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ORTHOGONALLY ADDITIVE AND ORTHOGONALLY INCREASING FUNCTIONS ON VECTOR SPACES

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A real-valued function $f: X \to R$ on an inner product space X is *orthogonally additive* if $f(x+y) = f(x) + f(y)$ whenever $x \perp y$. We extend this concept to more general spaces called orthogonality vector spaces. If X is an orthogonality vector space and if there exists an orthogonally additive function on X which satisfies certain natural conditions then there is an inner product on X which is equivalent to the original orthogonality and $f(x) = \pm \|x\|^2$ for all $x \in X$. We next consider a normed space X with James' orthogonality. A function $f: X \to R$ is *orthogonally increasing* if $f(x+y) \geqq f(x)$ whenever $x \perp y$. Orthogonally increasing functions on normed spaces are characterized.

1. Pythagoras' theorem. Pythagoras' theorem states that the function $f(x) = \|x\|^2$ is orthogonally additive, that is $f(x+y) = f(x) + f(y)$ whenever $x \perp y$ where x, y are vectors in the plane. One of the concerns of this paper is a converse of Pythagoras' theorem on an inner product space X. That is, if $f: X \to R$ is orthogonally additive, is $f(x) = c\|x\|^2$ for some $c \in R$? As it stands, the answer is no, since any linear functional is orthogonally additive.

Some natural additional conditions on f are:

(1) $f(x) \geqq 0$, *nonnegativity*;

(2) $f(x) = f(-x)$, *evenness*;

(3) $\lambda_i \to \lambda$ implies $f(\lambda_i x) \to f(\lambda x)$ for all $x \in X$, *hemicontinuity*.

We shall show that orthogonal additivity along with (1), or with (2) and (3) imply $f(x) = c\|x\|^2$ for some $c \in R$.

2. Orthogonality vector spaces. In this paper, vector spaces will be real and of dimension $\geqq 2$. In Theorem 2.2 we shall prove that Pythagoras' theorem characterizes inner product spaces in a certain sense.

A vector space X is an *orthogonality vector space* if there is a relation $x \perp y$ on X such that

(01) $x \perp 0$, $0 \perp x$ for all $x \in X$;

(02) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(03) if $x \perp y$, then $ax \perp by$ for all $a, b \in R$;

(04) if P is a two-dimensional subspace of X, then for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(05) if P is a two-dimensional subspace of X, then there exist nonzero vectors $u, v \in P$ such that $u \perp v$ and $u + v \perp u - v$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0$, $0 \perp x$ for all x, and for nonzero vectors x, y define $x \perp y$ iff x, y are linearly independent. Also an inner product space is such a space; we shall see that a normed space is one also with James' definition of orthogonality.

LEMMA 2.1. *Let $(X, \perp)$ be an orthogonality vector space and let $f: X \to R$ be orthogonally additive and hemi-continuous.* (a) *If f is odd, then f is linear.* (b) *If f is even, then $f(\alpha x) = \alpha^2 f(x)$ for all $\alpha \in R$, $x \in X$ and if $x \perp y$ and $x + y \perp x - y$, then $f(x) = f(y)$.*

Proof. Same as in [**2**; Lemmas 2, 3].

REMARK. The referee has pointed out to us that there is a mistake in the proof of Lemma 2 [**2**]. In that proof it is incorrectly stated that $F(2^r u) = 2^r F(u)$ for all rational r when, in fact, this is only proved for integral r. However, it is easily seen that $F(3^r u) = 3^r F(u)$ for all integral r. Indeed, in the notation of that proof

$$\begin{aligned} F(3u) - F(v) &= F(3u - v) = F(u + v + 2u - 2v) \\ &= F(u + v) + F(2(u - v)) = 3F(u) - F(v). \end{aligned}$$

Hence, by induction $F(2^p 3^q u) = 2^p 3^q F(u)$ for all integral p and q. Since these scalars $2^p 3^q$ are dense, continuity implies $F(\alpha u) = \alpha F(u)$.

An inner product $\langle \cdot, \cdot \rangle$ on $(X, \perp)$ is $\perp$*-equivalent* when $x \perp y$ iff $\langle x, y \rangle = 0$.

THEOREM 2.2. *If there exists an $f: (X, \perp) \to R$ which is orthogonally additive, even, hemicontinuous, and not identically* 0, *then there is a $\perp$-equivalent inner product $\langle \cdot, \cdot \rangle$ on $(X, \perp)$. In fact, $\langle x, y \rangle = \frac{1}{4}[f(x + y) - f(x - y)]$ and the induced norm satisfies $\|x\|^2 = f(x)$ for all $x \in X$, or $\|x\|^2 = -f(x)$ for all $x \in X$. Moreover, if $\langle \cdot, \cdot \rangle_1$ is another $\perp$-equivalent inner product on $(X, \perp)$, then there is a nonzero $c \in R$ such that $\langle \cdot, \cdot \rangle_1 = c\langle \cdot, \cdot \rangle$.*

Proof. We first show that f has constant sign. Let $0 \neq x \in X$ and suppose $f(x) > 0$. Let $0 \neq y \in X$. If $y = \alpha x$, then $f(y) = \alpha^2 f(x) > 0$. If y, x are linearly independent, let P be the generated 2-dimensional subspace. Then there exist $u, v \in X$ satisfying (05) and

(02). Hence $y = au + bv$, $x = cu + dv$ for $a, b, c, d \in R$. By Lemma 2.1 (b), $f(y) = (a^2 + b^2)f(u)$, $f(x) = (c^2 + d^2)f(u)$ so $f(y) > 0$. Similarly, $f(x) < 0$ implies $f(y) < 0$. For concreteness, suppose $f(x) \geqq 0$ for all $x \in X$. One can now show that $f(x)^{1/2}$ is a norm on X which satisfies the parallelogram law so X is an inner product space. If $x \perp y$ then $f(x + y) = f(x) + f(y)$ and so $\langle x, y \rangle = 0$. Conversely, suppose $x, y \neq 0$ and $\langle x, y \rangle = 0$. By (04) there is a $z \neq 0$ in the span of $\{x, y\}$ such that $x \perp z$. Hence $\langle x, z \rangle = 0$ and by (02) $y = ax + bz$ for some $a, b \in R$. From $\langle x, y \rangle = 0$ it follows that $a = 0$ so $x \perp y$. Corollary 3.4 concludes the proof.

If X is a normed linear space, James **[1]** defines $x \perp y$ iff $\|x + ky\| \geqq \|x\|$ for all $k \in R$. With this definition of $\perp$, $(X, \perp)$ is an orthogonality vector space. Indeed, (01), (02), (03) follows easily, (04) follows from **[1**; Corollary 2.3] and (05) follows from **[2**; Lemma 1].

The next result generalizes to inner product spaces a result of Sundaresan **[2]** whose proof relies on the completeness of Hilbert space.

COROLLARY 2.3. *Let X be a normed space and let $f: X \to R$ be an orthogonally additive, even, hemicontinuous function.* (a) *If X is not an inner product space, then $f \equiv 0$.* (b) *If X is an inner product space, then there is a $c \in R$ such that $f(x) = c\|x\|^2$ for all $x \in X$.*

We next prove a generalization of the Riesz representation theorem.

COROLLARY 2.4. *Let X be an inner product space and let $f: X \to R$ be orthogonally additive and satisfy $|f(x)| \leqq M\|x\|$ for all $x \in X$. Then f is a continuous linear functional and hence, if X is a Hilbert space, $f(x) = \langle x, z \rangle$ for some $z \in X$.*

Proof. We can assume $M > 0$. Clearly f is continuous at 0. Let $x \neq 0$. We first show that $\beta \to 1$ implies $f(\beta x) \to f(x)$. Let $\beta > 1$, $y \perp x$, $\|y\| = 1$ and $u = x + (\beta - 1)^{1/2}\|x\| y$. Then $(u - x) \perp x$ and $(u - \beta x) \perp u$. Thus $f(u) - f(x) = f(u - x)$ and $f(\beta x) - f(u) = f(\beta x - u)$. Hence

$$\begin{aligned} |f(x) - f(\beta x)| &\leqq |f(x) - f(u)| + |f(u) - f(\beta x)| \\ &\leqq M\|x\|[2(\beta - 1)^{1/2} + (\beta - 1)]. \end{aligned}$$

Now let $0 < \beta < 1$, $y \perp x$, $\|y\| = 1$ and

$$u = \beta x + (1 - \beta)^{1/2} \beta^{1/2} \|x\| y.$$

Then $(u - \beta x) \perp \beta x$ and $(x - u) \perp u$. Again $f(u) - f(\beta x) = f(u - \beta x)$, and $f(x) - f(u) = f(x - u)$, so that

$$\begin{aligned} |f(x) - f(\beta x)| &\leqq |f(x - u)| + |f(u - \beta x)| \\ &\leqq M \|x\| [(1 - \beta) + 2(1 - \beta)^{1/2} \beta^{1/2}]. \end{aligned}$$

It follows that $f(\beta x) \to f(x)$ as $\beta \to 1$. We now show that f is norm continuous. If $x_i \to x$, there exist $y_i \perp x$ such that $x_i = \alpha_i x + y_i$. Taking the inner product with x, we see that $\alpha_i \to 1$ and hence $y_i \to 0$. Since $f(x_i) = f(\alpha_i x + y_i) = f(\alpha_i x) + f(y)$, we have $f(x_i) \to f(x)$ as $x_i \to x$ and f is norm continuous. Applying Corollary 2.3 and Lemma 2.1(a), there is a continuous linear functional f_2 such that $f(x) = c \|x\|^2 + f_2(x)$. Hence $|c| \|x\| \leqq M + \|f_2\|$ for all $x \in X$, which implies $c = 0$.

3. Orthogonally increasing functions. In this section orthogonality on a normed space X will always be defined according to James' definition (see §2). A function $f: X \to R$ is *orthogonally increasing* iff $x \perp y$ implies $f(x + y) \geqq f(x)$. We shall later define other types of increasing functions.

In the last section we characterized orthogonally additive, hemicontinuous functions. We saw that they formed a very restricted class, being the sum of a linear functional and a constant times the norm squared. The orthogonally increasing functions form a much larger class. Indeed, if $g: R^+ \to R$, where $R^+ =$ nonnegative reals, is any nondecreasing function then $f(x) = g(\|x\|)$ is orthogonally increasing since $x \perp y$ implies $f(x + y) = g(\|x + y\|) \geqq g(\|x\|) = f(x)$. The main result of this section characterizes orthogonally increasing functions on a normed space and shows that they are essentially of this form.

Let X be a normed space. A function $f: X \to R$ is *radially increasing* if $\alpha > 1$ implies $f(\alpha x) \geqq f(x) \forall x \in X$, and f is *spherically increasing* if $\|x\| > \|y\|$ implies $f(x) \geqq f(y)$. It is clear that spherically increasing implies radially increasing and simple examples show that the converse need not hold. In a strictly convex (rotund) normed space, spherically increasing implies orthogonally increasing. Indeed, let f be a spherically increasing function on such a space and let $x \perp y$. Then $\|x + y\| \geqq \|x\|$. If $\|x + y\| > \|x\|$, then by spherical increasing

$$f(x + y) \geqq f(x).$$

Now suppose $\|x + y\| = \|x\|$. Then

$$\|x + \tfrac{1}{2} y\| = \|\tfrac{1}{2}(x + y) + \tfrac{1}{2} x\| \leqq \tfrac{1}{2}\|x + y\| + \tfrac{1}{2}\|x\| = \|x\|.$$

Since $x \perp y$, $\|x + \frac{1}{2}y\| \geqq \|x\|$ so $\|x + \frac{1}{2}y\| = \|x\|$. But a normed space is strictly convex if and only if $\|u\| = \|v\| = \|\frac{1}{2}(u + v)\|$ implies $u = v$, and so $\|x + y\| = \|x\| = \|x + \frac{1}{2}y\|$ implies $y = 0$. Hence $f(x + y) \geqq f(x)$ and f is orthogonally increasing. It is well known that any uniformly convex space is strictly convex, in particular an inner product space is strictly convex.

In a general normed space, spherically increasing need not imply orthogonally increasing. Indeed, let $X = (R^2, \|\cdot\|_\infty)$; that is, $X = R^2$ with $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$. Note that X is not strictly convex. Let $f: X \to R$ be defined as follows: $f(x) = \|x\|$ if $0 \leqq \|x\| < 1$, $f(x) = 2\|x\|$ if $\|x\| > 1$, $f(x) = 1$ if $\|x\| = 1$ and $x \neq (1, 0)$, and $f((1, 0)) = 2$. It is easy to check that f is spherically increasing. If $x = (1, 0)$ and $y = (0, 1)$ then $x \perp y$ but $f(x + y) = f((1, 1)) = 1 < 2 = f(x)$. Hence f is not orthogonally increasing. The next theorem shows that orthogonally increasing implies spherically increasing.

THEOREM 3.1. *Let X be a normed space with* $\dim X \geqq 2$ *and let $f: X \to R$ be orthogonally increasing. Then f is spherically increasing and there exists a countable number of spheres $S_1, S_2, \cdots$ such that f is norm continuous at w iff $w \notin \cup S_i$. Furthermore, there exists a nondecreasing function $g: R^+ \to R$ such that $f(w) = g(\|w\|)$ for every $w \notin \cup S_i$.*

Proof. We first show that f is radially increasing. Let $0 \neq y \in X$ and let $\alpha > 1$. By a modification of the proof of Lemma 1 **[2]** there exists $0 \neq x \in X$ such that $y \perp x$ and $(y + x) \perp [(\alpha - 1)y - x]$. Hence

$$f(\alpha y) = f[y + x + (\alpha - 1)y - x] \geqq f(y + x) \geqq f(y)$$

and f is radially increasing. We now show that f is norm continuous on a dense subset of X. Let $\|x_0\| = 1$ and let $V = \{\lambda x_0 \colon \lambda \in R^+\}$. Then f restricted to V is an increasing function and hence is continuous in V on a dense subset B of V. We shall show that f is norm continuous on $B - \{0\}$. Let $0 \neq x \in B$ and let $x_i \to x$. Now there exists y_i such that $x \perp y_i$ and $x_i = \alpha_i x + y_i$. Since

$$\|x_i - x\| = \|(\alpha_i - 1)x + y_i\| \geqq |\alpha_i - 1| \, \|x\|$$

we have $\alpha_i \to 1$. By the Hahn-Banach theorem, there exist continuous linear functionals f_{x_i} on X such that $f_{x_i}(x_i) = \|x_i\|^2$ and $\|f_{x_i}\| = \|x_i\|$. Now

$$|f_{x_i}(x_i) - f_{x_i}(x)| = |f_{x_i}(x_i - x)| \leqq \|x_i\| \, \|x_i - x\|$$

so $f_{x_i}(x) \to \|x\|^2$. Letting $k_i = \|x_i\|^2 / f_{x_i}(x)$ we see that $k_i \to 1$. Furthermore, for every $\alpha \in R$ we have

$$\|x_i + \alpha(k_i x - x_i)\| \geqq f_{x_i}[(1-\alpha)x_i + \alpha k_i x]/\|f_{x_i}\|$$
$$= [(1-\alpha)\|x_i\|^2 + \alpha k_i f_{x_i}(x)]/\|f_{x_i}\| = \|x_i\|.$$

Hence $x_i \perp (k_i x - x_i)$. Thus

$$f(k_i x) = f(x_i + k_i x - x_i) \geqq f(x_i) = f(\alpha_i x + y_i) \geqq f(\alpha_i x).$$

Since α_i, $k_i \to 1$ we have $f(k_i x)$, $f(\alpha_i x) \to f(x)$ so $f(x_i) \to f(x)$ and f is norm continuous on a dense subset of X. We next show that f is spherically increasing. Let $x, y \in X$ and suppose $\|y\| > \|x\|$. We shall show there exists $\lambda > 1$ and $x = x_0, x_1, \cdots, x_n \in X$ such that $y = \lambda x_n$ and $x_{i-1} \perp (x_i - x_{i-1})$, $i = 1, \cdots, n$. It would then follow that

$$f(y) = f(\lambda x_n) \geqq f(x_n) = f(x_{n-1} + x_n - x_{n-1}) \geqq f(x_{n-1}) \geqq \cdots \geqq f(x_0) = f(x).$$

To show such λ and x_i exist we proceed as follows. We can assume without loss of generality that $\|x\| = 1$, that x and y are linearly independent, and that the 2-dimensional subspace generated by $\{x, y\}$ is R^2 with $x = (1, 0)$. Let S be the unit sphere in R^2 corresponding to the unit sphere in X. Since the norm is a convex function, using polar coordinates, we can assume that S is given by $\rho = F(\theta)$ where F is a continuous function on $[0, 2\pi]$, which is periodic of period π, the right-hand derivative F' exists everywhere, and F' is bounded. Let S_0 be a unit sphere obtained by reflecting S about the x-axis. Then, in polar coordinates, S_0 is given by $\rho_0 = F_0(\theta)$ where $F_0(\theta) = F(2\pi - \theta)$. Denote orthogonality with respect to S and S_0 by $\perp$ and $\perp_0$ respectively, and the norm with respect to S and S_0 by $\|\cdot\|$ and $\|\cdot\|_0$ respectively. We now construct a polygonal path P starting at x and sweeping twice around the origin with vertices $x_0 = x, x_1, x_2, \cdots, x_{2n}$ as follows. The angle between x_{i-1} and x_i is $2\pi/n$, $x_{i-1} \perp (x_i - x_{i-1})$ for $i = 1, 2, \cdots, n$, and $x_{i-1} \perp_0 (x_i - x_{i-1})$ for $i = n+1, n+2, \cdots, 2n$. Now

$$\|x_{2n}\|_0 \geqq \|x_{2n-1}\|_0 \geqq \cdots \geqq \|x_n\|_0 = \|x_n\| \geqq \|x_{n-1}\| \geqq \cdots \geqq \|x\|.$$

Indeed, since $x_{2n} = x_{2n-1} + (x_{2n} - x_{2n-1})$ we have $\|x_{2n}\|_0 \geqq \|x_{2n-1}\|_0$ and the others follow in a similar way. Furthermore, $\|x_n\| \geqq \|w\|$ for any $w \in P$ which precedes x_n. Indeed, if w is on the edge with vertices x_n and x_{n-1} then $w = \lambda x_n + (1-\lambda)x_{n-1}$ for some $0 \leqq \lambda \leqq 1$ and hence $\|w\| \leqq \lambda\|x_n\| + (1-\lambda)\|x_{n-1}\| \leqq \|x_n\|$. A similar argument holds for other $w \in P$. Hence, if we can show that $\lim_{n\to\infty}\|x_{2n}\|_0 = 1$ we will be finished with this part of the proof. A simple calculation shows that the slope of S in the forward direction at angle θ is

$$[F(\theta)\cos\theta + F'(\theta)\sin\theta]/[F'(\theta)\cos\theta - F(\theta)\sin\theta].$$

Since $x \perp (x_1 - x)$ it follows that the slope of $x_1 - x$ equals the slope of S in the forward direction at $\theta = 0$. Letting ρ_1 be the ρ coordinate of x_1 we have

$$\rho_1 \sin(2\pi/n)/[\rho_1 \cos(2\pi/n) - 1] = [F'(0)]^{-1}.$$

Hence

$$\rho_1 = [\cos(2\pi/n) - F'(0)\sin(2\pi/n)]^{-1}$$

and this formula holds even if $F'(0) = 0$. In a similar way, a straightforward calculation gives

$$\rho_i = \rho_{i-1}\{\cos(2\pi/n) - [F'(2\pi i/n)/F(2\pi i/n)]\sin(2\pi/n)\}^{-1},$$

$i = 2, 3, \cdots, n$. A similar formula holds for ρ_{0i}, $i = n+1, n+2, \cdots, 2n$. Using the fact that $F_0(2\pi i/n) = F[2\pi(n-i)/n]$ and $F_0'(2\pi i/n) = -F'[2\pi(n-i)/n]$ we obtain

$$\begin{aligned}\rho_{02n} = {} & \{\cos^2(2\pi/n) - [F'(0)]^2\sin^2(2\pi/n)\}^{-1} \\ & \times \{\cos^2(2\pi/n) - [F'(2\pi/n)/F(2\pi/n)]^2\sin^2(2\pi/n)\}^{-1} \\ & \times \cdots \times \{\cos^2(2\pi/n) - [F'((n-1)2\pi/n)/F((n \\ & \qquad - 1)2\pi/n)]^2\sin^2(2\pi/n)\}^{-1}.\end{aligned}$$

Letting $M = \sup[F'(\theta)/F(\theta)]^2$ we have

$$\lim_{n\to\infty} \rho_{02n} \leqq \lim_{n\to\infty} [\cos^2(2\pi/n) - M\sin^2(2\pi/n)]^{-n}.$$

But L'Hospital's rule shows that

$$\lim_{x\to 0} (2\pi/x)\log[\cos^2 x - M\sin^2 x] = 0$$

so

$$\lim_{n\to\infty} \rho_{02n} = 1. \quad \text{Hence} \quad \lim_{n\to\infty} \|x_{2n}\|_0 = 1.$$

We next show that f is norm continuous except on a countable set of spheres. Let $\|x_0\| = 1$. Then from the above, f is norm continuous at δx_0 except for countably many δ's, say $\delta_1, \delta_2, \cdots$. Suppose f is continuous at $x = \delta x_0$ and $\|y\| = \|x\|$. If $\lambda > 1$ then $f(\lambda x) \geqq f(y)$, so letting $\lambda \to 1$ we have $f(x) \geqq f(y)$ and in a similar way we show that $f(x) \geqq f(y)$ so $f(x) = f(y)$. To show f is continuous at y, let $y_i \to y$. As $\|y\| = \|x\| > 0$, it is possible, for i sufficiently large, to find a sequence $a_i \in R$ such that $a_i \to 0$, $a_i > 0$ and $\|y_i\| - a_i > 0$. Let $x_i = (\|y_i\| + a_i)x/\|y\|$ and $z_i = (\|y_i\| - a_i)x/\|y\|$. Then $\|x_i\| > \|y_i\| > \|z_i\|$ so $f(z_i) \leqq f(y_i) \leqq f(x_i)$. Now $x_i \to x$, $z_i \to x$ and since f is continuous at x we have $f(y_i) \to f(x) = f(y)$. Hence f is continuous at y. If $S_i = \{x \in X; \|x\| = \delta_i\}$, it follows that f is continuous at w iff $w \notin \cup S_i$. Define $g: R^+ \to R$ by $g(\alpha) = f(\alpha x_0)$. Then g is a nondecreasing function and if $w \notin \cup S_i$ we have $f(w) = f(\|w\| x_0) = g(\|w\|)$.

Using Theorem 3.1 we can prove a result similar to Corollary 2.3 concerning nonnegative orthogonally additive functions.

COROLLARY 3.2. *Let X be a normed space with* $\dim X \geqq 2$ *and let $f: X \to R^+$ be orthogonally additive.* (a) *If X is not an inner product space, then $f \equiv 0$.* (b) *If X is an inner product space, then there is a $c \in R^+$ such that $f(x) = c\|x\|^2$ for all $x \in X$.*

In the rest of this section X will denote an inner product space with $\dim X \geqq 2$ and inner product $\langle \cdot, \cdot \rangle$.

COROLLARY 3.3. *If $f: X \to R^+$ is orthogonally additive, then there is a $c \in R^+$ with $f(x) = c\|x\|^2$.*

COROLLARY 3.4. *Let $\langle \cdot, \cdot \rangle_1$ be another inner product on X. If $x \perp y$ implies $x \perp_1 y$, then there is a $c > 0$ such that $\langle u, v \rangle_1 = c\langle u, v \rangle$ for all $u, v \in X$.*

Proof. Let $g(w) = \|w\|_1$. If $x \perp y$ then $x \perp_1 y$ so $g^2(x+y) = g^2(x) + g^2(y)$. Hence g^2 is orthogonally additive so there is a $c > 0$ with $\|w\|_1 = g(w) = c\|w\|$. Hence

$$\begin{aligned}\langle u, v \rangle_1 &= [\|u+v\|_1^2 - \|u-v\|_1^2]/4 = c^2[\|u+v\|^2 - \|u-v\|^2]/4 \\ &= c^2\langle u, v \rangle.\end{aligned}$$

COROLLARY 3.5. *If $f: X \to R$ is orthogonally additive and $f(x) \geqq -M\|x\|^2$ for all $x \in X$ for some $M \geqq 0$, then there is an $\alpha \in R$ such that $f(x) = \alpha\|x\|^2$.*

Proof. If $g(x) = f(x) + M\|x\|^2$, then $g: X \to R^+$ is orthogonally additive. Hence there is a $c \geqq 0$ such that $g(x) = c\|x\|^2$. Hence $f(x) = (c - M)\|x\|^2$.

In a similar way, Corollary 3.5 holds if $f(x) \leqq M\|x\|^2$, for all $x \in X$.

Let $x_0 \in X$, c, $d \in R^+$ and define $f(x) = c\|x - x_0\|^2 + d$. Then $f(x) \geqq f(x_0)$ and if $x \perp y$ we have

$$\begin{aligned} f(x+y) &= c\|x - x_0\|^2 - 2c\langle y, x_0\rangle + c\|y\|^2 + d \\ &= c\|x - x_0\|^2 + c\|y - x_0\|^2 - c\|x_0\|^2 + d \\ &= f(x) + f(y) - d - c\|x_0\|^2 = f(x) + f(y) - f(0). \end{aligned}$$

We now show that the converse holds.

COROLLARY 3.6. *Let $f: X \to R$ satisfy*: (a) *there is an $x_0 \in X$ such that $f(x) \geqq f(x_0)$ for all $x \in X$,* (b) *if $x \perp y$ then*

$$f(x+y) = f(x) + f(y) - f(0).$$

Then is a $c \geqq 0$ such that $f(x) = c\|x - x_0\|^2 + f(x_0)$ and if $c \neq 0$, x_0 is unique.

Proof. Let $g(x) = f(x + x_0) - f(x_0)$. Then $g: X \to R^+$. Let $x \perp y$ and write $x = x_1 + x_2 + x_3$ where x_1 is a multiple of x, x_2 is a multiple of y and x_3 is orthogonal to x and y. Then $g(x+y) = g(x) + g(y)$. Hence $g(x) = c\|x\|^2$ for some $c \geqq 0$ and $f(x + x_0) = c\|x\|^2 + f(x_0)$. Hence $f(x) = c\|x - x_0\|^2 + f(x_0)$. If $c \neq 0$ and $f(x) \geqq f(y_0)$ for all $x \in X$ then $f(y_0) = f(x_0)$ and $f(y_0) = c\|y_0 - x_0\|^2 + f(x_0)$. Thus $\|y_0 - x_0\| = 0$ so $y_0 = x_0$.

COROLLARY 3.7. *Let $f: X \to R$ be orthogonally additive. If there is an $x_0 \in X$ such that $f(x_0) = \|x_0\|^2$ and $|f(x)| \leqq \|x_0\|\,\|x\|$ for all $x \in X$, then $f(x) = \langle x, x_0\rangle$ for all $x \in X$.*

Proof. Let $g(x) = \|x\|^2 - 2f(x) + \|x_0\|^2$. Then

$$g(x) \geqq \|x\|^2 - 2\|x\|\,\|x_0\| + \|x_0\|^2 = (\|x\| + \|x_0\|)^2 \geqq 0 = g(x_0).$$

Also $x \perp y$ implies $g(x+y) = g(x) + g(y) - g(0)$. Hence by Corollary 3.6 there is a $c \geqq 0$ such that $g(x) = c\|x - x_0\|^2$. Therefore

$$\begin{aligned} 2f(x) = \|x\|^2 + \|x_0\|^2 - c\|x - x_0\|^2 = (1-c)\|x\|^2 \\ + (1-c)\|x_0\|^2 + 2c\langle x, x_0\rangle. \end{aligned}$$

Since $f(0) = 0$ we have $(1-c)\|x_0\|^2 = 0$. Thus either $c = 1$ or $x_0 = 0$. If $x_0 = 0$ then $|1 - c|\,\|x\|^2 = 2|f(x)| \leqq 0$ for all $x \in X$ so again $c = 1$. Hence $f(x) = \langle x, x_0\rangle$.

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