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A NOTE ON QUADRATIC FORMS OVER PYTHAGOREAN FIELDS

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A theorem of T. A. Springer states that if F is a field of characteristic not two and L is an extension field of F of odd degree then any anisotropic quadratic form over F remains anisotropic over L. A weaker version (and an immediate consequence) of this theorem says that the natural map $r: W(F) \to W(L)$, from the Witt ring of F to the Witt ring of L, is injective. This note investigates the relationship between these statements in the case that L is a finite Galois extension of a pythagorean field F. Specifically, it is shown that if r is injective then any anisotropic quadratic form over F remains anisotropic over L and if, in addition, L is pythagorean then the extension must be of odd degree. An example is provided of a Galois extension of even degree with r injective.

Notations and terminology in this paper will follow [**4**]. Thus by a field F we shall mean one of characteristic different from two and $W(F)$ will denote the Witt ring of anisotropic quadratic forms over F. If $F \subset L$ is an extension of field then $r_{L/F}: W(F) \to W(F)$ will denote the induced homomorphism of Witt rings. When there is no possibility of confusion we shall simply write r in place of $r_{L/F}$. In general, the mapping r will fail to be injective. However, if $F \subset L$ is an extension of odd degree then the above mentioned theorem of Springer will imply the injectivity of r [**4**, Chapter 7, §2]. In the case of ordered (= formally real) fields, information about the kernel of r can be used to yield information about extending orderings. Specifically, every ordering on F extends to an ordering on L if and only if Ker r is a nil ideal of $W(F)$ [**3**, Corollary 2.11]. One can use this, together with Springer's theorem, to recover the fact that if $F \subset L$ is an extension of odd degree with F formally real then every ordering on F extends to L. Moreover, if F is pythagorean then $W(F)$ has no nonzero nilpotent elements [**4**, Theorems 3.3 and 6.1, pp. 236 and 248] so for any extension L of F, $r: W(F) \to W(F)$ is injective if and only if every ordering on F extends to L.

PROPOSITION 1. *Let $F \subset L$ be a finite Galois extension of degree n with L pythagorean. If $r: W(F) \to W(L)$ is injective then n is odd.*

Proof. Let G be the Galois group of the extension $F \subset L$, let H be a 2-Sylow subgroup of G, and let $K = L^H$ be the fixed field of H. Then K is also pythagorean [**4**, Exercise 17, p. 254].

If F is not formally real then every element of K is a square in K (i.e. K is "quadratically closed"). Thus, from Galois theory, H must be trivial and hence G is a group of odd order.

Now assume F is formally real and let $<$ be an ordering on F. Since $r: W(F) \to W(L)$ is injective, $<$ extends to L (and to K). Moreover by [**2**, Exercise 2, p. 289], $<$ extends to exactly $[L:F]$ orderings on L and to $t \leqq [K:F]$ orderings on K (compare [**3**, Proposition 5.12]). Let $<_1, <_2, \cdots, <_m, m \leqq t$, be the orderings on K which extend $<$ and which also extend to L. Since $K \subset L$ is a Galois extension, it again follows that each $<_i$ extends exactly $[L:K]$ different ways to L. Thus $[L:F] = m[L:K]$, which implies that $m = [K:F]$. Hence $m = t$ so that every extension of $<$ to K also extends to L. But every ordering on K is the extension of some ordering on F, so it follows that every ordering on K extends to L. Since K is a pythagorean field, the mapping $r_{L/K}: W(K) \to W(L)$ is injective. If the Galois group H of the extension $K \subset L$ is not trivial then there will exist a nonsquare a in K with $\sqrt{a}$ in L. Then $\langle 1, -a \rangle$ is an anisotropic form over K whose class in $W(K)$ is a nonzero element in the kernel of $r_{L/K}$. Thus H is also trivial in this case, i.e. n is odd.

COROLLARY. *Let $F \subset L$ be a finite Galois extension of degree n with L pythagorean. If every ordering on F extends to L then n is odd.*

Proof. By [**4**, Exercise 17, p. 254], F is also pythagorean.

The following modification of a construction due to Manfred Knebusch shows that the hypothesis that L be pythagorean is essential in Proposition 1 and its corollary.

EXAMPLE. A Galois extension $F \subset L$ of formally real fields with F pythagorean (actually euclidean), $[L:F]$ even, and $r: W(F) \to W(L)$ injective.

Choose $n \geqq 5$ and let K be a formally real field on which the alternating group A_n acts as a group of automorphisms (e.g. $K = \boldsymbol{R}(x_1, \cdots, x_n)$). Let $k = K^{A_n}$ be the fixed field and let $\tilde{k}$ be the quadratic closure of k, i.e. the compositum of all Galois extensions of k with degree a power of 2 [**4**, p. 219]. Then $\tilde{k}$ is a Galois extension of k and since $[K:k]$ is not a power of two, K is not contained in $\tilde{k}$. Thus $\tilde{k} \cap K \neq K$ is a Galois extension of k so Galois theory and the simplicity of A_n imply that $\tilde{k} \cap K = k$.

Now let R be a real closure ([**2**], [**4**], [**5**]) of the formally real field K and let $F = R \cap \tilde{k}$. Then we also have $F \cap K = k$. Moreover, F is formally real and it is easy to see that any a in F is either a square in F

or the negative of a square in F. In particular, F is pythagorean and has exactly one ordering. From Sylvester's law of inertia we have $W(F) \cong Z$ (cf. [4, pp. 42–43]).

Let $L = FK$ be the compositum of F and K in R. Then L is a formally real Galois extension of F with Galois group A_n [5, Theorem 4, p. 196]. In particular, $[L:F]$ is even. Finally, any signature $\sigma_< : W(L) \to Z$ arising from an ordering $<$ on the formally real field L (see [4, pp. 42–43], [3, p. 211]) will provide a splitting for the map $r : W(F) \to W(L)$.

PROPOSITION 2. *Let F be a pythagorean field and L a finite Galois extension of F. Then the following statements are equivalent;*

(1) *$r : W(F) \to W(L)$ is injective.*

(2) *If q is an anisotropic quadratic form over F then $q_L = L \otimes_F q$ is anisotropic over L.*

Proof. (1) $\Rightarrow$ (2). If F is not formally real then F is quadratically closed so all anisotrpic forms over F are one dimensional. Hence the implication is obvious in this case.

Now assume F is formally real and let Tr_* denote Scharlau's transfer map relative to the F-linear trace map $Tr_{L/F}$ (which associates to each quadratic form q over L the F-quadratic form $Tr_{L/F} \circ q$) [4, Chapter 7, §1, 6], [3, §5]. Then for any anisotropic form q over F, there is an isometry $L \otimes_F Tr^*(q_L) q_L \perp \cdots \perp q_L \cong [L:F] \cdot q_L$, where $[L:F] \cdot q_L = q_L \perp \cdots \perp q_L$, $[L:F]$ times [4, Theorem 6.1, p. 212] compare [3, Corollary 5.10]). Since the mapping $r : W(F) \to W(L)$ is injective this means that $Tr_*(q_L)$ is isometric to $[L:F] \cdot q$ over F. But F is a formally real pythagorean field, so by (the proof of) [4, Theorem 3.3], $[L:F] \cdot q$ is anisotropic over F. Therefore $Tr_*(q_L)$ is anisotropic over F so that, in particular, q_L is anisotrpic over L.

The implication (2) $\Rightarrow$ (1) is immediate.

It seems to be an open question whether, for an arbitrary extension $F \subset L$, the injectivity of $r : W(F) \to W(L)$ implies that anisotropic forms over F remain so over L. However, for a certain class of pythagorean fields the answer is affirmative. Let F be a formally real field, let X be the set of orderings on F, and for a in F, let $V(a) = \{<\text{ in } X \mid a > 0\}$. Then the family $V(a)_{a \in F}$ generates a compact, Hausdorff, totally disconnected topology on X [3, Lemma 3.3, Theorem 3.18]. The field F satisfies the Strong Approximation Property (SAP) if given any two disjoint closed subsets U, V of X there is an element a in F which is positive at the orderings in U and negative at the orderings in V (cf. [1, Definition 1.4], [3, Corollary 3.21]).

PROPOSITION 3. *Let F be a formally real pythagorean field satisfying SAP and let L be any extension field of F. If $r : W(F) \to W(L)$ is* *injective then any anisotropic quadratic form over F remains anisotropic over L.*

Proof. In view of [**1**, Theorem 5.3 (1)], any anisotropic form q over F can be written $q = \langle a_1, \cdots, a_n \rangle$ where either all the a_i's are positive or all the a_i's are negative with respect to some ordering $<$ on F. If $r: W(F) \to W(L)$ is injective then $<$ extends to L so an equation $a_1x_1^2 + \cdots + a_nx_n^2 = 0$ with each x_i in L is impossible.

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