(A) = \mathcal{R}(A_n) \cap A$ when \mathcal{R} is hereditary. (Here we have identified A with the ring of scalar matrices.)

Let $0 \rightarrow Z \rightarrow X \rightarrow D \rightarrow 0$ be an exact sequence of abelian groups, where Z is the group of integers and D is torsion-free divisible. The functor $() \otimes X$ has a right adjoint and so $\mathcal{R}^* = \{G | G \otimes X \in \mathcal{R}\}$ is a radical class for every radical class \mathcal{R} of abelian groups [5]. The map $G \rightarrow G \otimes X$ defined by the isomorphism $G \cong G \otimes Z$ and the given exact sequence defines a natural transformation from the identity to $() \otimes X$. All requirements of §1 are satisfied.

PROPOSITION 3.5. *Every radical class \mathcal{R} of abelian groups satisfies (α) for the functor $() \otimes X$.*

Proof. If $G \in \mathcal{R}(G \otimes X)$, there is an epimorphism

$$G \otimes D \cong (G \otimes X)/G \longrightarrow (G \otimes X)/\mathcal{R}(G \otimes X) .$$

If \mathcal{R} contains only torsion groups, then G is torsion and so $G \otimes D = 0 \in \mathcal{R}$. If \mathcal{R} contains a nontorsion group, then it contains all divisible groups (see e.g. [3], Corollary 2.3) and so $G \otimes D \in \mathcal{R}$. Hence $(G \otimes X)/\mathcal{R}(G \otimes X) \in \mathcal{R}$ in all cases. Thus $G \otimes X \in \mathcal{R}$.

In all the examples considered so far, the natural transformation involved has arisen from a natural embedding $A \rightarrow A^\circ$. We conclude with a simple example in which the relevant map $A \rightarrow A^\circ$ need not be monic.

Let R_1 and R_2 be associative rings with identity, $R = R_1 \oplus R_2$

(ring direct sum) and let Φ be the functor defined on the category $\text{Mod}(R)$ of right unital R -modules by $M \mapsto MR_1$. The classes $\mathcal{R}_i = \{MR_i \mid M \in \text{Mod}(R)\}$, $i = 1, 2$, are actually hereditary radical classes and we have the situation analysed in Theorem 2.4 of Jans [6]. It is straightforward to show that Φ is exact and preserves unions of ascending chains and hence (see [5]) that $\mathcal{R}^* = \{M \mid \Phi(M) \in \mathcal{R}\}$ is a radical class for every radical class \mathcal{R} in $\text{Mod}(R)$. The projection $M \rightarrow MR_1$ defines a natural transformation with the properties we want. If N is a submodule of $MR_1 = M^e$ then $N^e = N \oplus MR_2$.

PROPOSITION 3.3. *Every radical class \mathcal{R} in $\text{Mod}(R)$ satisfies (α) for the functor defined by the correspondence $M \mapsto MR_1$.*

Proof. If $\mathcal{R}(MR_1) \oplus MR_2 = M = MR_1 \oplus MR_2$, then $MR_1 = \mathcal{R}(MR_1)$.

4. The question of "going up". We revert to our general situation to briefly mention a related problem: to determine when $\mathcal{R}(A^e) = \mathcal{R}^*(A)^e$. Since $\mathcal{R}^*(A) \in \mathcal{R}^*$, we always have $\mathcal{R}^*(A)^e \rightarrow A^e \subseteq \mathcal{R}(A^e) \rightarrow A^e$. The other inclusion seems to be more difficult. Amitsur [1] has given a (highly nontrivial) proof for $A^e = A[x]$ which is valid when \mathcal{R} is strongly hereditary or the Jacobson radical class. On the other hand, when $A^e = A_n$, it is relatively easy to check that $\mathcal{R}(A^e) = \mathcal{R}^*(A)^e$ for all rings A when \mathcal{R} is hereditary.

PROPOSITION 4.1. *Let A be an associative ring, \mathcal{R} a radical class of associative rings. If A has an identity or \mathcal{R} is hereditary, then $\mathcal{R}(A_n) = \mathcal{R}^*(A)_n$.*

Proof. If A has an identity, then $\mathcal{R}(A_n) = I_n$ for some ideal I of A . Since $I_n \in \mathcal{R}$ we have $I \in \mathcal{R}^*$; thus $I \subseteq \mathcal{R}^*(A)$ and so $\mathcal{R}(A_n) \subseteq \mathcal{R}^*(A)_n$. But $\mathcal{R}^*(A)_n \in \mathcal{R}$, so $\mathcal{R}(A_n) = \mathcal{R}(A)_n$. If A does not have an identity and \mathcal{R} is hereditary, we make use of the Dorroh extension A^1 of A . Because \mathcal{R} is hereditary (= normally-hereditary), so is \mathcal{R}^* , and thus we have $\mathcal{R}(A_n) = A_n \cap \mathcal{R}((A^1)_n) = A_n \cap \mathcal{R}^*(A^1)_n = [A \cap \mathcal{R}^*(A^1)]_n = \mathcal{R}^*(A)_n$.

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Received December 4, 1973.

UNIVERSITY OF TASMANIA, HOBART, TASMANIA, AUSTRALIA
AND
DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

