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MOMENT SEQUENCES IN l^p

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Let $p > 0$. Conditions are derived, each necessary and sufficient, for a moment sequence to be in l^p. It is shown that the moment sequences in l^p are dense in l^p. For $p = 2$, these results were obtained by G. G. Johnson.

G. G. Johnson obtained a necessary and sufficient condition for a moment sequence to be in l^2, and showed that the moment sequences in l^2 are dense in l^2. This paper shows that the same conclusions hold in any l^p space. The proofs are similar to and improvements of those in G. G. Johnson, Pacific J. Math., **46**(1973), 201–207.

LEMMA 1. *Let* $0 < p < 1$, $q > 0$. *If* $a_n = 1 - (n+1)^{-p}$, *then* $\{a_n^n\} \in l^q$.

Proof. $a_n^{nq} = \exp(qn \log(1-(n+1)^{-p})) < \exp(qn(-(n+1)^{-p})) = (\exp(qn(n+1)^{-p}))^{-1} < [\Sigma_{k=0}^N (qn(n+1)^{-p})^k/k!]^{-1}$, where N satisfies $N(1-p) > 1$. Then

$$\sum_{n=1}^{\infty} a_n^{nq} < \sum_{n=1}^{\infty} [(qn(n+1)^{-p})^N/N!]^{-1} = N!\, q^{-N} \sum_{n=1}^{\infty} [(n+1)^p/n]^N,$$

which converges if and only if $\Sigma_{n=1}^{\infty} n^{-(1-p)N}$ converges, and the latter is a convergent p-series.

THEOREM 1. *Let $p > 0, f \in BV[0,1]$, $\mu_n = \int_0^1 t^n df$. For each $\{a_n\}$ such that $0 \leqq a_n < 1$, and $\{a_n^n\} \in l^p$, the following are equivalent.*

(i) $\{\mu_n\} \in l^p$

(ii) $\left\{f(1) - (1-a_n^n)^{-1} \int_{a_n}^1 f(t)dt^n\right\}_{n=1}^{\infty} \in l^p.$

Lemma 1 shows such $\{a_n^n\}$ exist.

Proof. Split the integral for μ_n at a_n and integrate by parts to obtain, as in **[1]**, $\mu_n = a_n^n(\delta_n - \gamma_n) + (f(1) - \delta_n)$, where $\delta_n = (1-a_n^n)^{-1} \int_{a_n}^1 f(t)\,dt^n$ and $\gamma_n = (a_n^n)^{-1} \int_0^{a_n} f(t)dt^n$. Since $|\delta_n - \gamma_n|$ is bounded, $\{a_n^n(\delta_n - \gamma_n)\} \in l^p$, so that $\{\mu_n\} \in l^p$ if and only if $\{f(1) - \delta_n\}_{n=1}^{\infty} \in l^p$.

LEMMA 2. *If* $g(t) = 1 - (1-t)^\alpha$, $\alpha > 0$, *and* $\nu_n = \int_0^1 t^n dg$, *then* $\{\nu_n\} \in l^p$ *if and only if* $\alpha > 1/p$.

Proof. $\mu_n = \int_0^1 t^n dg = \Gamma(\alpha+1)\Gamma(n+1)/\Gamma(n+\alpha+1)$. Using Stirling's formula or Gauss's test, $\Sigma_n \mu_n^p$ converges if and only if $\alpha > 1/p$. [3, pp. 92–93].

Consequently no l^p space contains all of the moment sequences.

COROLLARY. *If there is* δ, $0 < \delta < 1$, $B > 0$ *and* α *such that*

(i) $\alpha > 1/p$ *and* $|f(1) - f(t)| \leqq B|1-t|^\alpha$ *for* t *in* $[\delta, 1]$, *then* $\{\mu_n\} \in l^p$

(ii) $\alpha \leqq 1/p$ *and* $f(1) - f(t) \geqq B(1-t)^\alpha$ *for* t *in* $[\delta, 1]$, *or* $f(t) - f(1) \geqq B(1-t)^\alpha$ *for* t *in* $[\delta, 1]$, *then* $\{\mu_n\} \notin l^p$.

Proof. of (i)

$$\begin{aligned}\left| f(1) - (1-a_n^n)^{-1}\int_{a_n}^1 f(t)dt^n \right| &= \left| (1-a_n^n)^{-1}\int_{a_n}^1 (f(1)-f(t))dt^n \right| \\ &\leqq (1-a_n^n)^{-1}\int_{a_n}^1 |f(1)-f(t)|dt^n \\ &\leqq (1-a_n^n)^{-1}\int_{a_n}^1 B(1-t)^\alpha dt^n \\ &= B\left[g(1) - (1-a_n^n)^{-1}\int_{a_n}^1 g(t)dt^n\right]\end{aligned}$$

which we shall call ψ_n. By Lemma 2 and Theorem 1, $\{\psi_n\} \in l^p$.

The proof of (ii) is analogous to that of (i).

For each integer $k > 0$, $m > 0$, define the moment sequence $c_{km} = \{c_{nkm}\}_{n=0}^\infty$ by $c_{nkm} = (-1)^m k^m m!^{-1} \Sigma_{r=0}^m \binom{m}{r} (-1)^r (r/k)^n = m!^{-1} \Delta_\omega^m x^n$, where $\Delta_\omega f(x) = [f(x+\omega) - f(x)]/\omega$, $\omega = k^{-1}$, and $x = 0$.

THEOREM 2. *Let* $p > 0$. *The moment sequences* c_{km} *belong to and have dense linear span in* l^p.

Proof. For $m > n$, $\Delta_\omega^m x^n = 0$. From [2, p. 13], with $f(x) = x^{n+m}$, $\Delta_\omega^m f(x) = f^{(m)}(\xi)$ for some ξ between 0 and $m\omega$, so that $|\Delta_\omega^m x^{n+m}| \leqq \max_{0 \leqq \xi \leqq m\omega} |f^{(m)}(\xi)| = (n+m)!\,(m\omega)^n/n!$.

Using these facts we have, for $0 < p \leqq 1$,

$$\sum_{\substack{n=0\\ n\neq m}}^{\infty} |c_{nkm}|^p = \sum_{n=m+1}^{\infty} |c_{nkm}|^p = \sum_{n=1}^{\infty} |c_{n+m,k,m}|^p = \sum_{n=1}^{\infty} |m!^{-1}\Delta_\omega^m x^{n+m}|^p$$

$$\leqq m!^{-p} \sum_{n=1}^{\infty} (n+m)!(mk^{-1})^{np}/n!$$

$$= m!^{1-p}[(1-(m/k)^p)^{-m-1} - 1].$$

Therefore the sum is finite and tends to 0 as $k \to \infty$.

Since $\Delta_\omega^m x^m = m!$, $c_{mkm} = 1$. For

$$0 < p \leqq 1,\ e^m = \{\delta_{jm}\}_{j=0}^{\infty},\ \| c_{km} - e^m \|_p = |c_{mkm} - 1|^p + \sum_{\substack{n=0\\ n\neq m}}^{\infty} |c_{nkm}|^p \to 0, \quad \text{as} \quad k \to \infty.$$

But the $\{e^m\}_{m=0}^{\infty}$ form a basis for l^p so that the c_{km} have dense linear span in l^p for $0 < p \leqq 1$.

For any $p' > p$, $l^{p'} \supset l^p$ and the $l^{p'}$ topology is weaker than that of l^p ([**4**, p. 203]). Therefore the $c_{km} \in l^{p'}$ and $c_{km} \to e^m$ in $l^{p'}$, so the c_{km} have dense linear span in each l^p space.

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