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DUAL MIXED VOLUMES

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A concept dual to the mixed volumes of Minkowski is introduced. Duals to the classical mixed volume inequalities of Minkowski, Fenchel and Aleksandrov are presented. As an application of this work a sharp isoperimetric inequality relating the mean width of a convex body and the cross-sectional measures of its polar body is obtained. This inequality implies that of all convex bodies of a given mean width the n-ball (centered at the origin) is the one whose polar body has minimal cross-sectional measures of any index. It further gives a sharp lower bound for the product of the mean widths of a convex body and its polar body.

The setting for this paper will be Euclidean n-space R^n. Convex, compact sets with nonempty interiors are called convex bodies. All convex bodies are assumed to contain the origin in their interiors and the space of all such convex bodies with the Hausdorff topology will be denoted by $\mathcal{K}_n$. Convex bodies will be denoted by capital letters such as A, B, K. Vectors in R^n are denoted by lower case letters such as a, u, x. Scalars are denoted by lower case Greek letters such as α, μ, λ.

The unit $n-1$ sphere and the unit n-ball are denoted by Ω and U, respectively. The volume of the unit n-ball is denoted by ω_n. For a convex body A the n-dimensional volume and surface area will be denoted by $V(A)$, and $S(A)$, respectively. The diameter and mean width of A will be denoted by $D(A)$ and $\bar{b}(A)$, respectively. The radial function of A ρ_A is defined on Ω by:

$$\rho_A(u) = \operatorname{Sup}\{\lambda > 0 \mid \lambda u \in A \quad u \in \Omega\}.$$

The polar body of A (with respect to the unit sphere centered at the origin) will be denoted by A^*.

Given n convex bodies $A_1, \cdots, A_n$ their mixed volume will be denoted by $V(A_1, \cdots, A_n)$. For convex bodies A and B we use $V_i(A, B)$ to denote

$$V(\underbrace{A, \cdots, A}_{n-i}, \underbrace{B, \cdots, B}_{i})$$

and $W_i(A)$ to denote $V_i(A, U)$. For reference see Bonnesen and Fenchel **[4]**.

We begin by defining the dual mixed volumes.

DEFINITION 1.

$$\tilde{V}(A_1, \cdots, A_n) = \frac{1}{n} \int_{\Omega} \rho_{A_1}(u) \cdots \rho_{A_n}(u)\, dS(u) \qquad [A_i \in \mathcal{K}_n]$$

where dS signifies the area element on Ω.

DEFINITION 2.

$$\tilde{V}_i(A, B) = \tilde{V}(\underbrace{A, \cdots, A}_{n-i}, \underbrace{B, \cdots, B}_{i}) \qquad [A, B \in \mathcal{K}_n].$$

The dual cross-sectional measures are the special dual mixed volumes defined by:

DEFINITION 3.

$$\tilde{W}_i(A) = \tilde{V}_i(A, u) \qquad [A \in \mathcal{K}_n].$$

By definition 1, $\tilde{V}$ is a map

$$\tilde{V} : \underbrace{\mathcal{K}_n \times \cdots \times \mathcal{K}_n}_{n} \to R.$$

We list some of its elementary properties.

(1) $\tilde{V}$ is continuous;

(2) $\tilde{V}(A_1, \cdots, A_n) > 0$;

(3) $\tilde{V}(\lambda_1 A_1, \cdots, \lambda_n A_n) = \lambda_1 \cdots \lambda_n \tilde{V}(A_1, \cdots, A_n) \qquad [\lambda_i > 0]$;

(4) If $A_i \subset B_i$, for all i, then $\tilde{V}(A_1, \cdots, A_n) \leqq \tilde{V}(B_1, \cdots, B_n)$ with equality iff $A_i = B_i$ for all i;

(5) $\tilde{V}(A, \cdots, A) = V(A)$.

By definition 3, $\tilde{W}_i$ is a map

$$\tilde{W}_i : \mathcal{K}_n \to R.$$

It is continuous, bounded, positive, rotation invariant, homogeneous of degree $n - i$ and monotone under set inclusion.

As defined in 2 and 3, $\tilde{V}_i$ and $\tilde{W}_i$ have indices i restricted to integer values between 0 and n. We now extend the definitions so that $\tilde{V}_i$ and $\tilde{W}_i$ are defined for all real indices. The extended definitions will be required to prove Theorem 4.

DEFINITION 2*.

$$\tilde{V}_i(A,B) = \frac{1}{n}\int_\Omega \rho_A^{n-i}(u)\,\rho_B^{i}(u)\,dS(u) \qquad [A,B \in \mathcal{K}_n\ i \in R].$$

DEFINITION 3*.

$$\tilde{W}_i(A) = \tilde{V}_i(A,U) \qquad [A \in \mathcal{K}_n\ i \in R].$$

The following simple extension of Hölder's Inequality will be required to prove our main theorems.

LEMMA 1. *If $f_0, f_1, \cdots, f_m$ are* (strictly) *positive continuous functions defined on Ω and $\alpha_1, \cdots, \alpha_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_\Omega f_0(u)f_1(u)\cdots f_m(u)dS(u) \leqq \prod_{i=1}^{m}\left[\int_\Omega f_0(u)f_i^{\alpha_i}(u)dS(u)\right]^{1/\alpha_i}$$

with equality iff there exist positive constants $\lambda_1, \cdots, \lambda_m$ such that $\lambda_1 f_1^{\alpha_1}(u) = \cdots = \lambda_m f_m^{\alpha_m}(u)$ for all $u \in \Omega$.

The following general inequality between mixed volumes is due to Aleksandrov [**1**] (or see [**5**]):

$$\prod_{i=0}^{m-1} V(A_1, \cdots, A_{n-m}, A_{n-i}, \cdots, A_{n-i}) \leqq V^m(A_1, \cdots, A_n) \qquad [1 < m \leqq n].$$

THEOREM 1.

$$\tilde{V}^m(A_1, \cdots, A_n) \leqq \prod_{i=0}^{m-1} \tilde{V}(A_1, \cdots, A_{n-m}, A_{n-i}, \cdots, A_{n-i}) \qquad [1 < m \leqq n]$$

with equality iff $A_{n-m+1}, A_{n-m+2}, \cdots, A_n$ are all dilations of each other (with the origin as the center of dilation).

To prove this we use Lemma 1 with

$$\alpha_i = m \quad [i = 1, \cdots, m], f_0 = \rho_{A_1} \cdots \rho_{A_{n-m}} \quad (f_0 = 1 \text{ if } m = n),$$

and $f_i = \rho_{A_{n-i+1}}$ $[i = 1, \cdots, m]$.

For $m = n$ Theorem 1 becomes:

COROLLARY 1.1.

$$\tilde{V}^n(A_1, \cdots, A_n) \leqq V(A_1) \cdots V(A_n)$$

with equality iff the A_i are all dilations of each other (with the origin as the center of dilation).

We combine this with the aforementioned inequality of Aleksandrov (for $m = n$) and obtain:

COROLLARY 1.2.

$$\tilde{V}(A_1, \cdots, A_n) \leqq V(A_1, \cdots, A_n)$$

with equality iff the A_i are all dilations of each other (with the origin as the center of dilation).

COROLLARY 1.3.

$$\tilde{V}_i(A, B) \leqq V_i(A, B) \qquad [0 < i < n]$$

with equality iff A is a dilation of B (with the origin as the center of dilation).

COROLLARY 1.4.

$$\tilde{W}_i(A) \leqq W_i(A) \qquad [0 < i < n]$$

with equality iff A is an n-ball (centered at the origin).

A special case of Corollary 1.1 are the

DUAL MINKOWSKI INEQUALITIES.

COROLLARY 1.5.

$$\tilde{V}_1^n(A,B) \leqq V^{n-1}(A)\,V(B) \text{ and } \tilde{V}_{n-1}^n(A,B) \leqq V(A)\,V^{n-1}(B)$$

with equality iff A is a dilation of B (with the origin as the center of dilation).

Our principal inequality between the $\tilde{V}_i$'s is given in:

THEOREM 2.

$$\tilde{V}_j^{k-i}(A,B) \leqq \tilde{V}_i^{k-j}(A,B)\ \tilde{V}_k^{j-i}(A,B) \qquad [i<j<k \ \ i,j,k \in R]$$

with equality iff A is a dilation of B (with the origin as the center of dilation).

To prove this we use Lemma 1 with $m=2$,

$$f_0 = \rho_A^{n-i}\rho_B^i, \quad f_1 = \rho_A^{i-j}\rho_B^{j-i}, \quad f_2 = 1, \quad \alpha_1 = (k-i)/(j-i)$$

and $\alpha_2 = (k-i)/(k-j)$.

We note that if the indices i,j,k are restricted to integer values between 0 and n, then the inequality of Theorem 2 is a special case of Theorem 1. However, the more general inequality of Theorem 2 will be required to prove Theorem 4.

The following are special cases of Theorem 2:

COROLLARY 2.1.

$$\tilde{V}_i(A,B) \leqq V^{(n-i)/n}(A)\,V^{i/n}(B) \qquad [0<i<n \quad i \in R]$$

with equality iff A is a dilation of B (with the origin as the center of dilation).

COROLLARY 2.2.

$$\tilde{W}_i(A) \leqq V^{(n-i)/n}(A)\,\omega_n^{i/n} \qquad [0<i<n \quad i \in R]$$

with equality iff A is an n-ball (centered at the origin).

Let U_0, be an n-ball (centered at the origin).

THEOREM 3.

$$\tilde{W}_i(A) \leqq \tilde{W}_i(U_0) \qquad [V(A) = V(U_0) \quad 0 < i < n \quad i \in R]$$

with equality iff $A = U_0$.

Proof. Consider $A \in \mathcal{K}_n$ such that $V(A) = V(U_0)$. From Corollary 2.2 we deduce:

$$\tilde{W}_i(A) \leqq V^{(n-i)/n}(A)\omega_n^{i/n} = V^{(n-i)/n}(U_0)\omega_n^{i/n} = \tilde{W}_i(U_0).$$

However, if A is not an n-ball (centered at the origin), then Corollary 2.2 states that

$$\tilde{W}_i(A) < V^{(n-i)/n}(A)\omega_n^{i/n}.$$

The following relation between $\bar{b}(A)$ and $W_{n-1}(A)$ is known (Hadwiger **[9]**):

$$W_{n-1}(A) = 2^{-1}\omega_n \bar{b}(A).$$

LEMMA 2.

$$\tilde{W}_{n+1}(A^*) = W_{n-1}(A) \qquad [A \in \mathcal{K}_n].$$

Proof.

$$\tilde{W}_{n+1}(A^*) = \frac{1}{n}\int_\Omega \rho_{A^*}^{-1}(u)dS(u) = \frac{1}{n}\int_\Omega H_A(u)dS(u) = 2^{-1}\omega_n \bar{b}(A),$$

where H_A is the support function of A.

As an application of our work we present the following isoperimetric inequality:

THEOREM 4.

$$\omega_n^{n-i+1} W_{n-1}^{i-n}(A) \leqq W_i(A^*) \qquad [0 \leqq i < n]$$

with equality iff A is an n-ball (centered at the origin).

Proof. From Theorem 2, with $(i, j, k) = (i, n, n+1)$ and $B = U$, we obtain:

(1) $$\omega_n^{n-i+1} = \tilde{W}_n^{n-i+1}(A^*) \leqq \tilde{W}_i(A^*)\tilde{W}_{n+1}^{n-i}(A^*)$$

with equality iff A is an n-ball (centered at the origin).
From Corollary 1.4 we have:

(2) $$\tilde{W}_i(A^*) \leqq W_i(A^*)$$

with equality if A is an n-ball (centered at the origin).
The desired result is obtained when we combine (1) and (2) and apply Lemma 2.

If we let $i = 0$ in Theorem 4 we obtain the

DUAL URYSOHN INEQUALITY.

COROLLARY 4.1.

$$2^n \omega_n \bar{b}(A)^{-n} \leqq V(A^*) \qquad [A \in \mathscr{K}_n]$$

with equality iff A is an n-ball (centered at the origin).

This immediately yields the

DUAL BIEBERBACH INEQUALITY.

COROLLARY 4.2.

$$2^n \omega_n D(A)^{-n} \leqq V(A^*) \qquad [A \in \mathscr{K}_n]$$

with equality iff A is an n-ball (centered at the origin).

Santaló **[10]** has shown that a convex body A can be repositioned in R^n so that $V(A)\,V(A^*) \leqq \omega_n^2$. This result can be combined with the Dual Urysohn Inequality and the Dual Bieberbach Inequality to yield the Urysohn Inequality **[12]** and the Bieberbach Inequality **[3]**.

Steinhardt **[11]** showed that for plane convex bodies

$$4\,\omega_2^2 \leqq S(A)\,S(A^*) \qquad [A \in \mathscr{K}_2].$$

If we let $i = n - 1$ in Theorem 4 we see that an n-space generalization of this is:

COROLLARY 4.3.

$$\omega_n^2 \leqq W_{n-1}(A)\, W_{n-1}(A^*) \qquad [A \in \mathcal{K}_n]$$

with equality iff A is an n-ball (centered at the origin).

This was obtained by Firey [7] for dimensions 2 and 3.

This inequality raises the question of finding

$$\operatorname*{Inf}_{A \in \mathcal{K}_n} W_i(A)\, W_i(A^*)$$

for values of i other than $n-1$. Contributions towards solving the problem for $i=0$ have been made by Dvoretzky and Rogers [**6**], Bambah [2] and recently by Guggenheimer [**8**]. However, a complete solution for $i=0$ is not yet available. For $0 < i < n-1$ the problem is open.

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