PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

THE INDEX OF A TANGENT 2-FIELD

MARK MAHOWALD

Thomas, using an obstruction theory approach, evaluated the index of a tangent 2-field on M^m, $m \equiv 1(4)$ if M is a spin manifold. Atiyah using the Atiyah-Singer index theorem evaluated the index for all orientable manifolds. The purpose here is to give a proof of Atiyah's result in the spirit of Thomas' work.

Let M be a connected closed smooth orientable manifold of dimension m. Let k be any integer and suppose M admits k vector fields which are linearly independent everywhere except possibly at a finite number of points. The obstruction to making the k vector field linearly independent everywhere is called the index of the k-field and it is an element of

$$H^m(M, \pi_{m-1}(V_{m,k})) \simeq \pi_{m-1}(V_{m,k}).$$

Suppose $m = 2r + 1$ and let

$$\hat{\chi}_2(M) = \left(\dim \bigoplus_{i=0}^{r} H^i(M, Z_2)\right) \bmod 2. \quad \text{In } [5],$$

Thomas proved:

THEOREM. *Let M be a closed connected spin manifold, $m \equiv 1(4)$, $m > 1$ with $W_{m-1}(M) = 0$. Then the index of any 2-field with singularities is*

$$\hat{\chi}_2(M) \in Z_2 = \pi_{m-1}(V_{m,2}).$$

Thomas' method was to calculate the secondary obstruction to a cross section of the association $V_{m,2}$ bundle to the tangent bundle. Atiyah **[1]** showed that if

$$b = \left(\dim \bigoplus_{i=0}^{r} H^i(M, \text{Reals})\right) \bmod 2$$

then the index of a 2-field for any orientable manifold with $W_{m-1}(M) = 0$ is b. Finally, Milnor, Lusztig, and Peterson **[3]** showed the relationship between these results by showing that

$$b + \hat{\chi}_2 = W_2 W_{m-2}.$$

It has always seemed that direct proof, in the spirit of Thomas, should be possible for the Atiyah result. In this paper we will provide such a proof, i.e., we will prove

THEOREM 1. *Let M be a closed connected orientable manifold $m \equiv 1(4)$, $m > 1$ with $W_{m-1}(M) = 0$. Then the index of any 2-field with finite singularities is*

$$(\hat{\chi}_2 + W_2 W_{m-2}) \in Z_2 = \pi_{m-1}(V_{m,2}).$$

2. *Proof of the theorem.* The proof has two key steps. The first is to show that a secondary operation on the Thom class involves the secondary obstruction; and the second step is to evaluate the cohomology operation.

Let $m \equiv 1 \bmod 4$. Then

$$Sq^2 Sq^{m-1} + Sq^m Sq^1 = Sq^{m+1}$$

and, thus, on m-dimensional integral classes $Sq^2 Sq^{m-1} = 0$. Let E be the fiber of the map

$$K(Z, m) \xrightarrow{Sq^{m-1}} K(Z_2, 2m-1).$$

Then the relation $Sq^2 Sq^{m-1} = 0$ defines a class $v \in H^{2m}(E, Z_2)$ which is defined up to a primary operation on the generator of $H^m(E, Z)$.

THEOREM 2.1. *Let $T(M)$ be the Thom complex of $\tau(M)$, where M is a manifold as in Theorem 1. There exists a map f: $T(M) \to E$ such that f^* in dimension m is an isomorphism and $f^*(v) = U \cup (O_2 + W_2 W_{m-2})$ where O_2 is the index of the 2-field.*

This is proved in **[4]**.

THEOREM 2.2. *For the data as in Theorem 2.1, $f^*(v) = \hat{\chi}_2(U \cup \mu)$ where μ generates $H^m(M, Z_2)$.*

This is the new result which we prove in §3. The main theorem is a direct consequence of these two results.

3. *Proof of Theorem 2.2.* Recall that the tangent bundle embeds in a natural way as a neighborhood of the diagonal in $M \times M =$

M^2. Let $j: M^2 \to T(\tau(M))$ be the obvious map. Let $\{\alpha_i; i = 1, \cdots, q\}$ a basis for

$$\bigoplus_{j=0}^{(m-1)/2} H^j(M, Z_2) \quad \text{and} \quad \{\beta\}$$

be the dual basis, i.e.,

$$\alpha_i \cup \beta_j = \delta_{ij}\mu.$$

PROPOSITION 3.1 [Theorem 2.6 **[5]**].

(a) $j^*U = A + tA$ *where* $A = \Sigma_{j=0}^{q}(\alpha_i \otimes \beta_i)$.

(b) $A \cup tA = \hat{\chi}_2(M)\mu \otimes \mu$.

Let $\tilde{\Omega}_m$ *be the secondary operation defined over* $K(Z_2, m)$ *based on* $Sq^2Sq^{m-1} + Sq^1(Sq^{m-1}Sq^1) = 0$.

PROPOSITION 3.2. [Thomas 2.6 **[5]**]. *If* $Sq^{m-1}U = 0$ *then* $Sq^{m-1}A = 0$.

Proof. An easy application of the Cartan formula shows that $Sq^{m-1}A \in H^{m-1}(M, Z_2) \otimes H^m(M, Z_2)$. Thus, $Sq^{m-1}A$ and $Sq^{m-1}(tA)$ are in different graded subgroups of $H^*(M^2)$ and so could add to zero only if each were zero separately.

PROPOSITION 3.3. $Sq^{m-1}Sq^1A = \langle V_r \cup Sq^1V_r[\dot{M}]\rangle\mu \times \mu$ *for any choice of basis* α_i *where* V_r *is the* r *dimensional Wu class.*

Proof. Since $Sq^{m-1}Sq^1A = Sq^{m-1}Sq^1(\Sigma\alpha_j \otimes \beta_j)$ $(\dim \alpha_j = r)$ it suffices to verify that if $H^r(M, Z_2)$ is a vector space of rank t and if N is a linear transformation taking the α_i to the new basis $\bar{\alpha}_i$ then

$$Sq^{m-1}Sq^1(\Sigma\, N\alpha_j \otimes N^*\beta_j) = Sq^{m-1}Sq^1(\Sigma\alpha_j \otimes \beta_j).$$

Moreover, N can be written as a composite of permutations (which obviously leave it invariant) and transformations of the form

$$N_{ij}\alpha_k = \begin{cases} \alpha_k & k \neq j \\ \alpha_i + \alpha_j & k = j \end{cases}.$$

So the lemma is true if it is true for N_{ij}. Now $N^*_{ij}\beta_k = \begin{cases} \beta_k & k \neq i \\ \beta_i + \beta_j & k = i \end{cases}$. Thus the difference between the two sums is easily seen to be 0.

Now notice that $Sq^{m-1}Sq^1(\alpha_i \otimes \beta_i) = V_r Sq^1 \alpha_i \otimes V_r \beta_i$ and since $V_r Sq^1 \alpha_i = (Sq^1 V_r)\alpha_i$, if $Sq^1 V_r = 0$ then the lemma is true. Assume then $Sq^1 V_r \neq 0$, and give a basis for $H^r(M, Z_2)$ by choosing $\alpha_1, \cdots, \alpha_{t-1}$ to span $\langle Sq^1 V_r \rangle^{\perp}$ and filling out to a basis by requiring α_t be dual to $Sq^1 V_r$. Then

$$Sq^{m-1}Sq^1(\Sigma \alpha_j \otimes \beta_j) = Sq^1 V_r \alpha_t \otimes V_r Sq^1 v_r$$

and the lemma follows.

Proposition 3.4. *Theorem* 2.2 *is true if* $Sq^{m-1}Sq^1 A = 0$, i.e., *if* $V_r \cup Sq^1 V_r = 0$.

With the additional hypothesis that $W_2 = 0$ this is exactly what Thomas proved in **[5]**. The proof which follows is the same as Thomas' up to the point where it is shown that the indeterminancy does not kill the argument.

Proof of 3.4. Let (E_1, u, v) be the universal example for $\tilde{\Omega}_m$, i.e., E_1 is a two stage Postnikov system with k-invariants Sq^{m-1} and $Sq^{m-1}Sq^1$ over a $K(Z_2, m)$. The class u is the image of the fundamental class of $H^m(K(Z_2, m))$ in $H^m(E_1)$. The class $v \in H^{2m}(E_1)$ is defined by the relation $Sq^2 Sq^m + Sq^1(Sq^{m-1}Sq^1) = 0$. The hypotheses imply that there is a commutative diagram

$$\begin{tikzcd} & E_1 \arrow[d, "p"] \\ M^2 \arrow[ur, "\bar{A}"] \arrow[r, "M"'] & K(Z_{2,m}) \end{tikzcd}$$

where $A^*(\kappa) = A$ and κ_m is the fundamental class of $K(Z_{2,m})$. Let $t\bar{A}$ be the composite $M^2 \xrightarrow{t} M^2 \xrightarrow{A} E$. Consider the diagram (not necessarily commutative)

$$\begin{tikzcd} T(M) \arrow[r, "\bar{U}"] & E_1 \\ M^2 \arrow[u, "j"] \arrow[r, "(\bar{A}, t\bar{A})"'] & E_1 \times E_1. \arrow[u, "\mu"'] \end{tikzcd}$$

The argument which Thomas used goes as follows: First

$$\mu^*(v) = v \otimes 1 + p^*(\kappa \otimes \kappa) + 1 \otimes v,$$

since v is not primitive, (see **[5]** or **[2]**). Then, since $(\bar{A}, t\bar{A})^*(v \otimes 1) = (\bar{A}, t\bar{A})^*(1 \otimes v)$, we see that

$$(\mu(\bar{A}, t\bar{A}))^* v = A \cup tA = \hat{\chi}_2(M)(\mu \otimes \mu).$$

Now $d(\mu(\bar{A}, t\bar{A}), \bar{U}j)$, the difference class, is a map into

$$K(Z_2, 2m-2) \times K(Z_2, 2m-1)$$

and thus is a pair of cohomology classes, (a, b). It follows from the definition of the secondary operation that

$$(\bar{U}_j)^* v = (\mu(\bar{A}, t\bar{A}))^* v + Sq^2 a + Sq^1 b.$$

Since M is orientable $Sq^1 b = 0$ and since in Thomas' case $Sq^2 W_2(M) = 0$, $Sq^2 a = 0$. What we need to show is that in the case of our diagram the same conclusion holds.

LEMMA 3.5. *Let $(a, b) \in H^{2m-2}(M^2, Z_2) \otimes H^{2m-1}(M^2, Z_2)$ be the pair of cohomology classes $(a, b) = d(v(\bar{A}, t\bar{A}), \bar{U}j)$. The class a is invariant under t^*.*

The proof is given in §4. We continue the proof of 3.4.

Thus if $(a, b) = d(v(\bar{A}, t\bar{A}), j\bar{U})$ then a is a symmetric class, i.e.,

$$a = a_1 \otimes \mu + a_2 \otimes a_2 + \mu \otimes a_1.$$

Now $Sq^2 a = 0$ if a is symmetric; and, therefore, if we use the diagram * with the maps as given we see that

$$(\bar{U}j) * v = \hat{\chi}_2(M)(\mu \otimes \mu).$$

This is 2.2 under the hypothesis of 3.4.

We now consider the case where $V_r \cup Sq^1 V_r \neq 0$. Let $A' = A - V_r \otimes Sq^1 V_r$. Then $j^* U = A' + tA' + Sq^1(V_r \otimes V_r)$. Let (E, u, v) be the universal example for the operation Ω based on the relation $Sq^2 Sq^{m-1} = 0$ which holds on integral classes. The class $u \in H^m(E, Z)$ is the fundamental class and $v \in H^{2m}(E, Z_2)$ is based on the relation. Let $f\colon M^2 \to E$ be such that $f^* u = A' + tA'$ and suppose $f = -tf$. Then $\Omega(A' + tA') = (\hat{\chi}(M) - 1)(\mu \otimes \mu)$. Note that Ω is also defined on $Sq^1(V_r \otimes V_r)$. Let E_2 be the fiber of the map $K(Z_2, m-1) \xrightarrow{\delta Sq^{m-3}} K(Z, 2m-3)$. Let u_2 be the fundamental class. Suppose a map defining Ω on $Sq^1(V_r \otimes V_r)$ factors $M^2 \xrightarrow{k} E_2 \xrightarrow{g} E$ where $g^* u = Sq^1 u_2$ and $k^* u_2 = V_r \otimes V_r$. The indeterminancy of the value of Ω via such factorization is $k^*(Sq^2 H^{2m-2}(E_2))$ but it is easy to see that $H^{2m-2}(E_2)$ is generated by primary operations on u_2 and primary operations on a symmetric class are symmetric and thus $k^*(Sq^2H^{2m-2}(E_2))=0$. Thus to complete the proof of 2.2 we need to show that k exists, (Lemma 3.6), and we need to evaluate Ω on such a factorization (Lemma 3.7).

LEMMA 3.6. $\delta Sq^{m-3}(V_r \otimes V_r)=0$

Proof. Since $W_{m-1}(M)=0$ and $W_{m-2}(M)$ is the reduction of an integer class $\delta_* W_{m-3}(M)$ we see that $W_{2m-4}(M\otimes M)=W_{m-2}(M)\otimes W_{m-2}(M)$ is the restriction of an integer class and so $\delta(W_{2m-4}(M\otimes M))=0$ but $\delta(W_{2m-4}(M\otimes M))=\delta Sq^{m-3}(V_r\otimes V_r)$.

LEMMA 3.7. *Let c be a class of dimension $m-1$ with $\delta Sq^{m-3}c=0$, where δ is the Bockstein $H^*(\ ,Z_2)\to H^{*+1}(\ ,Z)$. Then (E,u,v) is defined on Sq^1c and equals $Sq^{m-1}Sq^2c$ modulo a primary operation on Sq^1c.*

This is proved in §5.

This finishes the proof since $Sq^{m-1}Sq^2(V_r\otimes V_r)=Sq^rSq^1V_r\otimes Sq^rSq^1V_r$ and $Sq^rSq^1V=Sq^2Sq^{r-1}V_r$. Now $Sq^rSq^1V\neq 0$ iff $V_r\cup Sq^1V_r\neq 0$ and iff $Sq^2Sq^{r-1}V_r=Sq^2W_{m-2}\neq 0$ but $V_2=W_2$ and if $V_r\cup Sq^1V_r\neq 0$, $Sq^{m-1}Sq^2(V_r\otimes V_r)\neq 0$ and $W_2W_{m-2}\neq 0$. This completes the proof.

4. *Proof of* 3.5. Let $\bar{E}$ be the fiber of the map $K(Z_2,m)\xrightarrow[Sq^{m-1}]{} K(Z_2,2m-1)$. Let $[X]^k$ be a Z_2 homology skeleton of the space X, i.e., $i^*\colon H^j(X,Z_2)\to H^j([X]^k,Z_2)$ is an isomophism for $j\leqq k$ and $H^j([X]^k,Z_2)=$ for $j>k$. Then

$$[[M^2/[M^2]^{m-1}]^{2m-1},\bar{E}]\cong[\Sigma^{-2}([M^2/[M^2]^{m-1}]^{2m-2}),\Omega^2\bar{E}]$$

and

$$[M^2/[M^2]^{m-1},\bar{E}]\cong[[M^2/[M^2]^{m-1}]^{2m-2},\bar{E}].$$

Therefore

$$\Sigma^{-2}[M^2/[M^2]^{m-1}]\Omega^2\bar{E}\cong[M^2/[M^2]^{m-1},\bar{E}]=A.$$

This isomophism is not canonical since it depends on the particular desuspension used. Suppose we choose one so that j desuspends to

$$j'\colon \Sigma^{-2}([M^2/[M^2]^{m-1}])^{2m-2}\to\Sigma^{-2}([T(M)]^{2m-2}).$$

Since $\Omega^2\bar{E}=K(Z_2,m-2)\times K(Z_2,2m-4)$ we see that A is isomorphic to some extension of $H^m(M^2,Z_2)$ by $H^{m-2}(M^2,Z_2)$. The extension is determined by the loop multiplication in $\Omega^2\bar{E}$.

The following lemma is an easy calculation.

LEMMA 4.1. *For any class $a \in A$ represented by (a_1, a_2) with $a_1 \in H^m(M^2, Z_2)$, $2a$ is represented by $(0, Sq^{m-2}a_1)$.*

Since t^* on (Imj^*) is fixed and since $t^*Sq^{m-2}a = Sq^{m-2}t^*a$, the subset in $(H^m(M^2, Z_2), H^{2m-2}(M, Z_2))$ consisting of classes which are invariant under t^* is subgroup. Let $E_1 \xrightarrow{P} \bar{E}$ be the natural projection. Clearly $j\bar{U}P$ and $(\bar{A} + t\bar{A})P$ are maps in this subgroup and their difference is a where $(a, b) = d(j\bar{U}, \bar{A} + t\bar{A})$. Hence, $t^*a = a$.

5. *Proof of* 3.7. We will need to study several two stage Postnikov systems simultaneously and so some additional notation is needed. Let β be a vector of primary operations and $K(G)$ a generalized Eilenberg-MacLane space

$$K(G) = \Pi K(G_i, i).$$

Let $E_m(\beta, g)$ be the fiber of the map

$$K(Z_q, m) \xrightarrow{\beta} K(G).$$

For our purposes q is either 0 or 2. We will use u_m *to represent the characteristic class in* $H^m(E_m)$. If $\alpha\beta = 0$ is a relation on m-dimensional class then there is a class $v(\alpha) \in H^*(E_m)$ based on this relation. The triple $(E_m(\beta, q), u, v(\alpha))$, thus, represents the universal example for a secondary operation defined on a class $a \in H^m(X, Z_q)$ with $\beta a = 0$. Note also that $v(\alpha)$ could belong to different $E(\beta, q)$. For example $Sq^2Sq^{m-1} = 0$ and $Sq^2Sq^{m-2} = 0$ on $m-1$ dim integer classes so $v(Sq^2) \in H^*(E(S_q^{m-1}, 0))$ and a different $v(Sq^2) \in H^*(E(Sq^m, 0))$. It is usually clear from the context.

The proof of 3.7 uses the following diagram

5.1
$$\begin{tikzcd}
 & E_{m-1}(\delta Sq^{m-3}) \arrow[rr, "i_1"] \arrow[dd, "\iota_1"] & & K(Z_2, 2m-2) \arrow[dd, "\iota_2"] \\
\Sigma^2 K(Z, m-3) \arrow[ur, "i_1'"] \arrow[dr, "i'"'] & & & \\
 & K(Z, m-1) \arrow[rr, "j_1 \circ \bar{k}"] \arrow[dr, "\rho_1"'] & & E_{2m-3}(Sq^2, 0) \arrow[dl, "\rho_2"] \\
 & & K(Z, 2m-3) &
\end{tikzcd}$$

The maps are defined as follows:

$$j^*u_m = \delta u_{m-1};\ j_1^*u_{2m-3} = \delta Sq^{m-3};\ k^*u_{m-1} = u_{m-1}.$$

First we need to prove the existence of the diagram. The map j is the one induced from the diagram

5.2
$$\begin{tikzcd}
E_m(Sq^{m-1}, 0) \arrow[r] & K(Z_{2,m}) \arrow[r, "Sq^{m-1}"] & K(Z_2, 2m-1) \\
E_{m-1}(Sq^{m-1}Sq^1, 2) \arrow[u, "j"] \arrow[r] & K(Z_2, m-1). \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

The map j_1 is induced rom the diagram

5.3
$$\begin{tikzcd}
E_{m-1}(Sq^{m-1}Sq^1, 2) \arrow[r] \arrow[d, "j_1"] & K(Z_{2,m-1}) \arrow[r, "Sq^{m-1}Sq^1"] \arrow[d, "\delta Sq^{m-3}"] & K(Z_2, 2m-1) \\
E_{2m-3}(Sq^2, 0) \arrow[r] & K(Z, 2m-3) \arrow[ur, "Sq^2"'] &
\end{tikzcd}$$

together with the observation that $Sq^2\delta Sq^{m-3} = Sq^{m-1}Sq^1$ on $m-1$ dimensional classes, $m \equiv 1(4)$.

The map k exists because of the same relation. The map i is the double adjoint and since $i^*\delta Sq^{m-3}u = 0$ the lifting $\bar{\iota}$ exists.

Lemma 3.7 can be rephrased in this notation by the following.

PROPOSITION 5.4. *The class* $v(Sq^2)$ *can be chosen so that* $k^*j^*v(Sq^2) = Sq^{m-1}Sq^2u_{m-1}$.

The first formula we need is

$$j^*v(Sq^2) = j_1^*(v(Sq^2) + p_1^*(\gamma).$$

This follows directly from diagram 5.2 and 5.3. Indeed, either diagram allows one to define an operation in $E_{m-1}(Sq^{m-1}Sq^1, 2)$ based on the relation $Sq^2Sq^{m-1}Sq^1 = 0$. These two differ by some class in the base.

The second formula we need is $k^*j_1^*(v(Sq^2)) = 0$ modulo the indetermanancy, i.e., there is a choice of k such that the formula is true. This implies that $k^*j^*v(Sq^2) = k^*p_1^*\gamma$. We shall be finished when we evaluate

PROPOSITION 5.5. $\gamma = Sq^{m-1}Sq^2u_{m-1}$.

Proof. The map $K(Z, m-1) \xrightarrow[k']{} K(Z_2, m-1)$ lifts to a map $\bar{k}\colon K(Z, m-1) \to E_{m-1}(Sq^{m-1}Sq^1, 2)$. Clearly, $\bar{k}^* j^* v(Sq^2) = 0$. Thus, $\bar{k}^* j_1^* v(Sq^2) = \gamma u_{m-1}$. Note that anything which is lost in γ by evaluating it on an interger class is part of the ambiguity in defining $v \in H^{2m}(E_m(Sq^{m-1}, 0))$.

We have the following diagram

$$\begin{tikzcd}
 & E_{m-1}(\delta Sq^{m-3}) \arrow[rr, "i_1"] \arrow[dd, "t_1"] & & K(Z_2, 2m-2) \arrow[dd, "t_2"] \\
\Sigma^2 K(Z, m-3) \arrow[ur, "i'_1"] \arrow[dr, "i'"'] & & & \\
 & K(Z, m^{-1}) \arrow[rr, "j_1 \circ \bar{k}"] \arrow[dr, "\rho_1"'] & & E_{2m-3}(Sq^2, 0) \arrow[dl, "\rho_2"] \\
 & & K(Z, 2m-3) &
\end{tikzcd}$$

A direct check of the appropriate exact sequence shows that

$$i_1^* \kappa_{2m-2} = v(Sq^2).$$

It follows from [4] that $i_1'^*(v(Sq^2)) = \sigma^2(\kappa \cup Sq^2\kappa)$. Since $\iota_2^* v(Sq^2) = Sq^2 \kappa_{2m-2}$, we see that

$$\iota_1^*(j_1 \circ \bar{k})^* v(Sq^2) = Sq^2[v(Sq^2)].$$

Since $\ker i'^* = \ker \iota_1^*$ in this dimension we have

$$\begin{aligned} i'^*(j_1 \circ \bar{k})^* v(Sq^2) &= Sq^2(\sigma^2(\kappa \cup Sq^2\kappa) \\ &= Sq^{m-1}Sq^2(\sigma^2\kappa). \end{aligned}$$

Thus, $(j_1 \circ \bar{k})^* v(Sq^2) = Sq^{m-1}Sq^2\kappa_{m-1}$. This proves the proposition and completes the proof of the theorem.

It is interesting to note that the above argument proves the following theorem.

THEOREM 5.6. *In* $H^*(K(Z, m-1)), m \equiv 1(4)$, $\varphi_{1,1}(\delta Sq^{m-4}) = Sq^{m-1}Sq^2$ mod *the indeterminancy where* $\varphi_{1,1}$ *is the secondary operation defined on integer classes based on* $Sq^2Sq^2 = 0$.

REFERENCES

1. M. Atiyah, *Vector fields on manifolds*, Arberlsgemeinschaft fur Fosschuny des Landes Nordrhein Westfalen, Hept. 200.
2. E. Brown and F. Peterson, *Whitehead products and chomology operations*, **Quart. J. Math., 15** (1964), 116–120.
3. G. Lusztig, J. Milnor, and F. Peterson, *Semi-characteristic and cobordism*, Topology, **8** (1969), 357–360.
4. M. Mahowald and F. Peterson, *Secondary operations on the Thom class*, Topology, **2** (1964), 367–377.
5. E. Thomas, *The index of a tangent 2–field*, Comment. Math. Helv., **42** (1967), 86–110.

Received December 24, 1973. This work was supported in part by the NSF GP 25335.

NORTHWESTERN UNIVERSITY