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A NOTE ON BANACH SPACES OF LIPSCHITZ FUNCTIONS

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This note is divided into two sections. The first establishes some properties of extreme Lipschitz functions that, it is hoped, will lead to satisfactory ways of characterizing them in general. The second section shows how ideas due to Lindenstrauss can be used to establish the existence of Lipschitz spaces that fail to be injective and fail the approximation property.

Introduction. Our notation will follow essentially that of **[6]** and **[11]**. Given a metric space (S, d), $\mathrm{Lip}(S, d)$ denotes the Banach space of bounded real-valued functions on S with norm given by $\|f\| = \max(\|f\|_\infty, \|f\|_d)$, where

$$\|f\|_d = \sup\{|f(s) - f(t)|\, d^{-1}(s, t): s \neq t\} < \infty.$$

The closed subspace of functions f for which $|f(s) - f(t)| = o(d(s, t))$ is denoted by $\mathrm{lip}(S, d)$. If $A \subset S$, $\tilde{A}$ denotes its complement in S, and if $f: S \to R$ is a function, M_f denotes $\{s: |f(s)| = \|f\|_\infty\}$.

In **[11]**, Roy showed that a function f is an extreme point of the unit ball of $\mathrm{Lip}(S, d)$, with S the unit interval and d the usual metric, if and only if $|f'| = 1$ a.e. on $\tilde{M}_f$ and $\|f\| = \|f\|_\infty = 1$. (See **[10]** for more along these lines.) The purpose of §1 of this note is to discuss some results that we hope will provide clues to possible characterizations of these extreme points for more general metric spaces by presenting two ways in which the bond imposed by the condition "$|f'| = 1$ a.e. on $\tilde{M}_f$" may be broken.

In §2 we observe that if (S, d) is the unit ball of Enflo's space **[4]**, then $\mathrm{Lip}(S, d)$ fails the approximation property.

We also show that $\mathrm{Lip}(S, d)$, where (S, d) is the Hilbert cube, is not injective, ie., is not a $\mathcal{P}_\lambda$ space for any λ (see **[2**, p. 94]). This last result uses techniques due to Lindenstrauss **[8]**, and contrasts with the many examples where $\mathrm{Lip}(S, d)$ is isomorphic to the sequence space l_∞. (See the discussion preceding Proposition 2.2.) The results of §2 point out how large the class of spaces $\mathrm{Lip}(S, d)$ is, although they are always dual spaces **[6]** and, for S infinite, always contain a copy of l_∞ **[7]**.

1. We begin by giving a lemma, mainly for the sake of completeness, which is very similar to the technique of Phelps that was used in **[11**, p. 1159].

LEMMA 1.1. *Let E be a Banach space, F a dense subspace of E and U^* the unit ball of the dual E^* of E. If $\sup\{x^*(x): x^* \in \mathcal{A}\} = \|x\|$ for all $x \in F$, then U^* is the weak*-closed convex hull of $\mathcal{A}$.*

Proof. Let K be the weak*-closed convex hull of $\mathcal{A}$. If $x_0^* \in U^* \sim K$, then there is some x_0 in E with $\alpha = \sup\{x^*(x_0): x^* \in K\} < x_0^*(x_0) = \beta$. Choose $y_0 \in F$ with $\|x_0 - y_0\| < \frac{1}{2}(\beta - \alpha)$. Then $\sup\{x^*(y_0): x^* \in K\} \leqq \frac{1}{2}(\alpha + \beta) < x_0^*(y_0)$, a contradiction.

PROPOSITION 1.1. *Let (S, d) be a compact, connected metric space. Let $\mathcal{A}$ denote the collection of all functions f in the unit ball of* $\mathrm{Lip}(S, d)$ *with $\|f\|_\infty = 1$ such that there is a finite nonempty set $P \subset S$ with the property that given any $s \in \tilde{M}_f$, $|f(s) - f(t)| = d(s, t)$ for some $t \in P$. Then $\mathcal{A}$ is a sup-norm dense subset of the extreme points of the unit ball of* $\mathrm{Lip}(S, d)$.

Proof. Let $f \in \mathcal{A}$ and suppose $\|f \pm g\| \leqq 1$. If $s \in \tilde{M}_f \cap \tilde{P}$ then there is some $t \in P$ such that $|\tilde{f}(s, t)| = 1$ where $\tilde{f}(s, t) = f(s) - f(t)/d(s, t)$. Since $|\tilde{f}(s, t) \pm \tilde{g}(s, t)| \leqq 1$, $\tilde{g}(s, t) = 0$ and therefore $g(s) = g(t)$. If $s \in M_f$, $g(s) = 0$. Hence $g(S) = \{0\} \cup g(P)$. Since S is connected and P is finite, $g(S)$ is a singleton. But $\|f\|_\infty = 1$, so $0 \in g(S)$. Hence $g = 0$ and f is extreme. Thus $\mathcal{A}$ is a subset of the extreme points. Now, let F denote the linear span of the point evaluations in the dual of $\mathrm{Lip}(S, d)$. As was shown in [**6**, p. 157] the dual of F can be canonically identified with $\mathrm{Lip}(S, d)$ and w^*-convergence of bounded nets in $\mathrm{Lip}(S, d)$ is equivalent with uniform convergence. Hence, if we show that $\sup\{\phi(f): f \in \mathcal{A}\} = \|\phi\|$ for each $\phi \in F$, we will reach the desired conclusion by Lemma 1.1 and the $K^2 - M^3 - R$ theorem [**2**, p. 80]. To this end, let $\phi = \sum_{i=1}^n \lambda_i \epsilon_{s_i}$. (Here $\epsilon_s: f \to f(s)$.) If $P = \{s_1, \cdots, s_n\}$, then clearly $\|\phi\| \leqq \|\phi_{|P}\|$. By [**12**, Prop. 1.4], we can extend each function in $\mathrm{Lip}(P, d)$, without increase of norm, to an element of $\mathrm{Lip}(S, d)$. Thus, $\|\phi\| = \|\phi|_P\|$, where $\phi|_P$ is short for $\phi|_{\mathrm{Lip}(P,d)}$. Now $\phi_{|P}$ attains its norm on the unit ball of the finite dimensional space $\mathrm{Lip}(P, d)$ at an extreme point g. We extend g in a norm preserving way to S by the technique of Sherbert [**12**, Prop. 1.4] as follows: Let $g_0(s) = \max_{1 \leqq j \leqq n} g(s_j) - d(s, s_j)$ and let

$$f(s) = \begin{cases} g_0(s) & \text{if } \; -1 \leqq g_0(s) \leqq 1 \\ 1 & \text{if } \; g_0(s) > 1 \\ -1 & \text{if } \; g_0(s) < -1. \end{cases}$$

Since f extends g, we have $\|\phi\| = \|\phi_{|P}\| = \phi_{|P}(g) = \phi(f)$. If we show $f \in \mathcal{A}$, we are finished.

First, $\|f\| \leqq \|g\| = 1$ and since g is extreme $\|g\|_\infty = 1$. Thus, $\|f\|_\infty =$

1. Let $s \in \tilde{M}_f$. Then $|f(s)| < 1$, so $f(s) = g_0(s) = g(s_j) - d(s, s_j)$ for some $j = 1, \cdots, n$. But $f = g$ on P, so $f(s) = f(s_j) - d(s, s_j)$ or $|f(s) - f(s_j)| = d(s, s_j)$. Hence, $f \in \mathcal{A}$. This completes the proof.

REMARK. Define $\mathcal{B}$ to be the set of all functions f in the unit ball of $\mathrm{Lip}(S, d)$ such that there is a finite subset P of S with the property that for each $s \in S$, we have $|f(s) - f(t)| = d(s, t)$ for some $t \in M_f \cup P$. The same argument as above shows f is an extreme point of the unit ball. It is not difficult to see that the function defined in [**11**, Lemma 1.3] is in $\mathcal{B}$ ($P = \emptyset$ in this case) and that $\mathcal{A} \subset \mathcal{B}$.

Here we digress momentarily in order to show how the technique of the above proof can also be used to prove the following.

PROPOSITION 1.2. *Let $0 < \alpha < 1$ and (S, d) be a compact metric space. Then* $\bigcup_{\beta > \alpha} \mathrm{Lip}(S, d^\beta)$ *is dense in* $\mathrm{lip}(S, d^\alpha)$.

Proof. We apply the technique mentioned earlier ([**11**, p. 1159]). Let $\phi = \Sigma_{i=1}^n \lambda_i \epsilon_{s_i}$ be taken in the dual of $\mathrm{lip}(S, d^\alpha)$ with $\|\phi\| = 1$. As in the proof of Proposition 1.1, we consider $\phi = \phi_{|P}$ as an element of $\mathrm{Lip}(P, d^\alpha)^*$ where $P = \{s_1, \cdots, s_n\}$. Its norm is attained at an extreme point h of the unit ball. Given $\epsilon > 0$, choose $\beta > \alpha$ sufficiently near α so that $\inf_{i \neq j} d^{\beta - \alpha}(s_i, s_j)$ and $(\operatorname{diam} S)^{\beta - \alpha}$ are greater than $1/(1 + \epsilon)$. Define f_0 by

$$f_0(s) = \sup_j h(s_j) - d^\beta(s_j, s)$$

and then let f be the truncation of f_0 as in Proposition 1.1. Now, $\|h\|_{d^\beta} \leqq (1 + \epsilon)\|h\|_{d^\alpha} \leqq 1 + \epsilon$, by our choice of β. Since

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} = \frac{|f(s) - f(t)|}{d^\beta(s, t)}\, d^{\beta - \alpha}(s, t),$$

$\|f\|_{d^\alpha} \leqq \|f\|_{d^\beta} p^{\beta - \alpha} \leqq \|h\|_{d^\beta}(1 + \epsilon) \leqq (1 + \epsilon)^2$ where p is the diameter of S. Let $g = (1 + \epsilon)^{-2} f$. Then g belongs to $\mathrm{Lip}(S, d^\beta)$ and to the unit ball of $\mathrm{lip}(S, d^\alpha)$. Also $\phi(g) = (1 + \epsilon)^{-2}\phi(f)$ and $\phi(f) = \phi(h) = \|\phi_{|P}\| = \|\phi\|$ taken in $\mathrm{Lip}(S, d^\alpha)^*$. It follows from lemma 4.6 in [**6**] that the norm of ϕ taken in $\mathrm{Lip}(S, d^\alpha)^*$ is equal with its value in $\mathrm{lip}(S, d^\alpha)^*$. Therefore, $\phi(g) = (1 + \epsilon)^{-2}$. But $\epsilon > 0$ was arbitrary, so $1 = \|\phi\| = \sup \phi(g)$ where the supremum is taken over all g in the unit ball of $\mathrm{lip}(S, d^\alpha)$ that lie in $\bigcup_{\beta > \alpha} \mathrm{Lip}(S, d^\beta)$. It was proved by Jenkins [**5**] that the point evaluations span a norm dense subspace of $\mathrm{lip}(S, d^\alpha)^*$. Hence the unit ball of $\mathrm{lip}(S, d^\alpha)^*$ is the norm-closed convex hull of the subset consisting of members of $\bigcup_{\beta > \alpha} \mathrm{Lip}(S, d^\beta)$. This completes the proof.

The last result we include in this section is not very satisfying since we have been unable to establish it for metric spaces more general than an interval and also since it cannot, at least in its present form, be made to work if $\mathrm{lip}(S, d)$ is nontrivial (as an examination of the proof will show). The only reason we include it is that, as mentioned in the introduction, there have to our knowledge been no characterizations of extreme Lipschitz functions, other than that of Roy **[11]**, and no real clues as to how to extend his characterization. We only hope that the next proposition, as well as Proposition 1.1, may give some ideas as to possible directions to take.

In what follows $\mathcal{D}_s$ denotes the set of point derivations at s in $\mathrm{Lip}(S, d)^*$ (See **[12**, §8]). We now turn to

PROPOSITION 1.3. *Let f be in the unit ball of* $\mathrm{Lip}(I, d)$, *with* $\|f\|_\infty = 1$, *where I is the unit interval and d the usual metric. Then f is extreme if and only if the weak* closure of the linear span of* $\bigcup_{s \notin M_f} \{\phi \in \mathcal{D}_s : \|\phi\| = |\phi(f)| = 1\}$ *contains* $\bigcup_{s \notin M_f} \mathcal{D}_s$.

Proof. Suppose the condition holds. If $\|f \pm g\| \leqq 1$, then $\phi(g) = 0$ for all $\phi \in \bigcup_{s \notin M_f} \mathcal{D}_s$. Hence, by **[12**, Prop. 9.10], g is in $\mathrm{lip}(A, d)$ for each component A of $\tilde{M}_f$ and is therefore constant on each A. Since $g = 0$ on M_f, $g = 0$ on I. Thus, f is extreme. Now assume f is extreme. Then by Roy's characterization **[11**, Theorem 3.1], $|f'| = 1$ a.e. on $\tilde{M}_f$. Let $\mathcal{M}_s$ denote the set of multiplicative linear functionals x^* in the dual of $L_\infty(I)$ such that $x^*(g) = g(s)$ for each continuous function g on I. If $D: \mathrm{Lip}(I, d) \to L_\infty(I)$ is defined by $Df = f'$ (to be precise, the equivalence class containing f'), then $D^*\mathcal{M}_s$ is contained in $\{\phi \in \mathcal{D}_s : \|\phi\| = |\phi(f)| = 1\}$ for each $s \notin M_f$; here D^* denotes the adjoint of D. If we show $\bigcup_{s \notin M_f} D^*\mathcal{M}_s$ is weak* dense in $\bigcup_{s \notin M_f} \mathcal{D}_s$ we will be finished. Suppose $f \in \mathrm{Lip}(I, d)$ and $\phi(f) = 0$ for each $\phi \in \bigcup_{s \notin M_f} D^*\mathcal{M}_s$. Then $x^*(f') = 0$ for each $x^* \in \mathcal{M}_s$, $s \notin M_f$. We will show $f' = 0$ a.e. on $\tilde{M}_f$. Given $\epsilon > 0$, there is a compact $K \subset \tilde{M}_f$ with $\mu(\tilde{M}_f \sim K) < \epsilon$ (μ = Lebesgue measure on I.) Let h be continuous on I with $h = 1$ on K and $h = 0$ on M_f. Then $x^*(f'h) = 0$ for each $x^* \in \bigcup_{s \in I} \mathcal{M}_s$, the whole maximal ideal space of $L_\infty(I)$. Thus, $f'h = 0$ a.e., so $f' = 0$ a.e. on K. But $\mu(\tilde{M}_f \sim K) < \epsilon$, so $\mu\{s \in \tilde{M}_f : f'(s) \neq 0\} < \epsilon$. Hence $f' = 0$ a.e. on $\tilde{M}_f$ and thus f is constant on each component of $\tilde{M}_f$. Now, if $\phi \in \mathcal{D}_s$ and $s \in \tilde{M}_f$, then f is constant in a neighborhood of s and thus $\phi(f) = 0$ for each $\phi \in \mathcal{D}_s$. Since $\phi(f) = 0$ for all $\phi \in \bigcup_{s \notin M_f} D^*\mathcal{M}_s$ implies $\phi(f) = 0$ for all $\phi \in \bigcup_{s \notin M_f} \mathcal{D}_s$, we conclude that the former is weak*-dense in the latter. This completes the proof.

Let us remark that the above proposition holds for complex scalars. We are forced to use real scalars in Proposition 1.1 since our norm preserving extensions are not available otherwise without extra assumptions on (S, d) such as the Lipschitz 4-point property (see **[5]**).

2. In this section we present two applications of the paper of Lindenstrauss **[8]** which indicate just how general the class of spaces $\mathrm{Lip}(S, d)$ is.

The first is a relatively straightforward application of two powerful theorems to give the following.

PROPOSITION 2.1. *Let E be a Banach space without the approximation property* (see **[4]**, of course) *and let S denote its unit ball with d given by the norm. Then* $\mathrm{Lip}(S, d)$ *fails the approximation property.*

Proof. Let $\phi(x) = x$ if $x \in S$ and $\phi(x) = x/\|x\|$ if $x \notin S$. Then $\|\phi(x) - \phi(y)\| \leqq 2\|x - y\|$ for each $x, y \in E$. To see this, consider the cases (1) $x, y \notin S$ and (2) $x \in S$, $y \notin S$.

$$(1) \quad \|\phi(x) - \phi(y)\| \leqq \left\| \frac{x}{\|x\|} - \frac{x}{\|y\|} \right\| + \left\| \frac{x}{\|y\|} - \frac{y}{\|y\|} \right\|$$

$$= \frac{|\|y\| - \|x\||}{\|y\|} + \frac{1}{\|y\|}\|x - y\| \leqq \frac{2}{\|y\|}\|x - y\| \leqq 2\|x - y\|.$$

$$(2) \quad \|\phi(x) - \phi(y)\| \leqq \left\| x - \frac{x}{\|y\|} \right\| + \left\| \frac{x}{\|y\|} - \frac{y}{\|y\|} \right\|$$

$$\leqq \frac{\|y\| - 1}{\|y\|} + \frac{\|x - y\|}{\|y\|} \leqq \frac{\|y\| - \|x\|}{\|y\|} + \frac{\|x - y\|}{\|y\|}$$

$$\leqq \frac{2}{\|y\|}\|x - y\| \leqq 2\|x - y\|.$$

Now, $T: f \to f \circ \phi$ is easily seen to be an isomorphism ($=$ linear homeomorphism) of L_0 into $E^\#$ where $L_0 = \{f \in \mathrm{Lip}(S, d): f(0) = 0\}$ and $E^\# = \{f: E \to R: \|f\|_d < \infty, f(0) = 0\}$. Let Q be the restriction mapping of E^* into $\mathrm{Lip}(S, d)$, P the projection of $E^\#$ on E^* guaranteed by **[8**, Theorem 2], and J the mapping of $\mathrm{Lip}(S, d)$ into L_0 defined by $Jf = f - f(0)$. Then $QPTJ$ is a projection of $\mathrm{Lip}(S, d)$ on QE^*. Hence E^* is isomorphic to a complemented subspace of $\mathrm{Lip}(S, d)$. Since E fails the approximation property, so does E^* and hence $\mathrm{Lip}(S, d)$. This completes the proof.

We have as yet been unable to prove or disprove the existence of a compact metric space for which $\mathrm{Lip}(S, d)$ fails the approximation property. For a discussion of reformulations of this problem, see **[6**, p. 168].

We now turn our attention to another application of **[8]** in which we show that $\mathrm{Lip}(S, d)$, with (S, d) the Hilbert cube, is not injective. This contrasts with the case where (S, d) is the unit interval with the usual metric and the case where (S, d) is an infinite compact subset of Euclidean space with $d = |\cdot|^\alpha$, $0 < \alpha < 1$. In all these situations $\mathrm{Lip}(S, d)$ is isomorphic to the sequence space l_∞ and is hence injective. (For $0 < \alpha < 1$, see **[1]**; for the interval and $\alpha = 1$, the assertion follows from **[9]** and the fact that $\|f\|_d = \|f'\|_\infty$ in $L_\infty(I)$.)

Let us remark that it is not hard to show in addition that $\mathrm{Lip}(S, d)$ is isomorphic to l_∞ for any compact infinite subset of the line. Simply let I be a closed interval with S in its interior. Then for each f in $\mathrm{Lip}(S, d$ let Tf be the function obtained by extending f linearly in each component of $I \sim S$. T is an isomorphism of $\mathrm{Lip}(S, d)$ into $\mathrm{Lip}(I, d)$ whose composition TR with the restriction map is a projection on $T\,\mathrm{Lip}(S, d)$. But it is a well known fact due to Lindenstrauss that a complemented infinite dimensional subspace of l_∞ is isomorphic to l_∞.

It is tempting to conjecture that if (S, d) is a compact subset of Euclidean space, $\mathrm{Lip}(S, d)$ is injective. We have been unable to prove it however.

The proof of Proposition 2.2 is broken down into two lemmas. Lemma 2.1 is certainly well-known and is even a corollary to far stronger results. We sketch a proof only for completeness.

LEMMA 2.1. *An infinite dimensional reflexive subspace E of the continuous functions $C(X)$ on a compact Hausdorff space X is not complemented in $C(X)$.*

Proof. If P is a projection of $C(X)$ on E, P is weakly compact because E is reflexive. By [**3**, Corollary VI. 7.5], P is compact. But P is onto E, so E is finite dimensional.

In the sequel, C will denote the continuous functions on $[0, 1]$, *E an isometric copy of l_2 in C, and (S, d) the Hilbert cube in $E^* = E$.*

LEMMA 2.2. *There is no map $\alpha: S \to C^*$ and no constant $M > 0$ such that $\alpha(x^*)_{|E} = x^*$ and $\|\alpha(x^*) - \alpha(y^*)\| \leqq M\|x^* - y^*\|$ for each $x^*, y^* \in S$.*

Proof. (The idea for this proof is due to Professor Joram Lindenstrauss, to whom I express my thanks for permission to include it here.)

Suppose such a mapping exists. We will construct a map $\beta: E^* \to C^*$ with the same properties. First, we may assume, by replacing α by $\alpha - \alpha(0)$, that $\alpha(0) = 0$. Let $K_x^* = \{0\} \cup \{\mu \in C^* \mid \mu_{|E} = x^* \text{ and } \|\mu\| \leqq M\|x^*\|\}$ and define

$$\alpha_n(x^*) = \begin{cases} n\alpha\left(\dfrac{1}{n}x^*\right) & \text{if } x^* \in nS \\ 0 & \text{if } x^* \notin nS \end{cases}.$$

If Π is the product of those K_x^*'s with $x^* \in \bigcup_n nS$, then $\alpha_n \in \Pi$ for each n. In fact, for $x^*, y^* \in nS$, we have

$$\begin{aligned}
&\|\alpha_n(x^*) - \alpha_n(y^*)\| \\
&\quad = \left\| n\alpha\left(\frac{1}{n}x^*\right) - n\alpha\left(\frac{1}{n}y^*\right)\right\| \leqq nM\left\|\frac{1}{n}x^* - \frac{1}{n}y^*\right\| \\
&\quad = M\|x^* - y^*\|.
\end{aligned}$$

Now, each K_x^* is weak*-compact, so Π is compact in the product topology. Thus, there is a subnet $\{\alpha_{n_\gamma}\}$ of $\{\alpha_n\}$ converging to some member β of Π. Let $x^*, y^* \in \bigcup nS$. Pick n_0 so that $x^*, y^* \in n_0S$. Then there is an index γ_0 such that $n_\gamma \geqq n_0$ for all $\gamma \geqq \gamma_0$. If $x \in C$, $\|x\| \leqq 1$, we have $|\beta(x^*) - \beta(y^*))(x)| = \lim_{\gamma \geqq \gamma_0}|(\alpha_{n_\gamma}(x^*) - \alpha_{n_\gamma}(y^*))(x)|$. But $n_\gamma \geqq n_0$ for $\gamma \geqq \gamma_0$ and S is circled,so $x^*,y^* \in n_\gamma S$ for $\gamma \geqq \gamma_0$. Thus, $\|\alpha_{n_\gamma}(x^*) - \alpha_{n_\gamma}(y^*)\| \leqq M\|x^* - y^*\|$ for each $\gamma \geqq \gamma_0$, and $\|\beta(x^*) - \beta(y^*)\| \leqq M\|x^* - y^*\|$. Now, $\bigcup nS$ is dense in E^*, so β extends uniquely to all of E^* and $\|\beta(x^*) - \beta(y^*)\| \leqq M\|x^* - y^*\|$ for all $x^*, y^* \in E^*$. Since $\beta = \lim_\gamma \alpha_{n_\gamma}$, we have $\beta(x^*)_{|E} = x^*$ for each $x^* \in \bigcup nS$. But β is continuous on E^* and $\bigcup nS$ is dense in E^* so $\beta(x^*)_{|E} = x^*$ for all $x^* \in E^*$. Hence β is the required mapping. Now, define $T: C^{**} \to E^{**}$ by $TF(x^*) = F(\beta x^*)$ for $F \in C^{**}$, $x^* \in E^*$ (see the proof of Proposition 2.1 for the definition of $A^\#$). T is linear and continuous and by [**8**, Theorem 2] there is a projection P of E^{**} onto E^{**}. Consider $PT_{|C}: C \to E^{**}$. If $x \in E$, let $F_x(\mu) = \mu(x)$ for each $\mu \in C^*$. Then $TF_x(x^*) = F_x(\beta x^*) = (\beta x^*)(x) = x^*(x)$. This says $TF_x = F_x$, so $PTF_x = F_x$. Thus, identifying E and E^{**}, we obtain a projection of C onto E, a contradiction.

Proposition 2.2. *If (S, d) is the Hilbert cube,* Lip(S, d) *is not injective.*

Proof. A necessary and sufficient condition for a Banach space A to be injective is that for each Banach space F, subspace $F_0 \subset F$, and bounded linear operator $T: F_0 \to A$, there is a bounded linear operator $\tilde{T}: F \to A$ with $\tilde{T}_{|F_0} = T$ (see [**2**, p. 94]). We continue to use the notation of Lemma 2.2. Let $T: E \to \mathrm{Lip}(S, d)$ be defined by $Tx: x^* \to x^*(x)$, $x^* \in S$. If Lip(S, d) is injective, there is an operator $\tilde{T}: C \to \mathrm{Lip}(S, d)$ that extends T. Let $\alpha: S \to C^*$ be given by $\alpha(x^*): x \to (\tilde{T}x)(x^*)$. Then α satisfies the conditions in Lemma 2.2, an impossibility. This completes the proof.

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