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ON THE GROUP OF PERMUTATIONS WITH COUNTABLE SUPPORT

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Let S_X denote the group of permutations of the set X. If $\aleph_\alpha$ is an infinite cardinal, the set of permutations having support with cardinality less than or equal to $\aleph_\alpha$ is a normal subgroup of S_X. The principal result of this paper is a constructive proof that S_X is generated by its cycles, if X is countably infinite. Of particular interest is the corollary that for any set X, the cycles of S_X generate the subgroup of permutations with countable support.

If $f \in S_X$ and $x \in X$, then let $O_f(x)$ denote the orbit of x under f. The set X is the disjoint union of the distinct orbits of f [1]. In case $f(x) \neq x$, $O_f(x)$ is called a *nontrivial orbit* of f. Let $S(f)$ denote the support of the permutation f. If $S(f)$ consists of exactly one nontrivial orbit, then f is called a *cycle*. Let C_X be the subgroup of S_X consisting of all finite products of cycles. If X is finite, then $C_X = S_X$. For an uncountable set X, C_X is a proper subgroup of S_X. We now show that $C_X = S_X$ in the remaining case.

THEOREM. *If X is countably infinite, then S_X is generated by its cycles.*

Proof. Clearly, the subgroup C_X of S_X generated by its cycles is a normal subgroup. But the only normal subgroups of S_X are $\{1\}$, the set of even permutations of finite support, the set of all permutations of finite support, or S_X (see, e.g., [2]). Hence, $C_X = S_X$.

COROLLARY. *For any set X, the cycles of S_X generate the subgroup of permutations with countable support.*

Proof. Clear.

However, one can give a more constructive proof by means of the following lemma.

LEMMA. *Let $f \in S_X$ such that $S(f)$ is a countably infinite union of finite orbits, or a countably infinite union of countably infinite orbits. Then f is the product of two cycles in S_X.*

Proof. Suppose that $S(f) = \cup\{O_f(x_i) \mid i \in Z\}$, where $O_f(x_i)$ is finite for each integer i, and $O_f(x_i) \cap O_f(x_j) = \phi$ if $i \neq j$. Let $O_f(x_{-1}) =$

$\{a_1, a_2, \cdots, a_p\}$, $O_f(x_0) = \{b_1, b_2, \cdots, b_q\}$, and $O_f(x_1) = \{c_1, c_2, \cdots, c_r\}$. It follows that

$$\begin{aligned} f &= \cdots (a_1 a_2 \cdots a_p)(b_1 b_2 \cdots b_q)(c_1 c_2 \cdots c_r) \cdots \\ &= (\cdots a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q c_1 c_2 \cdots c_r \cdots)(\cdots c_1 b_1 a_1 \cdots). \end{aligned}$$

Now, suppose that $S(f)$ consists of orbits which are countably infinite. Chose a partition $A \cup B$ of $S(f)$ such that $A = \{x_i \mid i \in Z\}$ and $B = \{y_i \mid i \in Z \text{ and } i \geqq 0\}$. Let g denote the infinite cycle $(\cdots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 x_2 y_2 \cdots)$, and let

$$h = (\cdots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 y_4 y_5 x_2 y_6 y_7 x_3 \cdots).$$

Then

$$\begin{aligned} gh &= (\cdots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 \cdots)(\cdots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 \cdots) \\ &= (\cdots x_{-5} x_{-3} x_{-1} y_2 y_8 y_{20} \cdots)(\cdots x_{-6} x_{-4} x_{-2} y_0 y_4 y_{12} \cdots) \cdot \\ &\quad (\cdots x_3 x_1 x_0 y_1 y_6 y_{16} \cdots) \cdots (\cdots x_{2(2j)+1} x_{2j} y_{2j+1} y_{2(2j+1)+4} \cdots) \cdots . \end{aligned}$$

It is easy to see that gh fixes none of the elements in the set $A \cup B$. Hence $S(gh) = S(f)$. Since each cycle of gh contains at most one y with an odd subscript, gh has infinitely many cycles. Clearly, each of these cycles is infinite. Using the fact [2] that f and gh are conjugate in S_X if and only if f and gh have the same support structure, there exists a permutation t such that $f = t^{-1}(gh)t = (t^{-1}gt)(t^{-1}ht)$, where $t^{-1}gt$ and $t^{-1}ht$ are necessarily cycles in S_X. This completes the proof of the lemma.

The theorem follows from this, since if f is a permutation on X, then $f = f_1 f_2$, where f_1 agrees with f on its finite orbits and f_2 agrees with f on its infinite orbits.

REMARK. It is known [3] that if G is an abelian group, then G is isomorphic to a group of permutations on some set X, where each permutation has countable support. It follows that each abelian group is isomorphic to a subgroup of C_X, for some set X.

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