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CHARACTERS AND SCHUR INDICES OF THE UNITARY REFLECTION GROUP $[3\,2\,1]^3$

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The group G considered in this paper is the finite irreducible unitary reflection group in 6 dimensions denoted by $[3\,2\,1]^3$ in the notation of H. S. M. Coxeter. A character table for G is constructed. There are 169 irreducible characters and it is shown that each character has Schur index 1 over Q. Furthermore, each character has values in the field $Q(\sqrt{-3})$ and this field is a splitting field for G.

The concept of a reflection in Euclidean space was generalized by G. C. Shephard **[9]**, who first wrote $[2\,1;3]^3$ for $[3\,2\,1]^3$. A reflection in unitary space is a linear transformation of finite period with the property that all but one of its characteristic values are equal to 1. While reflections in Euclidean space must have period 2, a reflection in unitary space may have period m for any integer $m > 1$. Shephard and J. A. Todd **[10]** studied the finite groups generated by unitary reflections and classified the irreducible groups. The ordinary finite reflection groups, studied extensively by H. S. M. Coxeter (see **[2]** for references), are special cases of the finite unitary reflection groups. The group G considered in this paper is the largest finite primitive irreducible unitary reflection group which is not a Euclidean group. It is generated by reflections of period 2 and has order $2^9 \cdot 3^7 \cdot 5 \cdot 7$. The characters of all the other finite irreducible unitary reflection groups will be studied in a forthcoming paper by this author.

1. General remarks. The group G has a center of order 6 consisting of scalar transformations. The central quotient group is the collineation group in 5 dimensions which was studied by H. H. Mitchell **[7]**, C. M. Hamill **[6]**, and Todd **[11, 12]**. This group is denoted by G_1 throughout this paper. The element z denotes a generator of the center of G, fixed throughout. The groups G_2 and G_3 denote the quotient groups of G by $\langle z^2 \rangle$ and $\langle z^3 \rangle$, respectively.

The commutator subgroup G' of G has index 2 in G and contains the center of G. The quotient group G_1' is the simple group $PSU_4(3)$ of order $9 \cdot 9!$.

The group G contains a reflection subgroup H of type $[2\,2\,1]^3$ (see **[10]**). The elements of H leave invariant a 1-dimensional subspace of the 6-dimensional unitary space on which G operates and so H intersects the center of G trivially. Furthermore, H is isomorphic to a direct product of the simple group of order 25920 and the group of order 2. The character table for the simple group of order 25920 was given by J. S. Frame **[5]**.

The commutator subgroup of G consists of those elements which can be expressed as a product of an even number of reflections. The conjugacy classes falling inside G' are called even classes, while the others are called odd classes. In the tables, all the even classes are listed first. If χ is an (irreducible) character of G which does not vanish on all odd classes, then there is another (irreducible) character χ^* such that $\chi^*(g) = -\chi(g)$ if g belongs to an odd class and $\chi^*(g) = \chi(g)$ otherwise. The character χ^* is called the associate of χ. Only one character of each pair of associates is listed in the character tables. The principal character 1 of G has the associate 1^*, the linear character with kernel G'. Also, for any χ, $\chi^* = \chi 1^*$.

The conjugacy classes of G_1 were determined by Hamill. There are 34 classes, 18 of which are even. Each class of G_1 may give rise to up to 6 classes of G. The classes of G are listed in cliques $C_1, \cdots, C_{31}$ corresponding to the classes of the quotient group G_1. This numbering of cliques coincides with the numbering of the classes of G_1 assigned by Hamill. Each clique is represented in the tables by a class containing elements of smallest order. Since representatives of the other classes can be found by multiplying by appropriate elements of the center, then the orders and character values of all elements are easily deduced from the information provided in the tables. The conjugacy class consisting of the 126 reflections of G is contained in the clique C_2. The pairs $\{C_{9a}, C_{9b}\}$, $\{C_{18a}, C_{18b}\}$, and $\{C_{24a}, C_{24b}\}$ correspond to classes of G_1 which are pair-wise algebraically conjugate under complex conjugation. Hence the classes of C_{9a} (C_{18a}, C_{24a}, respectively) are algebraically conjugate to the classes of C_{9b} (C_{18b}, C_{24b}, respectively). Only C_{9a}, C_{18a}, and C_{24a} are listed in the tables.

The automorphism of G which sends each transformation into its adjoint interchanges z and z^{-1}. Hence there are essentially 2 ways in which G may be represented as a unitary reflection group in 6 dimensions. The character ψ (or ψ_6) is the character of that representation such that $\psi(z) = -6\omega$, where ω is the primitive cube root of unity satisfying $\omega - \bar{\omega} = \sqrt{-3}$. The values of ψ and the characteristic values for the elements in the underlying representation are given by Todd **[11]**. The characteristic polynomial for an element of smallest order in each clique is listed in Table I. Obviously the orders of elements are determined by their characteristic values.

The characteristic values corresponding to an element gz can be obtained by multiplying the characteristic values of g by $-\omega$. Since conjugate elements must have the same characteristic values, it is clear that 24 of the 34 cliques consist of 6 conjugacy classes each. Furthermore, the cliques C_{11}, C_{16}, and C_{29} each contains either 2 or 6 classes of G, and C_{12}, C_{31}, C_5, C_{25}, and C_{26} each contains either 3 or 6 classes. In §§3 and 4 it is shown that each of these 8 cliques contains exactly the minimum number of cliques listed and that the cliques C_{22} and C_{23} each contains 2 classes.

The irreducible characters of G are found in this paper by decomposing reducible characters constructed from known irreducible characters of G and H. The orthogonality relations are the primary tools, but Schur's method for decomposing Kronecker powers of characters is also used in 2 instances. Schur's method provides a technique by which the nth power of an irreducible character can be partitioned into constituents with the use of the characters of the symmetric group of degree n. For the full general linear group the constituents produced are irreducible (see **[8]**), but for arbitrary groups the constituents may be reducible. Given a character χ of an arbitrary group and a positive integer n, there exists characters χ_i, one for each irreducible character α_i of $\mathrm{Sym}(n)$, such that for each group element g and for each $\sigma \in \mathrm{Sym}(n)$ with cycle structure $\{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$,

$$\chi(g)^{m_1}\chi(g^2)^{m_2}\cdots\chi(g^n)^{m_n} = \sum_i \alpha_i(\sigma)\chi_i(g).$$

The system of equations, one for each conjugacy class of $\mathrm{Sym}(n)$, can be solved for the χ_i to get

$$\chi_i(g) = \sum_\sigma \frac{1}{c(\sigma)}\alpha_i(\sigma)\chi(g)^{m_1}\cdots\chi(g^n)^{m_n}$$

where the sum ranges over representatives σ of all conjugacy classes of $\mathrm{Sym}(n)$ and $c(\sigma)$ is the order of the centralizer of σ.

2. Schur indices and splitting fields.

For any irreducible character χ of a group and any field F (of characteristic 0), $F(\chi)$ denotes the field generated over F by the values of χ. The Schur index $m_F(\chi)$ of χ over F is the smallest positive integer m such that $m\chi$ is afforded by a $F(\chi)$-representation. A field F is a splitting field for a group if each character of the group is afforded by an F-representation. Hence $Q(\sqrt{-3})$ is a splitting field for G if each irreducible character has values in $Q(\sqrt{-3})$ and Schur index 1 over Q.

If χ is an irreducible character of a group and η is any character afforded by an F-representation, then $m_F(\chi)$ divides (χ, η). This result may be found in §11 of **[4]**; other results on Schur indices and splitting fields may be found in Chapter X of **[3]**.

If χ_1 and χ_2 are 2 characters afforded by representations $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, then the character $\chi_1\chi_2$ is afforded by $\mathcal{R}_1 \times \mathcal{R}_2$. Hence if χ_1 and χ_2 are afforded by F-representations, then so is $\chi_1\chi_2$. In particular, $m_F(\chi^*) = m_F(\chi)$ for any irreducible character χ of G.

The characters of G_1 were given by Todd [12]. The irreducible characters are denoted by the symbols ϕ_n, where n is the degree. The nonassociate characters of degree 35 are denoted by ϕ_{35} and ϕ'_{35}. Similar notation is used for multiple characters of other degrees. It is easy to show that $Q(\sqrt{-3})$ is a splitting field for G_1. Each irreducible character of G_1, with the exception of the 3 characters of degree 560, has multiplicity 1 in some permutation character given by Todd. Thus each of these characters has Schur index 1 over Q. Each of the other 3 characters has multiplicity 2 in a permutation character, so each has Schur index $\leqq 2$. Furthermore, ϕ_{729} is rational-valued and thus is afforded by a rational representation, and each character of degree 560 has multiplicity 45 in $(\phi_{729})^2$. Hence each of the characters of degree 560 has Schur index 1 over Q. Since each character of G_1 has values in $Q(\sqrt{-3})$, then $Q(\sqrt{-3})$ is a splitting field for G_1.

The character ψ of G mentioned previously is afforded by a $Q(\sqrt{-3})$-representation. Such a representation is given by Shephard and Todd ([10]), p. 298).

Each character of H has Schur index 1 over Q. The simple group of order 25920 is isomorphic to the commutator subgroup W' of the Weyl group W of type E_6. Each of the characters of W can be afforded by a rational representation ([1]). Since $|W: W'| = 2$, then for each irreducible character χ of W' there exists a character η of W such that $(\chi, \eta_{W'}) = 1$. Thus $m_Q(\chi) = 1$ for each irreducible character of the simple group of order 25920. Since each character of a group of order 2 has a rational representation, then each character of H has Schur index 1 over Q.

3. The group G_2. The faithful irreducible characters of G_2, the quotient group of G by $\langle z^2 \rangle$, are denoted by θ_n, with the same conventions as for the characters ϕ_n. Schur's method of partition characters applied to ψ^3 yields 3 characters of degrees 20, 56, and 70. In particular, $\psi^3 = \theta_{20} + \theta_{56} + 2\theta_{70}$, where

$$\theta_{20}(g) = \tfrac{1}{6}(\psi(g)^3 - 3\psi(g)\psi(g^2) + 2\psi(g^2))$$

$$\theta_{56}(g) = \tfrac{1}{6}(\psi(g)^3 + 3\psi(g)\psi(g^2) + 2\psi(g^2))$$

$$\theta_{70}(g) = \tfrac{1}{3}(\psi(g)^3 - \psi(g^3))$$

for all $g \in G$. The orthogonality relations show that these 3 characters are irreducible. Only θ_{56} is rational-valued, but $\bar{\theta}_{20} = \theta^*_{20}$. However, $\bar{\theta}_{70} \neq \theta^*_{70}$, so this provides another irreducible character.

The subgroup H of G has an isomorphic image H_2 in G_2 since $H \cap \langle z^2 \rangle = \langle 1 \rangle$. Let α denote the character of G_2 induced from the principal character of H_2. H_2 has a faithful irreducible character ρ of degree 6 (see **[5]**), and β denotes the character of G_2 induced by ρ. More irreducible characters of G_2 are found from the following equations.

$$
\begin{aligned}
\theta'_{70} &= \alpha - \phi_1 - \phi_{35} - \phi_{90} - \theta'^*_{56} \\
\theta_{420} &= \theta_{20}\phi_{21} \\
\theta_{540} &= \theta_{20}\phi_{35} - \theta_{20} - \theta_{70} - \overline{\theta^*_{70}} \\
\theta_{140} &= \theta_{20}\phi'_{35} - \theta_{20} - \theta_{540} \\
\theta_{1280} &= \theta_{20}\phi_{140} - \theta^*_{20} - \theta_{420} - 2\theta^*_{540} \\
\theta_{560} &= \theta_{70}\phi'_{35} - \bar{\theta}^*_{70} - \theta_{540} - \theta_{1280} \\
\theta'_{56} &= \beta - \phi^*_{21} - \phi'_{315} - \phi^*_{420} - \theta_{140} - \theta_{560} \\
\theta_{504} &= \theta_{56}\phi_{21} - \theta_{56} - \theta'^*_{56} - \theta^*_{560} \\
\theta_{896} &= \theta_{70}\phi_{21} - \theta'_{70} - \theta'^*_{504} \\
\theta'_{504} &= \theta_{56}\phi'_{35} - \theta_{560} - \theta^*_{896} \\
\theta_{630} &= \theta'_{70}\phi_{21} - \theta_{70} - \bar{\theta}_{70} - \theta_{140} - \theta^*_{560}
\end{aligned}
$$

Since the sum of the squares of the degrees of all irreducible characters must equal $|G_2|$, then the remaining characters must have degrees whose squares total 28800. The character $\theta_{56}\phi_{140}$, after subtracting the irreducible constituents which are already known (see decomposition below), yields a reducible character ζ. The character ζ has degree 240 and is the sum of 2 distinct irreducible characters. Because of degree limitations, then these characters must each have degree 120. This completes the list of characters for G_2; there are 63 irreducible characters. The clique C_{22} contains either 2 or 6 classes of G, so it must yield 2 classes of G_2. Since all but the last 2 faithful characters vanish on C_{22}, then the last 2 must have nonzero values there and must be associates. The values of θ_{120} are found by applying the orthogonality relations. All faithful characters of G_2 vanish on the cliques C_{12}, C_{31}, C_5, C_{25}, and C_{26}, so each of these provides one conjugacy class of G_2.

All faithful irreducible characters of G_2 are rational-valued except for the characters θ_{20}, θ_{70}, $\bar{\theta}_{70}$, θ_{540}, and their associates. These 8 characters all have values in $Q(\sqrt{-3})$. Since ψ is afforded by a $Q(\sqrt{-3})$-representation and since θ_{20} has multiplicity 1 in ψ^3, then θ_{20} and θ_{20}^* each has Schur index 1 over Q. Since θ_{70} and θ_{540} each has multiplicity 1 in $\theta_{20}\phi_{35}$, then each of these characters (along with each associate and complex conjugate) has Schur index 1 over Q.

To compute the Schur indices for the rational-valued characters, the following 6 reducible characters are needed.

$$\alpha = \phi_1 + \phi_{35} + \phi_{90} + \theta_{56} + \theta'_{70}$$

$$\beta = \phi^*_{21} + \phi'_{315} + \phi^*_{420} + \theta'_{56} + \theta_{140} + \theta_{560}$$

$$\theta_{56}\phi_{140} = \theta'^*_{56} + \theta_{120} + \theta^*_{120} + \theta_{504} + \theta'_{504} + \theta'^*_{504} + 3\theta^*_{560} + 2\theta_{896} + 2\theta_{1280}$$

$$\theta'_{70}\phi_{21} = \theta_{70} + \bar{\theta}_{70} + \theta_{140} + \theta^*_{560} + \theta_{630}$$

$$\theta'_{70}\phi_{35} = \theta'_{56} + \theta'^*_{70} + \theta_{420} + \theta'_{504} + \theta'^*_{504} + \theta^*_{896}$$

$$\theta_{120}\phi_{21} = \theta_{120} + \theta_{560} + \theta^*_{560} + \theta_{1280}.$$

Each irreducible character of H_2 has Schur index 1 over Q and ρ is rational-valued, so ρ is afforded by a rational representation. Hence α and β are both afforded by rational representations. From the decompositions of α and β, each of the irreducible characters θ_{56}, θ'_{70}, θ'_{56}, θ_{140}, and θ_{560} has Schur index 1 over Q. Therefore each of the next 3 reducible characters listed has a rational representation. So each of the characters θ_{120}, θ_{504}, θ'_{504}, θ_{630}, θ_{420}, and θ_{896} has Schur index 1 over Q. Finally, then, the character $\theta_{120}\phi_{21}$ has a rational representation so θ_{1280} has Schur index 1 over Q. Therefore each irreducible character of G_2 has Schur index 1 over Q.

4. The group G_3. An irreducible character of G_3 of degree n with the property that its value at z is $n\omega$ is denoted by λ_n, with the same conventions as used for the characters of G_1 and G_2. The partitioning of the character $\bar{\psi}^2$ yield irreducible characters λ_{15} and λ_{21}, where

$$\lambda_{15}(g) = \tfrac{1}{2}(\overline{\psi(g)^2} - \overline{\psi(g^2)})$$

$$\lambda_{21}(g) = \tfrac{1}{2}(\overline{\psi(g)^2} + \overline{\psi(g^2)})$$

for all $g \in G$. Other irreducible characters are found as follows.

$$\lambda_{315} = \lambda_{15}\phi_{21}$$

$$\lambda_{105} = \psi\theta_{20} - \lambda^*_{15}$$

$$\lambda'_{105} = \psi\theta'_{70} - \lambda_{315}$$

$$\lambda_{210} = \psi\theta_{56} - \lambda_{21} - \lambda'_{105}$$

$$\lambda''_{105} = \psi\theta_{70} - \lambda_{105} - \lambda_{210}$$

$$\lambda_{384} = \psi\bar{\theta}_{70} - \lambda_{15} - \lambda_{21}$$

$$\lambda_{336} = \psi\theta'_{56}$$

$$\lambda_{720} = \psi\theta_{120}$$

$$\lambda_{420} = \lambda^*_{15}\phi'_{35} - \lambda_{105}$$

$$\lambda_{630} = \lambda_{21}\phi'_{35} - \lambda''^*_{105}$$

$$\lambda_{756} = \lambda_{15}\theta_{90} - \lambda_{384} - \lambda_{210}$$

$$\lambda_{729} = \lambda_{21}\phi_{90} - \lambda_{21} - \lambda'_{105} - \lambda''_{105} - \lambda_{210} - \lambda^*_{336} - \lambda_{384}$$

$$\lambda_{945} = \lambda_{15}\phi_{140} - \lambda_{105} - \lambda_{420} - \lambda^*_{630}.$$

These irreducible characters, together with their associates and complex conjugates, are all the faithful irreducible characters of G_3.

All faithful irreducible characters of G_3 have values in $Q(\sqrt{-3})$ and none are rational-valued. Since ψ is afforded by a $Q(\sqrt{-3})$-representation and since $\bar{\psi}^2 = \lambda_{15} + \lambda_{21}$, then each of the characters λ_{15} and λ_{21} has Schur index 1 over Q. Furthermore, as can be seen from the equations given above, each of the other faithful irreducible characters of G_3 has multiplicity 1 in some character afforded by a $Q(\sqrt{-3})$-representation, so each irreducible character of G_3 has Schur index 1 over Q.

The faithful irreducible characters of G_3 vanish on the cliques C_{11}, C_{16}, C_{29}, C_{22}, and C_{23}, so each of these cliques yields only one class of G_3. Since each of these cliques provides 2 classes of G_2, then each contains exactly 2 classes of G. Also, each of the cliques C_{12}, C_{31}, C_5, C_{25}, and C_{26} provides 1 class of G_2 and 3 classes of G_3, so each contains exactly 3 classes of G. Hence G has 169 conjugacy classes.

5. The faithful characters. An irreducible character of G of degree n with the property that its value at z is $-n\omega$ is denoted by the symbol ψ_n. The character ψ_6 has already been mentioned. Other faithful irreducible characters are found as follows

$$\psi_{126} = \psi_6\phi_{21}$$

$$\psi_{210} = \psi_6\phi'_{35}$$

$$\psi_{84} = \theta_{20}\lambda_{15} - \psi_6 - \psi_{210}$$

$$\psi_{336} = \theta_{20}\lambda_{21} - \psi_{84}$$

$$\psi_{840} = \theta'_{56}\lambda_{15}$$

$$\psi_{120} = \psi_6\phi_{35} - \psi_6 - \psi_{84}$$

$$\psi_{420} = \psi_6\phi_{90} - \psi_{120}$$

$$\psi_{630} = \psi_6\phi_{140} - \psi^*_{210}$$

$$\psi'_{840} = \psi_6\phi_{210} - \psi_{420}$$

$$\psi_{384} = \psi_6\phi'_{280} - \psi_{120} - \psi^*_{336} - \psi'_{840}$$

$$\psi_{540} = \theta'_{70}\lambda_{15} - \psi_{126} - \psi_{384}$$

$$\psi_{1260} = \theta_{120}\lambda_{15} - \psi_{540}$$

These characters, together with their associates and complex conjugates, form the complete set of faithful irreducible characters of G.

All faithful irreducible characters have values in $Q(\sqrt{-3})$ and none are rational-valued. Also, as can be seen from the equations above, each of these characters has multiplicity 1 in some character afforded by a $Q(\sqrt{-3})$ representation. Thus each of these characters has Schur index 1 over Q, and in particular, $Q(\sqrt{-3})$ is a splitting field for G.

6. The tables. There are 169 conjugacy classes of G. Table I lists one class for each of the 34 cliques except C_{18b}, C_{9b}, and C_{24b}. The elements of C_{18b}, C_{9b}, and C_{24b} are inverses of the elements of C_{18a}, C_{9a}, and C_{24a}, respectively. The even classes are listed first, and except where noted otherwise, each clique contains 6 classes. The order and characteristic polynomial given for each clique is for an element of smallest order. Representatives for other classes in a clique can be found by multiplying by appropriate powers of z, where z has characteristic polynomial $(x+\omega)^6$.

The character table for G is given in 4 parts. Table II gives the characters ϕ_n of the central quotient group G_1. The 34 irreducible characters of G_1 are the 16 given characters plus 1, $\bar{\phi}_{280}$, and the 16 possible associates. The 29 faithful irreducible characters of G_2 are the 15 characters θ_n given in Table III plus $\bar{\theta}_{70}$ and the 13 possible associates. The characters λ_n given in Table IV satisfy $\lambda_n(z) = n\omega$. The 58 faithful irreducible characters of G_3 are the 15 given characters plus their complex conjugates and the 28 possible associates. The characters ψ_n given in Table V are faithful characters of G satisfying $\psi_n(z) = -n\omega$. The final 48 irreducible characters are the 13 given characters plus their complex conjugates and the 22 possible associates.

The following notation is used in the character table to simplify the display: $a = 1+2\omega$, $b = 1+3\omega$, $c = 1+4\omega$, $d = 1+6\omega$.

TABLE I. CONJUGACY CLASSES OF G

Clique	Smallest Order	Class Size	Centralizer	Characteristic Polynomial
(Even classes)				
1	1	1	$2^9 \cdot 3^7 \cdot 5 \cdot 7$	$(x-1)^6$
3	2	2835	$2^9 \cdot 3^3$	$(x-1)^4(x+1)^2$
4	3	3360	$2^4 \cdot 3^6$	$(x-1)^3(x^3-1)$
8	3	560	$2^5 \cdot 3^7$	$(x^2+x+1)^3$
10	6	90720	$2^4 \cdot 3^3$	$(x^2-1)^2(x^2+x+1)$
13	6	45360	$2^5 \cdot 3^3$	$(x^2+x+1)(x^4+x^2+1)$
14	5	653184	$2^2 \cdot 3 \cdot 5$	$(x-1)(x^5-1)$
15	4	34020	$2^7 \cdot 3^2$	$(x-1)^2(x^2+1)^2$
17	6	90720	$2^4 \cdot 3^3$	$(x-1)^2(x^4+x^2+1)$
18a	9	120960	$2^2 \cdot 3^4$	$(x-1)^2(x-\omega)(x^3-\bar{\omega})$
28	12	272160	$2^4 \cdot 3^2$	$(x^2-x+1)(x^4-x^2+1)$
30	7	933120	$2 \cdot 3 \cdot 7$	$x^6+x^5+x^4+x^3+x^2+x+1$
11 (2 classes)	3	120960	$2^2 \cdot 3^4$	$(x^3-1)^2$
12 (3 classes)	4	408240	$2^5 \cdot 3$	$(x^2-1)(x^4-1)$
16 (2 classes)	3	10080	$2^4 \cdot 3^5$	$(x^3-1)^2$
29 (2 classes)	9	725760	$2 \cdot 3^3$	x^6+x^3+1
31 (3 classes)	8	816480	$2^4 \cdot 3$	$(x^2+1)(x^4+1)$
(Odd classes)				
2	2	126	$2^8 \cdot 3^5 \cdot 5$	$(x-1)^5(x+1)$
6	6	60480	$2^3 \cdot 3^4$	$(x-1)(x^2-1)(x^3-1)$
7	4	68040	$2^6 \cdot 3^2$	$(x^2-1)(x^4-1)$
9a	6	5040	$2^5 \cdot 3^5$	$(x-\bar{\omega})^2(x^2-\bar{\omega})(x^2+x+1)$
19	6	30240	$2^4 \cdot 3^4$	$(x-1)^2(x-\omega)^2(x^2-\omega)$
20	12	544320	$2^3 \cdot 3^2$	$(x^2+x+1)(x^4-1)$
21	10	653184	$2^2 \cdot 3 \cdot 5$	$(x+1)(x^5-1)$
24*a*	18	362880	$2^2 \cdot 3^3$	$(x-\bar{\omega})(x^2-1)(x^3-\omega)$
27	12	272160	$2^4 \cdot 3^2$	$(x-\omega)(x+\bar{\omega})(x^4-x^2+1)$
5 (3 classes)	2	11340	$2^7 \cdot 3^3$	$(x^2-1)^3$
22 (2 classes)	6	1088640	$2^2 \cdot 3^2$	x^6-1
23 (2 classes)	6	272160	$2^4 \cdot 3^2$	x^6-1
25 (3 classes)	8	816480	$2^4 \cdot 3$	$(x^2-1)(x^4+1)$
26 (3 classes)	4	68040	$2^6 \cdot 3^2$	$(x^2+1)(x^4-1)$

TABLE II. CHARACTERS OF G_1

Clique															
1	21	35	35	90	140	189	210	280	315	315	420	560	729	896	560
3	5	3	3	10	12	−3	2	−8	11	11	4	−16	9	0	−16
4	3	8	−1	9	−4	0	3	10	18	−9	6	2	0	−4	2
8	−6	8	8	9	5	27	21	10	−9	−9	−39	−34	0	32	20
10	−1	0	3	1	0	0	−1	−2	2	−1	−2	2	0	0	2
13	2	0	0	1	−3	3	5	−2	−1	−1	1	2	0	0	−4
14	1	0	0	0	0	−1	0	0	0	0	0	0	−1	1	0
15	1	3	3	−2	4	5	−2	0	−1	−1	4	0	−3	0	0
17	−1	3	0	1	0	0	−1	1	−1	2	−2	2	0	0	−4
18a	0	2	−1	0	−1	0	0	b	0	0	0	−1	0	−1	2
28	−2	0	0	1	1	−1	1	0	−1	−1	1	0	0	0	0
30	0	0	0	−1	0	0	0	0	0	0	0	0	1	0	0
11	3	−1	−1	0	5	0	3	1	0	0	−3	2	0	−4	2
12	1	−1	−1	2	0	1	−2	0	−1	−1	0	0	1	0	0
16	3	−1	8	9	−4	0	3	1	−9	18	6	2	0	−4	20
29	0	−1	2	0	−1	0	0	1	0	0	0	−1	0	−1	−1
31	−1	−1	−1	0	0	1	0	0	1	1	0	0	−1	0	0
2	9	15	−5	30	20	9	30	40	75	−15	60	0	81	64	0
6	3	0	1	3	2	0	3	−2	0	−3	0	0	0	−2	0
7	1	3	−1	2	0	−3	−2	0	3	1	0	0	−3	0	0
9a	0	6	4	3	−7	9	3	−2d	3	3	−3	6a	0	−8	0
19	−3	3	−2	3	2	0	3	1	−3	0	−6	0	0	4	0
20	1	0	−1	−1	0	0	1	0	0	1	0	0	0	0	0
21	−1	0	0	0	0	−1	0	0	0	0	0	0	1	−1	0
24a	0	0	1	0	−1	0	0	ω	0	0	0	a	0	1	0
27	0	2	0	−1	1	1	−1	0	−1	−1	1	0	0	0	0
5	1	−1	−5	6	4	9	−10	−8	3	9	−4	0	9	0	0
22	1	−1	1	0	1	0	−1	1	0	0	−1	0	0	0	0
23	1	−1	−2	3	−2	0	−1	1	−3	0	2	0	0	0	0
25	−1	−1	−1	0	0	1	0	0	1	−1	0	0	−1	0	0
26	−3	−1	3	2	4	1	2	0	−1	5	4	0	−3	0	0

TABLE III. FAITHFUL CHARACTERS OF G_2

Clique															
1	20	56	56	70	70	120	504	504	540	560	630	896	140	420	1280
3	−4	8	8	2	2	8	8	8	−12	16	−14	0	4	−20	0
4	2	11	2	7	7	−6	18	−9	0	2	9	−4	−4	6	−16
8	−7	2	2	−11	16	12	18	18	−27	−34	−18	32	−22	42	−16
10	2	−1	2	−1	−1	2	2	−1	0	−2	1	0	4	−2	0
13	−1	2	2	−1	−4	−4	2	2	3	−2	−2	0	−2	−2	0
14	0	1	1	0	0	0	−1	−1	0	0	0	1	0	0	0
15	4	0	0	2	2	0	0	0	4	0	−6	0	4	4	0
17	2	2	−1	2	−1	2	−1	2	0	−2	1	0	−2	−2	0
18a	−1	2	−1	b	1	0	0	0	0	−1	0	−1	2	0	2
28	−1	0	0	1	−2	0	0	0	−1	0	0	0	2	2	0
30	−1	0	0	0	0	1	0	0	1	0	0	0	0	0	−1
11	2	2	2	−2	−2	3	0	0	0	2	0	−4	−4	6	2
16	2	2	11	−2	7	−6	−9	18	0	2	9	−4	14	6	−16
29	−1	−1	2	1	1	0	0	0	0	−1	0	−1	−1	0	2
2	0	24	−16	20	20	0	96	24	0	−80	60	64	0	0	0
6	0	3	−4	−1	−1	0	0	3	0	−2	−3	−2	0	0	0
7	0	4	0	2	2	0	0	−4	0	0	−2	0	0	0	0
$9a$	$3\bar{a}$	6	2	$-d$	2	0	6	6	$9\bar{a}$	10	6	−8	0	0	0
19	0	0	−1	−4	5	0	3	0	0	4	−3	4	0	0	0
20	0	1	0	−1	−1	0	0	−1	0	0	1	0	0	0	0
21	0	−1	−1	0	0	0	1	−1	0	0	0	−1	0	0	0
$24a$	a	0	−1	$-\omega$	−1	0	0	0	0	1	0	1	0	0	0
27	$\bar{a}$	0	0	$\bar{a}$	0	0	0	0	a	0	0	0	0	0	0
22	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0
23	0	0	−3	0	3	0	−3	0	0	0	3	0	0	0	0

TABLE IV. FAITHFUL CHARACTERS OF G_3

Clique															
-F	1	15	21	105	105	105	210	315	336	384	420	630	729	756	945
3		−1	5	−7	9	9	2	−5	16	0	4	6	9	−12	−15
4		3	6	3	12	3	15	9	6	12	−6	−9	0	0	0
8		6	3	15	−12	15	3	−36	−6	24	33	9	0	27	−27
10		−1	2	−1	0	3	−1	1	−2	0	−2	3	0	0	0
13		2	−1	−1	0	3	−1	4	−2	0	1	−3	0	3	−3
14		0	1	0	0	0	0	0	1	−1	0	0	−1	1	0
15		3	1	5	1	1	−2	3	0	0	4	2	−3	−4	1
17		2	2	2	0	0	2	−2	−2	0	−2	0	0	0	0
18a		$\bar{a}$	$\bar{\omega}a$	$\bar{\omega}\bar{a}$	$\omega\bar{a}$	$\bar{\omega}\bar{a}$	a	0	ωa	$\bar{a}$	$\omega\bar{a}$	0	0	0	0
28		0	1	−1	−2	1	1	0	0	0	1	−1	0	−1	1
30		1	0	0	0	0	0	0	0	−1	0	0	1	0	0
12		−1	1	1	1	1	−2	−1	0	0	0	−2	1	0	1
31		1	−1	−1	1	1	0	−1	0	0	0	0	−1	0	1
2		5	11	5	35	25	50	45	−64	64	20	−30	81	36	45
6		−1	2	−1	2	1	−1	−3	−4	−2	2	3	0	0	0
7		1	3	1	3	1	2	1	0	0	0	−2	−3	0	−3
9a		-4ω	$\bar{c}$	$-3+8\omega$	$2\omega b$	$-\bar{c}$	$4\bar{\omega}-5$	0	$2\omega b$	-8ω	$3+8\omega$	$3\bar{c}$	0	9	−9
19		$2\bar{\omega}$	$2\bar{\omega}$	$2\bar{\omega}$	$-4\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	$-6\bar{\omega}$	$2\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	0	0	0	0
20		1	0	1	0	1	−1	1	0	0	0	1	0	0	0
21		0	1	0	0	0	0	0	1	−1	0	0	1	1	0
24a		−1	$-\omega$	$-\omega$	$-\bar{\omega}$	ω	−1	0	$-\bar{\omega}$	1	$-\bar{\omega}$	0	0	0	0
27		−2	−1	1	0	1	1	0	0	0	−1	1	0	1	−1
5		−3	3	−3	3	9	−6	−3	0	0	−12	−6	9	−12	−3
25		−1	1	−1	−1	1	0	1	0	0	0	0	−1	0	1
26		−1	1	−1	−3	−1	2	3	0	0	4	2	3	−4	−5

TABLE V. FAITHFUL CHARACTERS OF G

Clique													
1	6	84	120	126	210	336	384	420	630	840	840	540	1260
3	2	-4	8	10	6	-16	0	12	18	-8	-8	-12	4
4	3	6	15	9	-3	6	12	12	-9	6	-3	0	-18
8	-3	-15	-6	18	-24	-6	24	-21	9	12	-42	54	18
10	-1	2	-1	1	-3	2	0	0	3	-2	1	0	-2
13	-1	-1	2	-2	0	2	0	-3	3	4	-2	-6	-2
14	1	-1	0	1	0	1	-1	0	0	0	0	0	0
15	2	4	0	2	6	0	0	-4	2	0	0	4	-4
17	2	2	2	-2	0	2	0	0	0	-2	-2	0	4
18a	$\omega\bar{a}$	a	$\bar{\omega}a$	0	ωa	$-\omega\bar{a}$	$\bar{a}$	$\bar{\omega}\bar{a}$	0	a	$\bar{\omega}a$	0	0
28	1	-1	0	-2	0	0	0	1	1	0	0	2	-2
30	-1	0	1	0	0	0	-1	0	0	0	0	1	0
2	4	16	40	36	-20	-16	64	80	60	-80	40	0	0
6	1	-2	1	3	1	2	-2	2	3	4	1	0	0
7	2	0	4	2	-2	0	0	0	-2	0	-4	0	0
9a	$2\bar{\omega}-7$	$2\bar{\omega}-t$	$2\bar{c}$	0	$4\omega b$	$8\omega-6$	-8ω	$2\omega-3$	$-3\omega b$	-8ω	$2\bar{c}$	0	0
19	$-2\bar{\omega}$	$-2\bar{\omega}$	$-2\bar{\omega}$	$6\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	$4\bar{\omega}$	$-4\bar{\omega}$	0	$-2\bar{\omega}$	$-2\bar{\omega}$	0	0
20	-1	0	1	-1	1	0	0	0	1	0	-1	0	0
21	-1	1	0	1	0	-1	-1	0	0	0	0	0	0
24a	$\bar{\omega}$	1	ω	0	$\bar{\omega}$	$-\bar{\omega}$	1	$-\omega$	0	1	ω	0	0
27	a	a	0	0	0	0	0	$\bar{a}$	a	0	0	0	0

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