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BOUNDS FOR DISTORTION IN PSEUDOCONFORMAL MAPPINGS

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1. Introduction. When considering a conformal mapping of a domain, say[1] B^2, of the z-plane, it is useful to introduce a metric which is invariant with respect to conformal transformations. The line element of this metric is given by

$$ds_B^2(z) = K_B(z, \bar{z})|dz|^2 , \quad B \equiv B^2 , \tag{1.1}$$

where $K_B(z, \bar{z})$ is the kernel function of B^2. (In the case of $[|z| < 1]$ the metric (1.1) is identical with the hyperbolic metric introduced by Poincaré.) In addition to the invariant metric one can also introduce scalar invariants, for instance,

$$J_B(z) = -\frac{1}{C_B(z)}, \; C_B(z) = -\frac{2}{K^3}\begin{vmatrix} K & K_{0\bar{1}} \\ K_{10} & K_{1\bar{1}} \end{vmatrix}, \; K_{10} = \frac{\partial K}{\partial z} , \tag{1.2}$$
$$K_{0\bar{1}} = \frac{\partial K}{\partial \bar{z}} .$$

($C_B(z)$ is the curvature of the metric (1) at the point z.)

Using the kernel function $K_{\mathfrak{B}}(z, \bar{z})$, $z = (z_1, \cdots, z_n)$, one can generalize this approach to the theory of PCT's (pseudoconformal transformations), i.e., to the mappings of $2n$ dimensional domains by n analytic functions of n complex variables (with a nonvanishing Jacobian). It is of interest to obtain bounds for the invariant $J_{\mathfrak{B}}(z)$, see (3.1), depending on quantities which are in a simple way connected with the domain, for instance, the maximum and minimum (euclidean) distances between the point z and the boundary of the domain.

In the present paper we shall determine such bounds in the case of pseudoconformal mapping of the domain $\mathfrak{B} = \mathfrak{B}^4$ of the z_1, z_2-space by pairs

$$w_k = f_k(z_1, z_2) , \qquad k = 1, 2 , \tag{1.3}$$

of analytic functions of two complex variables (with nonvanishing Jacobian). The generalization of our procedure to the case of pseudoconformal mappings of domains $\mathfrak{B}^{2n}$ by n functions of n complex variables, $3 \leqq n < \infty$, is immediate and will not be discussed in the following.

2. The minima $\lambda_{\mathfrak{B}}^{\cdots}(z)$. To obtain the desired bound we use

[1] The upper index at a set indicates its dimension.

the minimum values $\lambda_{\mathfrak{B}}^{\cdots}(z)$ of the integral

$$\int_{\mathfrak{B}} |f(\zeta)|^2 d\omega,\ \zeta = (\zeta_1, \zeta_2)\ , \tag{2.1}$$

($d\omega =$ the volume element), under some additional conditions for f at the point $z = (z_1, z_2)$.

As indicated in [1, pp. 183 and 198 ff.], many invariant quantities arising in the theory of PCT's can be expressed in terms of the minima $\lambda_{\mathfrak{B}}^{\cdots}(z)$. For instance,

$$K_{\mathfrak{B}}(z, \bar{z}) = \frac{1}{\lambda_{\mathfrak{B}}^{1}(z)}\ ,\quad J_{\mathfrak{B}}(z) = \frac{\lambda_{\mathfrak{B}}^{01}(z)\lambda_{\mathfrak{B}}^{001}(z)}{[\lambda_{\mathfrak{B}}^{1}(z)]^3}\ . \tag{2.2}$$

Here $\lambda_{\mathfrak{B}}^{X_{00}}(z)$ is the minimum of (2.1) under the condition $f(z) = X_{00}$, $z \in \mathfrak{B}$, $\lambda_{\mathfrak{B}}^{X_{00}X_{10}}$ is the minimum under the condition $f(z) = X_{00}$, $(\partial f(z)/\partial z_1) = X_{10}$ and $\lambda_{\mathfrak{B}}{}^{X_{00}X_{10}X_{01}}(z)$ is the minimum under the condition $f(z) = X_{00}$, $(\partial f(z)/\partial z_1) = X_{10}$, $(\partial f(z)/\partial z_2) = X_{01}$. ($K$ is a *relative* invariant, see (25), p. 180, of [1].)

Using (23b), p. 179 of [1], one obtains the representations for the $\lambda_{\mathfrak{B}}^{\cdots}(z)$ in terms of the kernel function $K \equiv K_{\mathfrak{B}}$ and their partial derivatives $K_{10\bar{0}\bar{0}} = (\partial K/\partial z_1)$, $K_{01\bar{0}\bar{0}} = (\partial K/\partial z_2)$, $K_{00\bar{1}\bar{0}} = (\partial K/\partial \bar{z}_1)$, $K_{00\bar{0}\bar{1}} = \partial K/\partial \bar{z}_2$. Obviously it holds

LEMMA 2.1. *Suppose that* $z \in \mathfrak{B} \subset \mathfrak{G}$, *then*

$$\lambda_{\mathfrak{B}}^{\cdots}(z) \leqq \lambda_{\mathfrak{G}}^{\cdots}(z)\ . \tag{2.3}$$

Here it is assumed that the minima $\lambda^{\cdots}(z)$ *on both sides of* (2.3) *are taken under the same conditions.*

Choosing for $\mathfrak{G}$ a domain for which the kernel function $K_{\mathfrak{G}}$ is a simple expression of the equation of its boundary (e.g., choosing for $\mathfrak{G}$ a sphere or certain Reinhardt circular domains, see [2, p. 21]), we obtain the desired inequality.

Using the above method, we shall derive in the next section an inequality for the invariant $J_{\mathfrak{B}}(z)$.

3. Derivation of bounds for $J_{\mathfrak{B}}(z)$. Let $\mathfrak{B}$ be a connected domain of the (four-dimensional) z_1, z_2-space, $z_k = x_k + iy_k$, $k = 1, 2$. Let

$$J_{\mathfrak{B}}(z, \bar{z}) \equiv J_{\mathfrak{B}} = \frac{K}{T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2}\ ,\quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n}\ , \tag{3.1}$$

denote the invariant respect to PCT's, see (37a), p. 183 of [1]. Here with K is the kernel function of $\mathfrak{B}$ and $T_{m\bar{n}}$ are the coefficients of the line element

$$ds^2_{\mathfrak{B}} = \sum_{m=1}^{z} \sum_{n=1}^{2} T_{m\bar{n}} dz_m d\bar{z}_n \tag{3.2}$$

of the metric which is invariant with respect to PCT's, see [1, p. 182 ff.].

Theorem I. *Suppose that r is the maximum distance of the point z, $z \in \mathfrak{B}$, to the boundary $\partial\mathfrak{B}$, and ρ is the corresponding minimum distance. Then*

$$H(\rho, r) \leqq J_{\mathfrak{B}}(z) \leqq H(r, \rho) , \tag{3.3}$$

$$H(\rho, r) = \frac{2r^6[P(\rho)]^9}{9\rho^6[P(r)]^9\pi^2} , \quad P(\rho) = \rho^2 - z_1\bar{z}_1 - z_2\bar{z}_2 .$$

Proof. By (97), p. 198 of [1],

$$J_{\mathfrak{B}}(z) = \frac{\lambda^{01}_{\mathfrak{B}}(z)\lambda^{001}_{\mathfrak{B}}(z)}{[\lambda^1_{\mathfrak{B}}(z)]^3} \tag{3.4}$$

and in accordance with (2.3) for $\mathfrak{J} \subset \mathfrak{B} \subset \mathfrak{A}$ the inequality

$$\frac{\lambda^{01}_{\mathfrak{J}}(z)\lambda^{001}_{\mathfrak{J}}(z)}{[\lambda^1_{\mathfrak{A}}(z)]^3} \leqq J_{\mathfrak{B}}(z) \leqq \frac{\lambda^{01}_{\mathfrak{A}}(z)\lambda^{001}_{\mathfrak{A}}(z)}{[\lambda^1_{\mathfrak{J}}(z)]^3} \tag{3.5}$$

holds. If r is the maximum distance of the point z from the boundary $\partial\mathfrak{B}$, and ρ is the minimum distance of z from $\partial\mathfrak{B}$, then one can use for $\mathfrak{A}$ the hypersphere $|z_1|^2 + |z_2|^2 < r^2$ and for $\mathfrak{J}$ the hypersphere $|z_1|^2 + |z_2|^2 < \rho^2$. By (23b)[2], p.179 of [1] and by (5a), p. 22 of [2] it holds for the hypersphere $|z_1|^2 + |z_2|^2 < r^2$,

$$\lambda^{01}_{\mathfrak{A}}(z)\lambda^{001}_{\mathfrak{A}}(z) = \frac{\pi^4[P(r)]^8}{36r^6} , \tag{3.6}$$

$$\lambda^1_{\mathfrak{A}}(z) = \frac{1}{K_{\mathfrak{A}}(z, \bar{z})} = \frac{\pi^2[P(r)]^3}{2r^2} . \tag{3.7}$$

Analogous formulas hold for $\lambda^{01}_{\mathfrak{J}}(z)\lambda^{001}_{\mathfrak{J}}(z)$ and $\lambda^1_{\mathfrak{J}}(z)$. Consequently (3.3) holds.

4. An application of Theorem I.

A domain which admits the group

$$z^*_k = z_k e^{i\varphi_k} , \quad 0 \leqq \varphi_k \leqq 2\pi , \quad k = 1, 2 , \tag{4.1}$$

[2] In the last term of the expression for $\lambda^{X_{00}X_{10}X_{01}}(t)$ of (23b) are misprints, in the denominator $\begin{vmatrix} K & K_{00\bar{0}\bar{0}} \\ K_{\bar{1}000} & K_{10\bar{1}\bar{0}} \end{vmatrix}$ should be replaced by $\begin{vmatrix} K & K_{00\bar{1}\bar{0}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} \end{vmatrix}$. In the nominator of the last term of (23b) the last term $K_{01\bar{0}\bar{1}}$ in the third row should be replaced by $K_{01\bar{1}\bar{0}}$. In the denominator the first term $K_{01\bar{0}\bar{1}}$ of the third row should be replaced by $K_{01\bar{0}\bar{0}}$.

of PCT's onto itself (automorphisms) is called a Reinhardt circular domain (see [3], pp. 33–34).

A domain, say $\mathfrak{R}$, bounded by the hypersurface

$$|z_2| = r(|z_2|) \ , \tag{4.2}$$

where $y_2 = r(x_1)$ is a convex curve, is a Reinhardt circular domain. Its kernel function is

$$K_{\mathfrak{R}}(z, \bar{z}) = B_{00} + B_{10}z_1\bar{z}_1 + B_{01}z_2\bar{z}_2 + B_{02}z_1^2\bar{z}_1^2 + B_{11}z_1\bar{z}_1z_2\bar{z}_2 + \cdots \ , \tag{4.3}$$

$$B_{mp}^{-1} = \int_{\mathfrak{R}} |z_1|^{2m} |z_2|^{2p} d\omega \ , \tag{4.4}$$

$d\omega$ volume element (B_{mp} are the inverse of moments of $\mathfrak{R}$), see [2], p. 20 ff.

LEMMA. *The kernel function $K_{\mathfrak{R}}$ and its derivatives at the center* 0 *of $\mathfrak{R}$ equal*

$$\begin{aligned} &K_{\mathfrak{R}} \equiv K = B_{00} \ , \\ &K_{10\bar{0}\bar{0}} \equiv K_{z_1}(0) = 0 \ , \quad K_{10\bar{1}\bar{0}} \equiv \frac{\partial^2 K}{\partial z_1 \partial \bar{z}_1} = B_{10} \ , \quad K_{01\bar{0}\bar{0}} = 0 \ , \\ &K_{01\bar{0}\bar{1}} = B_{01}, \ \cdots \ . \end{aligned} \tag{4.5}$$

Therefore

$$J_{\mathfrak{R}}(0) = \frac{K}{\begin{vmatrix} K & K_{00\bar{1}\bar{0}} & K_{00\bar{0}\bar{1}} \\ K_{10\bar{0}\bar{0}} & K_{10\bar{1}\bar{0}} & K_{10\bar{0}\bar{1}} \\ K_{01\bar{0}\bar{0}} & K_{01\bar{1}\bar{0}} & K_{01\bar{0}\bar{1}} \end{vmatrix}} = \frac{B_{00}^4}{\begin{vmatrix} B_{00} & 0 & 0 \\ 0 & B_{10} & 0 \\ 0 & 0 & B_{01} \end{vmatrix}} = \frac{B_{00}^3}{B_{10}B_{01}} \tag{4.6}$$

(see [1], p. 183, (37a)).

THEOREM II. *Let $\mathfrak{B} = \boldsymbol{B}(\mathfrak{R})$ be a pseudoconformal image of a Reinhardt circular domain $\mathfrak{R}$, and let r and ρ be the maximum and minimum distances from the boundary, respectively, of the image $z^0 = (z_1^0, z_2^0) = \boldsymbol{B}(0)$ of the center* 0 *of $\mathfrak{R}$ in $\mathfrak{B}$. Then*

$$H(\rho, r) \leqq \frac{B_{00}^3}{B_{10}B_{01}} \leqq H(r, \rho) \ . \tag{4.7}$$

Here B_{mn} are the inverse moments (introduced in (4.4)) *of $\mathfrak{R}$.*

Proof. Since $J_{\mathfrak{R}}$ is invariant and $\mathfrak{B}$ is a pseudoconformal image of $\mathfrak{R}$

$$J_{\mathfrak{A}}(0) = J_{\mathfrak{B}}(z^0) = \frac{B_{00}^3}{B_{10}B_{01}} . \tag{4.8}$$

By Theorem I it follows that for $J_{\mathfrak{B}}(z^0)$ the inequality (4.7) holds.

Similar results as above can be obtained for other interior distinguished points, for instance, for critical points of $J_{\mathfrak{B}}(z, \bar{z})$.

REMARK. One obtains a generalization of Theorem I by assuming that $\mathfrak{J}$ and $\mathfrak{A}$ are domains $|z_1|^{2/m} + |z_2|^2 < \rho^2$ and $|z_1|^{2/M} + |z_2|^2 < r^2$, respectively. The kernel function for the above domains is given in (5), p. 21, of [2].

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