PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

WHITNEY CONTINUA IN THE HYPERSPACE $C(X)$

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Let $C(X)$ denote the hyperspace of subcontinua of the continuum X, and let $\mu: C(X) \to [0,1]$ be a Whitney function. The purpose of this paper is to investigate whether certain properties of X are inherited by the continua $\mu^{-1}(t)$.

Let $C(X)$ denote the hyperspace of subcontinua of the continuum X. H. Whitney has defined a continuous function $\mu: C(X) \to [0,1]$ satisfying (1) $\mu(X) = 1$, (2) $\mu(\{x\}) = 0$, for all x in X, and (3) $\mu(A) < \mu(B)$, for $A \subset B$ and $A \neq B$. We may think of the function μ as measuring the size of a continuum. The map μ is monotone; that is, $\mu^{-1}(t)$ is a continuum for each t. The purpose of this paper is to investigate how properties of the continuum X are reflected in the continua $\mu^{-1}(t)$, which we call the Whitney continua of X.

Before giving the results of our investigation, let us note that $\mu^{-1}(1)$ is the degenerate subcontinuum X of $C(X)$ and that $\mu^{-1}(0)$, the set of singleton subcontinua of X, is homeomorphic to X. We, therefore, need only investigate the structure of $\mu^{-1}(t)$ for $0 < t < 1$, and we state our results only for these values of t.

There have been several papers which have contained information on the continua $\mu^{-1}(t)$. J. L. Kelley **[9]** showed that if X is a hereditarily indecomposable continuum, then so is $\mu^{-1}(t)$. This, recall, wwas Kelley's technique for constructing infinite-dimensional, hereditarily indecomposable continua, assuming the existence of hereditarily indecomposable continua of dimension greater than one (the difficult question of the existence of the latter continua was solved by Bing about 10 years later). C. Eberhart and S. Nadler **[5]** have observed that Kelley's result, together with a result of Bing **[2]**, implies that $\mu^{-1}(t)$ is a pseudo-arc whenever X is. In a recent paper, J. Krasinkiewicz **[10]** has proved that if X is arc-like, then so is $\mu^{-1}(t)$ and if X is circle-like but not arc-like, then so is $\mu^{-1}(t)$.

There are two original techniques for hyperspace proofs introduced in this paper. In the fourth section of this paper, we define, for an arbitrary continuum X, an upper semi-continuous, continuum-valued map γ_t of X onto $\mu^{-1}(t)$ and an upper semi-continuous, continuum-valued map σ_t of $\mu^{-1}(t)$ onto X. In the special case that X is a circle-like continuum, these multi-valued maps can be applied to show that X and $\mu^{-1}(t)$ are cohomologically equivalent continua.

This result yields as corollaries the following facts. If X is a nonplanar, circle-like continuum, then so is $\mu^{-1}(t)$. If X is a planar, circle-like continuum, then $\mu^{-1}(t)$ is a plane continuum (which is either arc-like or circle-like). This latter result answers a question of Krasinkiewicz [10].

We also prove a partial converse to Krasinkiewicz's results. We show that if X is a decomposable, circle-like continuum that is also arc-like, then there exists a t such that $\mu^{-1}(t)$ is not circle-like.

Since each nonplanar, circle-like continuum X is indecomposable, our results imply that $\mu^{-1}(t)$ is indecomposable for such continua X. We provide examples, however, of planar, indecomposable, circle-like continua such that $\mu^{-1}(t)$ is decomposable, for all t between 0 and 1. On the other hand, we will find for the pseudo-circle (which, by definition, is planar) that $\mu^{-1}(t)$ is again a pseudo-circle and thus hereditarily indecomposable.

A second central idea of this paper is a technique developed in the third section for locating arcs in the continua $\mu^{-1}(t)$. Besides being crucial in later examples, this technique is applied in the second section to prove that $\mu^{-1}(t)$ is arcwise-connected whenever X is. The examples mentioned in the previous paragraph show that if X is an indecomposable continuum or a circle-like continuum, then $\mu^{-1}(t)$ need not be such a continuum unless additional conditions are imposed. In the last section, we give an example of an atriodic, unicoherent continuum such that $\mu^{-1}(t)$ is neither atriodic nor unicoherent, for some t between 0 and 1. We also give an example of a hereditarily unicoherent, one-dimensional continuum such that $\mu^{-1}(t)$ does not have either of these properties, for some t.

Turning to algebraic properties of continua, we find that the groups associated with $\mu^{-1}(t)$ may be quite different from those associated with X, whether these groups be Čech homology, Čech cohomology, singular homology, singular cohomology, or homotopy. Hence circle-like continua are rather special in the fact that X is (Čech) cohomologically equivalent to $\mu^{-1}(t)$.

This paper continues the author's investigation of the analogies between cones and hyperspaces. In previous papers [12, 13], we have shown that the cone over a continuum X and $C(X)$ are actually homeomorphic in some instances. In fact, the first example [12] of a continuum X such that $C(X)$ does not have the fixed-point property was obtained via such a homeomorphism. The cone $K(X)$ over a continuum X has a projection function $p\colon K(X)\to[0,1]$ defined by $p(x,t)=t$; hence p measures height in the cone. The corresponding role in the hyperspace $C(X)$ is played by the Whitney function μ. The projection function p has the property that $p^{-1}(t)$ is a continuum homeomorphic to X, for $0\leqq t<1$. Investigating the class of continua $\{\mu^{-1}(t)\}$, we will find that very rarely is $\mu^{-1}(t)$ homeomorphic to X, for $0\leqq t<1$. Even in the case that $C(X)$ is homeomorphic to $K(X)$, we find that often $\mu^{-1}(t)$ is not homeomorphic to X. We will see, however, that $\mu^{-1}(t)$ is homeomorphic to X, $0 \leqq t < 1$, if X is an arc, a circle, a pseudo-arc, or a pseudo-solenoid. Let us note here the curious result [12] that in case X is a pseudo-arc or a pseudo-solenoid, $C(X)$ is not homeomorphic to $K(X)$, even though $\mu^{-1}(t)$ is homeomorphic to X for each t between 0 and 1.

After a few preliminary remarks and definitions in Section One, we develop some functorial properties of hyperspaces in the second section. The main theorems of the paper appear in the third and fourth sections. In the last section we give some results and examples about some continua that are neither arc-like nor circle-like.

1. Basic facts about hyperspaces. A continuum is a compact, connected, nonvoid, metric space. $C(X)$, the hyperspace of subcontinua of a continuum X, is the space of all subcontinua of X with the topology induced by the Hausdorff metric ρ, where $\rho(A,B) = \text{g.l.b.}\{\epsilon\}$, for all ϵ such that A is contained in the ϵ-neighborhood of B and B is contained in the ϵ-neighborhood of A. We write X for the point in $C(X)$ corresponding to the continuum X. We note that the subspace X' of $C(X)$ consisting of the degenerate subcontinua of X is isometric to X.

An arc is a continuum homeomorphic to the closed interval $[0,1]$. A circle or simple closed curve is a continuum homeomorphic to S^1. A continuum is arc-like if it is homeomorphic to the inverse limit of an inverse sequence $\{X_i, f_i\}$ of arcs with surjective bonding maps. The definition of circle-like continuum is similar. We know from [14, Lemma 10] that without loss of generality a circle-like continuum X has an inverse limit representation that satisfies one of the following:

(1) $\deg f_n^{n+1} = 0$, for all n.
(2) $\deg f_n^{n+1} = 1$, for all n
(3) $\deg f_n^{n+1} > 1$, for all n.

These three cases correspond, respectively, to (1) the class of circle-like continua that are also arc-like, hence planar, (2) the class of planar circle-like continua that separate the plane, and (3) the class of nonplanar, circle-like continua. If we let $H^n(X)$ be the nth Čech cohomology group of X, then we find that these three cases correspond, respectively, to (1) $H^1(X) = 0$, (2) $H^1(X) = Z$ and (3) $H^1(X) \neq 0$ and $H^1(X) \neq Z$.

A continuum is acyclic if $H^n(X) = 0$, for $n \geqq 1$. Otherwise, it is cyclic.

A map is a continuous function. We will call any map $\mu: C(X) \to I$ satisfying conditions (1), (2), and (3) of the introductory paragraph a Whitney function.

2. The hyperspace functor. Let K denote the category the objects of which are continua and the morphisms of which are continuous functions. Then there is a hyperspace functor $C: K \to K$ that assigns to each object X of K the continuum $C(X)$ and to each morphism $f: X \to Y$ of K the morphism $C(f): C(X) \to C(Y)$ of K defined by $C(f)(A) = f(A)$, for A belonging to $C(X)$. If (X_i, f_i) is an inverse sequence of continua with bonding maps $f_i^{i+1}: X_{i+1} \to X_i$, then C induces another inverse sequence $(C(X_i), C(f_i))$ with bonding maps $C(f_i^{i+1}): C(X_{i+1}) \to C(X_i)$. Jack Segal **[15]** has proved that the functor C is continuous with respect to inverse limits, that is, if $X = \lim(X_i, f_i)$, then $C(X)$ is homeomorphic to $\lim(C(X_i), C(f_i))$.

Whenever one has a functorially-induced inverse sequence, one must check to see whether or not the induced bonding maps are surjective. Consider the following definition, due to M. K. Fort, Jr. and J. Segal [7].

DEFINITION 2.1. A continuum X is said to have a hyper-onto representation if there is an inverse sequence (X_i, f_i), where each X_i is a polyhedron, such that X is homeomorphic to $\lim(X_i, f_i)$ and each induced map $C(f_i^{i+1}): C(X_{i+1}) \to C(X_i)$ is surjective.

Clearly, as remarked by Fort and Segal, each arc-like continuum has a hyper-onto representation. We shall show next that each circle-like continuum also has such a representation.

PROPOSITION 2.2. *Each circle-like continuum has a hyper-onto representation.*

Proof. Let X be a circle-like continuum. If $H^1(X) = 0$, then X, being arc-like, has a hyper-onto representation as an inverse limit of arcs. If $H^1(X) \neq 0$, then let (X_i, f_i) be an inverse limit representation of X such that each factor space X_i is a circle and each bonding map $f_i^{i+1}: X_{i+1} \to X_i$ is essential. Suppose that for some i, A is an element of $C(X_i)$ and that no subcontinuum B of X_{i+1} has the property that $f_i^{i+1}(B) = A$. We shall show that this leads to a contradiction by defining a homotopy between f_i^{i+1} and a null-homotopic map.

Since f_i^{i+1} is a surjective function, A must be a nondegenerate subcontinuum of X_i. Let a_1 and a_2 be the endpoints of A. Let D be a component of $(f_i^{i+1})^{-1}(A)$. Then $D = [d_1, d_2]$. Without loss of generality, suppose that $f_i^{i+1}(d_i) = a_1$. Then $f_i^{i+1}(d_2) = a_1$, for otherwise

$f_i^{i+1}(D) = A$. The homotopy H will be defined on D to have the following properties:

(1) $H_0(x) = f_i^{i+1}(x)$
(2) $H_1(x) = a_1$
(3) $H_t(d_1) = H_t(d_2) = a_1$, for each t.

Define H similarly on the other components of $(f_i^{i+1})^{-1}(A)$. If $f_i^{i+1}(x)$ is not an element of A, then define $H(x,t) = f_i^{i+1}(x)$, for all t, $0 \leqq t \leqq 1$. Then $H_1(X_{i+1})$ contains no point of (a_1, a_2), so H_1 is nullhomotopic. This contradicts the fact that $f_i^{i+1} = H_0$ is essential.

Notice the important distinction between the manner of proof of the preceding proposition for cyclic and acyclic circle-like continua. Each acyclic, circle-like continuum has a hyper-onto representation as an inverse limit of arcs, but such a representation may not be obtainable from an inverse sequence of circles; indeed this is the crux of the matter when some later theorems that hold for cyclic, circle-like continua are false for acyclic circle-like continua.

If A and B are elements of $C(X)$ and $A \subset B$, then there is a maximal collection $\{A_t : 0 \leqq t \leqq 1\}$ of subcontinua of X such that

(1) $A_0 = A$
(2) $A_1 = B$
(3) if $0 \leqq s \leqq t \leqq 1$, then $A_s \subset A_t$

and such that $\{A_t\}$ forms an arc in $C(X)$ from A to B, provided $A \neq B$. Such a set is called a segment from A to B and is denoted $[A, B]$. There may be many distinct segments between A and B.

The question naturally arises as to whether the image $C(f_i)([P,Q])$ of a segment in $C(X)$ is a segment in $C(X_i)$. The following proposition gives an affirmative answer.

PROPOSITION 2.3. *If $[P,Q]$ is a segment in $C(X)$, then $C(f_i)$ $([P,Q])$ is a segment in $C(X_i)$.*

Proof. Suppose that R and S are points of $[P,Q]$ and that $R \supset S$. Then $f_i(R) \supset f_i(S)$ in X_i. If $C(f_i)(R) = C(f_i)(S)$, then $f_i(R) = f_i(S)$ and hence $C(f_i)([S,R]) = f_i(R)$, where $[S,R]$ is the subsegment of $[P,Q]$ with endpoints R and S. Therefore, $C(f_i)([P,Q])$ is the monotone image of an arc and hence is an arc or a point.

Suppose that M and N are distinct points of $C(f_i)([P,Q])$. Let M' and N' be a pair of points in $[P,Q]$ that are mapped by $C(f_i)$ onto M and N, respectively. Assume that M' is a proper subcontinuum of

N'. Then $f_i(M') \subset f_i(N')$; that is, $M \subset N$. So with proper parametrization, $C(f_i)([P, Q])$ is a segment in $C(X_i)$.

3. Finding arcs in $\mu^{-1}(t)$. In this section, we develop a technique for recognizing arcs and circles in $\mu^{-1}(t)$. This technique provides a very simple proof of the fact that if X is an arc (a circle), then $\mu^{-1}(t)$ is an arc (a circle). This fact was originally proved by Krasinkiewicz **[10]** through considerably more complicated means. The section culminates in a proof of the theorem that if X is arcwise-connected, so is $\mu^{-1}(t)$.

A point a in a space X is said to be accessible from a subset B of X if for each point b in B, there exists an arc A in X with endpoints a and b such that $A - a \subset B$. If C is a subset of X and if each point of C is arcwise accessible from B, then C is said to be arcwise accessible from B. Arcwise accessibility provides the following useful criterion for recognizing simple closed curves in S^2.

Proposition 3.1. **[17, p. 67]**. *A necessary and sufficient condition that a subset M of S^2 should be an S^1 is that it be a common boundary of two disjoint domains, from each of which M is arcwise accessible.*

We shall use two variations of the preceding proposition in dealing with hyperspaces.

Proposition 3.2. *Let A be a continuum that lies in the interior of a 2-cell D. Suppose that $D - A$ consists of two components and that A is arcwise accessible from each. Then A is a simple closed curve.*

Proposition 3.3. *Let A be a subcontinuum of a 2-cell D such that $A \cap \partial D$ contains at least two points. Suppose that A separates D into two components, each containing some point of ∂D, and that A is arcwise accessible from each. Then A is an arc with endpoints a and b belonging to ∂D.*

Now we apply Propositions 3.2 and 3.3 to determine the structure of the continuum $\mu^{-1}(t)$, for X an arc or a simple closed curve. Note that if $X = S^1$, then $C(X)$ is homeomorphic to the disk D in the plane defined by $D = \{(r, \theta): r \leqq 1\}$. The homeomorphism $H: C(X) \to D$ is given by mapping a positively oriented arc $[\phi_1, \phi_2]$ in S^1 to the point $(r, \phi) = (1 - (\phi_2 - \phi_1)/2\pi, \phi_1)$ in D, a point $[\phi_1, \phi_1]$ in S^1 to the point $(1, \phi_1)$ in D, and S^1 itself to the origin.

Theorem 3.4. *If X is a simple closed curve, then $\mu^{-1}(t)$, $0 \leqq t < 1$, is a simple closed curve.*

Proof. If $t = 0$, then $\mu^{-1}(0)$ is the set of singleton continua in X and hence is homeomorphic to X. If $0 < t < 1$, then $\mu^{-1}(t)$ is a continuum in the interior of the 2-cell $C(X)$, for the boundary points of the 2-cell are precisely the singleton subcontinua of X. Furthermore, $C(X) - \mu^{-1}(t)$ consists of 2 components, $\mu^{-1}(t, 1]$ and $\mu^{-1}[0, t)$. We see from the structure of segments that $\mu^{-1}(t)$ is arcwise accessible from $\mu^{-1}(t, 1]$. Also $\mu^{-1}(t)$ is arcwise accessible from $\mu^{-1}[0, t)$, since $\mu^{-1}(0) \cong S^1$ is arcwise connected. Therefore, by Proposition 3.2, A is a simple closed curve.

If $X = [0, 1]$, then $C(X)$ is again a 2-cell. Let D be the triangular disk in the plane with vertices at $(0, 0)$, $(0, 1)$, and $(1, 1)$. Then there is a homeomorphism $G\colon C(X) \to D$ defined by letting G map a subcontinuum $[a, b]$ of C to the point (a, b) in D.

THEOREM 3.5. *If X is an arc, then $\mu^{-1}(t)$, $0 \leqq t < 1$, is an arc.*

The proof of this theorem is similar to that of Theorem 3.4, using Proposition 3.3 in place of Proposition 3.2. Hence we omit it.

Next we use Propositions 3.3 to find arcs in $\mu^{-1}(t)$.

THEOREM 3.6. *Suppose that P and Q are distinct points of $C(X)$ such that $\mu(P) = \mu(Q) = t$ and such that P and Q are not disjoint subcontinua of X. Then there is an arc in $\mu^{-1}(t)$ with endpoints P and Q.*

Proof. Let A be a component of $P \cap Q$. Let S_P be a segment in $C(P)$ from A to P, and let S_Q be a segment in $C(Q)$ from A to Q. Let D be the set of all subcontinua of X that can be expressed as the union of two subcontinua of X, one belonging to S_P and one belonging to S_Q. D is represented in Diagram 1. Then each point of D may be assigned a pair of coordinates from the segments S_P and S_Q, and one may use these coordinates to prove that D is a 2-cell.

We shall show that $\mu^{-1}(t) \cap D$ is an arc with endpoints P and Q. The points P, Q, A, and $P \cup Q$ belong to ∂D. Since $A \neq P$ and $P \neq P \cup Q$, it follows from the structure of segments that $\mu^{-1}(t) \cap D$ separates D into two arc components, one containing A and one containing $P \cup Q$. Each point of $\mu^{-1}(t) \cap D$ is accessible by a segment from both A and $P \cup Q$. Hence the hypotheses of Proposition 3.3 are satisfied, and P and Q are the endpoints of the arc $\mu^{-1}(t) \cap D$.

THEOREM 3.7. *Suppose that P and Q are points of $C(X)$ such that $\mu(P) = \mu(Q) = t$ and such that P and Q are disjoint subcontinua of X. If there exists an arc in X that intersects both P and Q, then there exists an arc in $\mu^{-1}(t)$ with endpoints P and Q.*

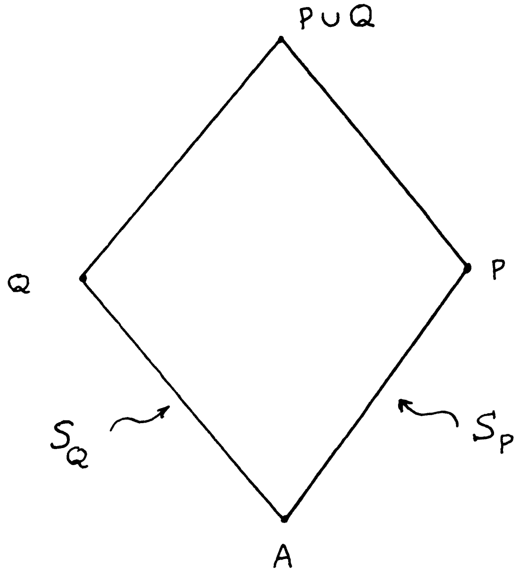

DIAGRAM 1

Proof. Let A be an arc in X irreducible from P to Q with end points p and q. Let S_P be a segment in $C(P)$ from p to P, and let S_Q be a segment in $C(Q)$ from q to Q. We will identify a 2-cell D in $C(X)$ that is the union of four 2-cells, D_1, D_2, D_3 and D_4.

D_1 is $C(A)$. D_2 is the set of all subcontinua of $P \cup A$ that can be expressed as the union of a subcontinuum of A and a subcontinuum of P that is a point of S_P. Each point of D_2 may be assigned two coordinates, and D_2 is clearly a disk. Similarly D_3 is the set of all subcontinua of $Q \cup A$ that can be expressed as the union of a subcontinuum of A and a subcontinuum of Q that is a point of S_Q. Finally, let T_P be the segment in D_2 from A to $P \cup A$, and let T_Q be the segment in D_3 from A to $Q \cup A$. Define D_4 to be the set of all subcontinua of X that can be expressed as the union of two continua, one a point of T_P and the other a point of T_Q.

As indicated in Diagram 2, $D = D_1 \cup D_2 \cup D_4 \cup D_4$ is a 2-cell. The existence of an arc from P to Q now follows from reasoning similar to that of the proof of the previous theorem; we omit the details.

The next theorem is an immediate corollary of the previous two theorems.

THEOREM 3.8. *If X is an arcwise-connected continuum, so is* $\mu^{-1}(t)$, $0 \leqq t \leqq 1$.

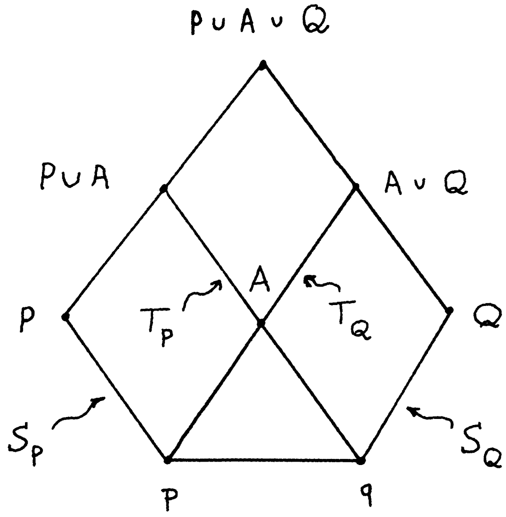

DIAGRAM 2

4. Continuum-valued maps and a cohomological equivalence for circle-like continua. In this section we define for an arbitrary continuum X an upper semi-continuous, continuum-valued map γ_t of X onto $\mu^{-1}(t)$ and an upper semi-continuous, continuum-valued map σ_t of $\mu^{-1}(t)$ onto X. We then apply techniques of Howard Cook **[4]** to show that these maps yield a cohomological equivalence between X and $\mu^{-1}(t)$ in the case that X is a circle-like continuum. At the end of the section, various corollaries of this theorem are derived.

Perhaps it is desirable, before defining the multivalued maps, to give an example showing that the existence of single-valued maps having these properties is out of the questtion.

EXAMPLE 4.1. Let X be the arc-like plane continuum defined by

$$\begin{array}{ll} y = \sin 1/x, & 0 < x \leqq 1/2\pi \\ -1 \leqq y \leqq 3, & x = 0 \\ y = 2 + \sin 1/x, & -1/2\pi \leqq x \leqq 0 \end{array}$$

X is pictured in Diagram 3. Let ϵ be a small, positive number. Then we may choose μ such that $\mu^{-1}(\epsilon)$ is homeomorphic to the arc-like continuum pictured in Diagram 4, while $\mu^{-1}(1-\epsilon)$ is an arc.

Then $\mu^{-1}(1-\epsilon)$ cannot be mapped onto X, and X cannot be mapped onto $\mu^{-1}(\epsilon)$.

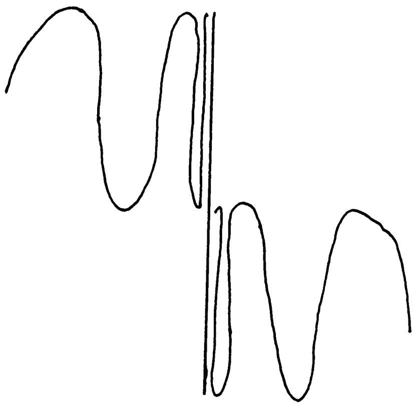

DIAGRAM 3

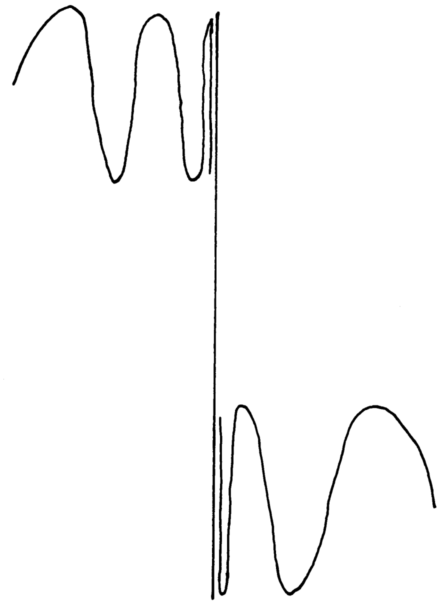

DIAGRAM 4

The next thing we will do is to define the continuum-valued function $\gamma_t : X \to \mu^{-1}(t)$. For each point p in X, let $C_p^t = \{A \in \mu^{-1}(t) \colon p \in A\}$.

THEOREM 4.2. *C_p^t is an arcwise-connected continuum.*

Proof. First we show that C_p^t is closed. Let A be a limit point of C_p^t. If p is not in A, let $\epsilon = d(p, A)$. If $p \in B \in C_p^t$, then $\rho(A, B) \geqq \epsilon$.

Finally we show that C_p^t is arcwise-connected. Let A and B be points of C_p^t. Let D be the component of $A \cap B$ containing p. Let I_D^A be a segment from D to A and I_D^B a segment from D to B. Then I_D^A and I_D^B span a disk E and $\mu^{-1}(t) \cap E$ is an arc from A to B, by arguments similar to those of the preceding section.

Define $\gamma_t : X \to \mu^{-1}(t)$ by $\gamma_t(p) = C_p^t$.

THEOREM 4.3. *The continuum-valued function $\gamma_t : X \to \mu^{-1}(t)$ is upper semi-continuous.*

Proof. Let $\{p_n\}$ be a sequence of points of X converging to p, and let Y be an element of lim sup $C_{p_n}^t$. Then, by taking a subsequence if necessary, there exists a sequence $\{Y_n\}$ of continua such that Y_n converges to Y in $C(X)$ and $Y_n \in C_{p_n}^t$. Since $Y_n \to Y$ and $p_n \to p$, we have $p \in Y$ and then $Y \in C_p^t$.

The γ_t functions yield the following interesting characterization of hereditarily indecomposable continua.

THEOREM 4.4. *The following statements are equivalent*:
(1) *X is hereditarily indecomposable.*
(2) *$\gamma_t: X \to \mu^{-1}(t)$ is single-valued, for each t.*
(3) *$\mu^{-1}(t)$ contains no arc, for each t.*

Proof. (1) $\Rightarrow$ (3). Kelley **[9]** has shown that if X is hereditarily indecomposable, then so is each $\mu^{-1}(t)$. No hereditarily indecomposable continuum contains an arc.

(3) $\Rightarrow$ (2). This is clear, since each C_p^t is arcwise-connected.

(2) $\Rightarrow$ (1). Suppose X contains the decomposable continuum $A \cup B$. Let p be a point of $A \cap B$, and let D be the component of $A \cap B$ containing p. Proceed similarly to the proof of Theorem 3.6. to find an arc in some $\mu^{-1}(t)$ that is contained in C_p^t. This contradicts the assumption that γ_t is single-valued.

In order to apply the function γ_t to circle-like continua, we need to know that the values of γ_t are proper subcontinua of $\mu^{-1}(t)$, provided $t < 1$. We accomplish this via the following proposition.

PROPOSITION 4.5. *If p is a point of the non-arc-like, circle-like continuum X, then for any $\epsilon > 0$, there exists a subcontinuum Y of $X \backslash \{p\}$ with the property that $\rho(Y, X) < \epsilon$.*

This proposition is a consequence of Proposition 2.2. Perhaps it is more easily seen by an appeal to chaining arguments rather than inverse limits. We omit the details of the proof.

COROLLARY 4.6. *If X is a non-arc-like, circle-like continuum, then C_p^t is a proper subcontinuum of $\mu^{-1}(t)$, for any point p of X and real number $t < 1$.*

Proof. Let ϵ be the distance in $C(X)$ between the sets $\mu^{-1}(t)$ and $\{X\}$. By Proposition 4.5, there exists a continuum Y contained in $X \backslash \{p\}$ with the property that $\rho(X, Y) < \epsilon$. Let Z be a subcontinuum of Y that is an element of $\mu^{-1}(t)$. Then Z does not belong to C_p^t.

Next we define the multi-valued map $\sigma_t: \mu^{-1}(t) \to X$. For $A \in \mu^{-1}(t)$, define $\sigma_t(A) = A$. Clearly σ_t is an upper semi-continuous, continuum-valued map. Furthermore, $\sigma_t(A)$ is a proper subcontinuum of X unless $A = X$.

We now are ready to state and prove the main theorem of this section.

THEOREM 4.7. *If X is a non-arc-like, circle-like continuum, then $\mu^{-1}(t)$ is a non-arc-like, circle-like continuum that has the same Čech cohomology groups as X.*

Proof. Krasinkiewicz **[10]** has proved that $\mu^{-1}(t)$ is a non-arc-like, circle-like continuum. We have the upper semi-continuous mapping γ_t of X onto $\mu^{-1}(t)$ satisfying the condition $\gamma_t(x)$ is a proper subcontinuum of $\mu^{-1}(t)$, for each x in X. We also have the upper semi-continuous map σ_t of $\mu^{-1}(t)$ onto X, and each value of σ_t is a proper subcontinuum of X. One can now show that X and $\mu^{-1}(t)$ are cohomologically equivalent, using a proof similar to that of Theorem 9 of **[4]**.

COROLLARY 4.8. *If X is a nonplanar, circle-like continuum, so is $\mu^{-1}(t)$.*

Proof. Results of Bing **[1]** as reformulated by McCord **[11]** state that a circle-like continuum Y is planar if and only if $H^1(X) = 0$ or $H^1(X) = Z$.

COROLLARY 4.9. *If X is a nonplanar, circle-like continuum, then $\mu^{-1}(t)$ is indecomposable.*

Proof. Tom Ingram **[8]** has shown that each nonplanar, circle-like continuum is indecomposable.

The next corollary answers a problem of J. Krasinkiewicz **[10**, p. 13**]**.

COROLLARY 4.10. *If X is a planar, circle-like continuum, then $\mu^{-1}(t)$ is a planar continuum.*

Proof. If X is also arc-like, then $\mu^{-1}(t)$ is arc-like **[10]** and hence planar **[2]**. If X is not arc-like, then $H^1(X) = Z$ and thus $H^1(\mu^{-1}(t)) = Z$, and so $\mu^{-1}(t)$ is a planar, circle-like continuum.

A pseudo-solenoid is a hereditarily indecomposable, circle-like continuum that is not a pseudo-arc. A pseudo-circle is a planar pseudo-solenoid.

COROLLARY 4.11. *If X is a pseudo-solenoid, then $\mu^{-1}(t)$ is homeomorphic to X. In particular, if X is a pseudo-circle, then $\mu^{-1}(t)$ is a pseudo-circle.*

Proof. Kelley **[9]** has shown that $\mu^{-1}(t)$ is hereditarily indecomposable whenever X is. From Theorem 4.7, we find that $\mu^{-1}(t)$ is a circle-like continuum cohomologically equivalent to X. L. Fearnley **[6]** has shown that pseudo-solenoids are classified by cohomology, so $\mu^{-1}(t)$ is homeomorphic to X.

Other continua having the property that X and $\mu^{-1}(t)$ are homeomorphic are the arc, the circle, and the pseudo-arc.

COROLLARY 4.12. *If X is a circle-like continuum, then $\mu^{-1}(t)$ has the same Čech cohomology groups as X.*

Proof. If X is arc-like, then so is $\mu^{-1}(t)$, so both continua have trivial cohomology groups. Otherwise, the corollary follows immediately from Theorem 4.7.

5. Some properties of X that are not inherited by $\mu^{-1}(t)$. In this section, we first prove that if X is a decomposable, circle-like continuum with trivial cohomology groups, then $\mu^{-1}(t)$ is not circle-like, for some t. We also define a class of indecomposable, circle-like continua X satisfying $H^1(X) = Z$ such that $\mu^{-1}(t)$ is decomposable, for all t between 0 and 1. Therefore, neither the property of being circle-like nor of being indecomposable is necessarily inherited from X by $\mu^{-1}(t)$. We also give examples of a one-dimensional, hereditarily unicoherent, tree-like continuum such that $\mu^{-1}(t)$ possesses none of these properties and of an atriodic, unicoherent continuum such that $\mu^{-1}(t)$ possesses neither of these properties.

Consider the class of circle-like continua X satisfying $H^1(X) = 0$. This is precisely the class of continua that are simultaneously arc-like and circle-like. C. E. Burgess [3] has studied such continua and divided them into two types. First, each indecomposable, arc-like continuum is circle-like. Second, each decomposable, arc-like continuum that is also circle-like is the union of two indecomposable, arc-like continua. We will find next that continua of the latter type form a singularity; there always exists a real number t such that $\mu^{-1}(t)$ is not circle-like.

THEOREM 5.1. *Let X be a decomposable continuum that is both arc-like and circle-like. Then there exists t_0 in $[0, 1)$ such that, for $t > t_0$, $\mu^{-1}(t)$ is not circle-like.*

Proof. According to Burgess [3], X is 2-indecomposable; in particular, X is the union of two indecomposable chainable continua X_1 and X_2, and $X_1 \cap X_2$ is a chainable continuum that is an end continuum of both X_1 and X_2. Therefore there is a unique segment in $C(X_i)$ from $X_1 \cap X_2$ to X_i. Thus, the subcontinua of $C(X)$ that contain $X_1 \cap X_2$ form a 2-cell in $C(X)$. By arguments similar to those previous, one finds that for $1 > t > \mu(X_1 \cap X_2)$, $\mu^{-1}(t)$ intersects this 2-cell in an arc. We know that $\mu^{-1}(t)$ intersects $C(X_i)$ in a chainable continuum or a point. Therefore, for $1 > t > \mu(X_1 \cap X_2)$, $\mu^{-1}(t)$ is an arc, a chainable continuum formed by the union of an arc and another chainable continuum, or a chainable continuum formed by the union of 2 chainable continua and an arc. In none of these cases is $\mu^{-1}(t)$ circle-like. This proves the theorem.

Consider next certain quotient spaces of indecomposable continua.

THEOREM 5.2. *If p and q are two points of the indecomposable continuum X, then the continuum Y obtained from X by identifying p and q is also indecomposable.*

Proof. Y is irreducible between each two points of an uncountable set.

We are particularly interested in the structure of $\mu^{-1}(t)$ for those circle-like continua that are obtained by identifying a pair of opposite endpoints of an arc-like continuum. Recall that p and q are opposite endpoints of an arc-like continuum X provided there is a representation of X as an inverse limit of unit intervals such that p has coordinates all zero and q has coordinates all one. This definition is equivalent to Bing's definition that p and q are opposite endpoints of an arc-like continuum X provided that for each positive number ϵ, there is an ϵ-chain covering X such that only the first link contains p and only the last link contains q.

We have the following theorem concerning such continua.

THEOREM 5.3. *If Y is a circle-like continuum obtained by identifying a pair of opposite endpoints of an arc-like continuum X, then $\mu^{-1}(t)$ contains an arc with nonempty interior, for all t satisfying $0 < t < 1$.*

Proof. Let p_1 and p_2 be the pair of opposite endpoints of X that are identified to the point p of Y. According to Bing [**2**, p. 660], if each of two subcontinua of X contain p_i, one of the subcontinua of X contains the other. This implies that there is a unique segment $[p_i, X]$ in $C(X)$ from p_i to X. Since X is irreducible between p_1 and p_2, we find that $[p_1, X] \cap [p_2, X] \times \{X\}$.

We shall show in this paragraph that the set $C_p(Y)$ of all subcontinua of Y that contains p forms a 2-cell in $C(Y)$. Each point of $C_p(Y)$ is of the form $A_t \cup B_s$, where $A_t \in [p_1, X]$ and $B_s \in [p_2, X]$. Consider the function $H: I \times I \to C_p(Y)$ that assigns to the ordered pair (t, s) the continuum $A_t \cup B_s$. Then H is a cellular map of $I \times I$ onto $C_p(Y)$. The only non-degenerate point-inverse is

$$H^{-1}(Y) = \{(1, s): 0 \leqq s \leqq 1\} \cup \{(t, 1): 0 \leqq t \leqq 1\},$$

which is an arc on the boundary of $I \times I$. Therefore, $C_p(Y)$ is a 2-cell.

Finally, it is clear that if Z is any continuum of $C_p(Y)$ and Z can be expressed as $A_t \cup B_s$, where $A_t \in (p_1, X)$ and $B_s \in (p_2, X)$, then there exists a neighborhood N of Z in $C(X)$ such that $N \subset C_p(Y)$. Furthermore, one can use Proposition 3.3, as before, to show that $\mu^{-1}(t) \cap C_p(X)$ is an arc. Therefore $\mu^{-1}(t)$ contains an arc with nonempty interior, for $0 < t < 1$. In particular, $\mu^{-1}(t)$ is locally connected at a continuum of points.

COROLLARY 5.4. *If Y is as above, then $\mu^{-1}(t)$ is decomposable, for* $0 < t < 1$.

EXAMPLE 5.5. Let X be a simple triod (a continuum homeomorphic to the capital letter T.) Then a little reflection shows us that the set of all continua containing the junction point forms a 3-cell and that $C(X)$ is homeomorphic to a 3-cell with three 2-cell "fins". In particular, $C(X)$ is 3-dimensional at the point X, so $\mu^{-1}(t)$ must be 2-dimensional, for t near 1. As a matter of fact, it is simple to choose μ so that $\mu^{-1}(t)$ is a 2-cell, for t near 1. Although X is one-dimensional and hereditarily unicoherent, $\mu^{-1}(t)$ possesses neither of these properties. Clearly $\mu^{-1}(t)$ is not tree-like.

EXAMPLE 5.6. Let X be the standard sin $1/x$-continuum, and let Y be the continuum obtained from X by identifying the points $(0, -1)$ and $(0, 1)$ to a point p. Then Y is an atriodic, unicoherent continuum. For t close to 0, however, $\mu^{-1}(t)$ is homeomorphic to the continuum Z pictured in Diagram 5, a continuum that is neither atriodic nor unicoherent.

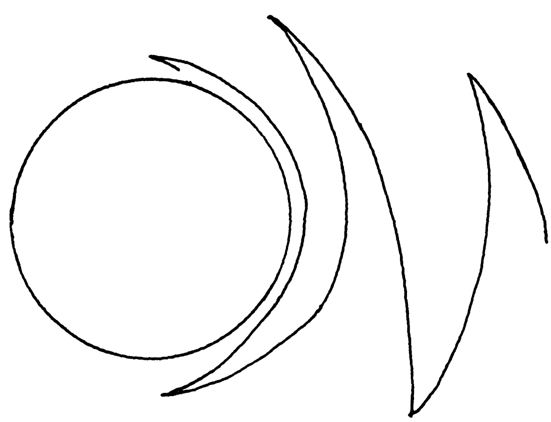

DIAGRAM 5

For t close to 1, $\mu^{-1}(t)$ is an arc. Hence all homology, cohomology, and homotopy groups of $\mu^{-1}(t)$ are trivial. This is obviously not the case for Y.

6. Question. We still have much to learn about the continua $\mu^{-1}(t)$. We think the following questions have special interest.

Question 1. Which continua X have the property that X is homeomorphic to $\mu^{-1}(t)$?

Question 2. If the arc-like continuum X is indecomposable, is $\mu^{-1}(t)$ indecomposable? An affirmative answer to this question, together with the results of this paper, would imply that if X is a circle-like continuum, then $\mu^{-1}(t)$ is circle-like if and only if X is indecomposable or $H^1(X) \neq 0$.

Question 3. How are the other properties of continua reflected in the continua $\mu^{-1}(t)$?

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Received April 19, 1974. During the preparation of this paper, the author was partially supported by a grant from the Tulane Committee on Research.

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