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# A CONDITIONAL ENTROPY FOR THE SPACE OF PSEUDO-MENGER MAPS

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**Let $X$ be a set and $T: X \to X$ be a bijection. Consider the space $\mathscr{M}$ of pseudo-Menger maps on $X$ which induce a compact topology on $X$ for which $T$ is a homeomorphism. The lattice properties of $\mathscr{M}$ are investigated and a bivariate nonnegative function of $\mathscr{M}$ is defined which possesses certain properties analogous to those of the usual conditional entropy function defined on the space of measurable partitions of a probability space.**

**1. Introduction.** A pseudo-Menger map on a set $X$ is, roughly speaking, an assignment of a distribution function to every pair of points in $X$ in a manner consistent with the axioms of a pseudo-metric space. Each such map induces a topology on $X$ as defined by Schweizer and Sklar [9]. Let $T$ be a bijection of $X$ onto itself. Let $\mathscr{M}$ denote the space of all pseudo-Menger maps which induce a compact topology on $X$ for which $T$ is a homeomorphism. If $\theta \in \mathscr{M}$ let $h(T, \theta)$ denote the topological entropy of $T$ with respect to the topology on $X$ induced by $\theta$. In [7] it was shown that $h(T, \cdot)$ is left-continuous in the sense that if $\theta, \theta_n \in \mathscr{M}$, $\theta_n \geqq \theta$ and $\theta_n(p, q) \to \theta(p, q)$ in distribution for all $p, q \in X$ then $h(T, \theta_n) \to h(T, \theta)$. In an effort to extend this result one is led to ask the question whether $\mathscr{M}$ is closed under the operations of Max and Min and, if so, what can one say about the entropy of $T$ acting on the topology engendered by the maps obtained as a result of such operations. We now proceed to provide precise definitions and notation.

**2. Preliminaries.** Let $I$ denote the closed unit interval, $\boldsymbol{R}$ the real numbers and $\boldsymbol{Z}^+$ the positive integers. Let $\mathscr{D}$ be the set of all left-continuous monotone increasing functions $F: \boldsymbol{R} \to I$ satisfying $F(0) = 0$ and $\sup_x F(x) = 1$. Endowed with the Lévy metric $\mathscr{D}$ is a complete metric space. If $F_n, F \in \mathscr{D}$ then $F_n \xrightarrow{\mathscr{D}} F$ will denote convergence with respect to the Lévy topology. It is well-known that $F_n \xrightarrow{\mathscr{D}} F$ if and only if $F_n(x) \to F(x)$ for each $x \in \boldsymbol{R}$ at which $F$ is continuous. If $F, G \in \mathscr{D}$ then $F \geqq G$ will mean $F(x) \geqq G(x)$ for all $x \in \boldsymbol{R}$. Let $H \in \mathscr{D}$ be the function defined by: $H(t) = 0$ for $t \leqq 0$ and $H(t) = 1$ for $t > 0$.

Let $X$ be a fixed set. Let $\mathscr{F}(X)$ denote the collection of all functions $\theta: X \times X \to \mathscr{D}$. For convenience we shall often write $\theta_{pq}$ in place of $\theta(p, q)$. A *statistical pseudo-metric space* is an ordered pair $(X, \theta)$ where $\theta \in \mathscr{F}$ satisfies:

(SM 1) $\theta_{pq} = \theta_{qp}$ for all $p, q \in X$

(SM 2) $\theta_{pq}(a + b) = 1$ whenever $\theta_{pr}(a) = \theta_{rq}(b) = 1$ for some $r \in X$

(SM 3) $\theta_{pp} = H$ for all $p \in X$.

If, in addition, $\theta$ satisfies:

(SM 4) $\theta_{pq} = H$ only if $p = q$

then $(X, \theta)$ is a *statistical metric space*. Let $\mathscr{S}(X)$ denote the collection of all $\theta$ for which $(X, \theta)$ is a statistical pseudo-metric space.

A *triangular norm* is a function $\Delta: I \times I \to I$ which is associative, commutative, non-decreasing in each variable and satisfies $\Delta(y, 1) = y$ for each $y \in I$. A continuous *Menger space* [*pseudo-Menger space*] is a statistical metric space [statistical pseudo-metric space] $(X, \theta)$ for which there exists a continuous triangular norm $\Delta$ satisfying:

(SM 5) $\theta_{pr}(a + b) \geqq \Delta(\theta_{pq}(a), \theta_{qr}(b))$ for all $p, q, r \in X$ and $a, b \in \boldsymbol{R}$.

Let $\mathscr{M}(X)$ denote the set of all $\theta$ for which $(X, \theta)$ is a continuous pseudo-Menger space. If $\theta_n, \theta \in \mathscr{M}(X)$ and $\theta_n(p, q) \xrightarrow{\mathscr{D}} \theta(p, q)$ for all $p, q \in X$ then we will write $\theta_n \xrightarrow{\mathscr{D}} \theta$. Similarly, if $\theta, \Gamma \in \mathscr{M}(X)$ and $\theta_{pq} \geqq \Gamma_{pq}$ for all $p, q \in X$ then we write $\theta \geqq \Gamma$. Let $\Xi \in \mathscr{M}(X)$ be defined by $\Xi_{pq} = H$ for all $p, q \in X$.

If $\theta \in \mathscr{F}(X)$ let $X$ be endowed with the topology, denoted $\tau(\theta)$, generated by all sets of the form $N(p, \varepsilon, \lambda, \theta) = \{q \in X: \theta_{pq}(\varepsilon) > 1 - \lambda\}$ where $p \in X$, $\varepsilon > 0$, $\lambda > 0$. Let $T: X \to X$ be a bijection. Let $\mathscr{M}(X, T) = \{\theta \in \mathscr{M}(X)$: $T$ is a self-homeomorphism of $(X, \tau(\theta))$ and $\tau(\theta)$ is compact$\}$.

If $\theta \in \mathscr{M}(X, T)$ we will let $h(T, \theta)$ denote the topological entropy of $T$ with respect to the $\tau(\theta)$ topology. We will follow the notation and definitions of topological entropy developed in [1]. The only exception is the understanding that if $T$ is a self-homeomorphism of $(X, \tau)$ where $\tau$ is not a compact topology and $\mathscr{U} \subset \tau$ is a cover of $X$ which possesses a finite subcover then $h(T, \mathscr{U}) = \lim_{k\to\infty} (1/k)H(\bigvee_{j=0}^{k-1} T^j\mathscr{U})$. We let $\mathscr{M}_F(X, T) = \{\theta \in \mathscr{M}(X, T): h(T, \theta) < \infty\}$.

Let $\rho$ be a pseudo-metric on a set $Y$ and let $\mathscr{U}$ be an open cover of $(Y, \rho)$. Then $\rho$-diam $\mathscr{U}$ will mean the sup $\{\rho\text{-diam } A: A \in \mathscr{U}\}$. Let $\tau(\rho)$ denote the topology on $Y$ determined by $\rho$. In addition, if $\varepsilon > 0$ and $a \in Y$ then let $B(Y, a, \rho, \varepsilon) = \{q \in Y: \rho(q, a) < \varepsilon\}$. If $D$ is another pseudo-metric on $Y$ then $\rho \geqq D$ will mean $\rho(a, b) \geqq D(a, b)$ for all $a, b \in Y$. Finally, if $X$ is a pseudo-metric space then $X^*$ will denote the (unique up to uniform isomorphism) completion for which $X^* \sim X$ is Hausdorff. Such an $X^*$ will be called the *pseudo-metric space completion of* $X$.

**3. Lattice operations.** If $\theta$, $\Psi \in \mathscr{M}(X, T)$ define $\theta \vee \Psi = \operatorname{Min}(\theta, \Psi)$ and $\theta \wedge \Psi = \operatorname{Max}(\theta, \Psi)$. It is easy to construct examples in which $\theta \wedge \Psi$ fails even to belong to $\mathscr{S}(X)$. However, we will show that $\theta \vee \Psi$ admits a canonical extension to a map belonging to $\mathscr{M}(X^*, T^*)$ where $X^*$ is the completion of $(X, \tau(\theta \vee \Psi))$ and $T^*$ is the extension of $T$ to $X^*$.

PROPOSITION 1. *If $\theta, \Psi \in \mathscr{M}(X)$ then $\theta \vee \Psi \in \mathscr{M}(X)$.*

*Proof.* Let $\Delta_1$ and $\Delta_2$ be continuous triangular norms for $\theta$ and $\Psi$ respectively which satisfy:

$$\theta_{pq}(a + b) \geqq \Delta_1(\theta_{pr}(a), \theta_{rq}(b))$$

and

$$\Psi_{pq}(a + b) \geqq \Delta_2(\Psi_{pr}(a), \Psi_{rq}(b)) \text{ for all } a, b \in \boldsymbol{R} \text{ and all } p, q, r \in X .$$

It is easy to check that $\Delta_3 = \operatorname{Min}(\Delta_1, \Delta_2)$ is a continuous triangular norm. Using the monotonicity of $\Delta_1$ and $\Delta_2$ we verify the triangle inequality for $\theta \vee \Psi$ with respect to $\Delta_3$:

$$\begin{aligned}(\theta \vee \Psi)_{pr}(a + b) &= \operatorname{Min}(\theta_{pr}(a + b), \Psi_{pr}(a + b)) \\ &\geqq \operatorname{Min}(\Delta_1(\theta_{pq}(a), \theta_{qr}(b)), \Delta_2(\Psi_{pq}(a), \Psi_{qr}(b))) \\ &\geqq \operatorname{Min}(\Delta_1, \Delta_2)(\operatorname{Min}(\theta_{pq}(a), \Psi_{pq}(a)), \operatorname{Min}(\theta_{qr}(b), \Psi_{qr}(b))) \\ &= \Delta_3((\theta \vee \Psi)_{pq}(a), (\theta \vee \Psi)_{qr}(b)) .\end{aligned}$$

LEMMA 1. *Assume $\theta, \Psi \in \mathscr{M}(X)$ determine compact topologies $\tau(\theta)$ and $\tau(\Psi)$ respectively. Let $d_1$ and $d_2$ be pseudo-metrics on $X$ which generate the topologies $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then the pseudo-metric $D = d_1 + d_2$ determines the topology $\tau(\theta \vee \Psi)$.*

*Proof.* As a consequence of Lemma 4 of [7] and the above proposition we know that $\tau(\theta \vee \Psi) \supset \tau(\theta) \cup \tau(\Psi)$. Thus it suffices to show that $\tau(\theta \vee \Psi)$ is generated by $\{A \cap C: A \in \tau(\theta) \text{ and } C \in \tau(\Psi)\}$. Let $p \in X$, $\varepsilon > 0$ and $\lambda > 0$ be given. Let $q \in N(p, \varepsilon, \lambda, \theta \vee \Psi)$. For each $n \in \boldsymbol{Z}^+$ choose $A_n = N(q, 1/n, 1/n, \theta)$ and $C_n = N(q, 1/n, 1/n, \Psi)$. Suppose for each $n$ there exists $y_n \in A_n \cap C_n$ such that $y_n \notin N(p, \varepsilon, \lambda, \theta \vee \Psi)$. Then $\theta_{qy_n}(1/n) > 1 - (1/n)$ and $\Psi_{qy_n}(1/n) > 1 - (1/n)$ from which $(\theta \vee \Psi)_{qy_n} \to H$. Since $(\theta \vee \Psi)_{pq}$ is left-continuous and $(\theta \vee \Psi)_{pq}(\varepsilon) > 1 - \lambda$, there exists a $\delta > 0$ for which $(\theta \vee \Psi)_{pq}(\varepsilon - \delta) > 1 - \lambda$. Now $(\theta \vee \Psi)_{py_n}(\varepsilon) \geqq \Delta((\theta \vee \Psi)_{pq}(\varepsilon - \delta), (\theta \vee \Psi)_{qy_n}(\delta)) \to (\theta \vee \Psi)_{pq}(\varepsilon - \delta) > 1 - \lambda$ from which one draws the contradiction that $y_n \in N(p, \varepsilon, \lambda, \theta \vee \Psi)$ for large $n$.

PROPOSITION 2. *Let $\theta, \Psi \in \mathscr{M}(X)$ and suppose $\tau(\theta)$ and $\tau(\Psi)$ are each compact. Then $\tau(\theta \vee \Psi)$ is totally bounded.*

*Proof.* Let $d_1, d_2$ be pseudo-metrics on $X$ which determine $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then $D = d_1 + d_2$ is a pseudo-metric for $\tau(\theta \vee \Psi)$. Let $\varepsilon > 0$ be given. Let $p_i, q_j \in X$, $1 \leqq i \leqq N$, $1 \leqq j \leqq M$, be chosen such that $\bigcup_{i=1}^N B(X, p_i, d_1, \varepsilon/2) = \bigcup_{j=1}^M B(X, q_j, d_2, \varepsilon/2) = X$. Then it is easy to verify that, for each $i$ and $j$, $B(X, p_i, d_1, \varepsilon/2) \cap B(X, q_j, d_2, \varepsilon/2) \subset B(X, z_{ij}, D, \varepsilon)$ for any $z_{ij} \in B(X, p_i, d_1, \varepsilon/2) \cap B(X, q_j, d_2, \varepsilon/2)$ provided this intersection is nonempty.

THEOREM 1. *Let $\theta, \Psi \in \mathscr{M}(X, T)$ and let $(X^*, \tau^*)$ denote the pseudo-metric space completion of $(X, \tau(\theta \vee \Psi))$. Then:*

1. *$T$ admits a unique extension to a self-homeomorphism $T^*$ of $(X^*, \tau^*)$.*
2. *$\theta \vee \Psi$ admits a unique extension to a map $(\theta \vee \Psi)^*: X^* \times X^* \to \mathscr{D}$*
3. *$(\theta \vee \Psi)^* \in \mathscr{M}(X^*, T^*)$*
4. *$\tau^* = \tau((\theta \vee \Psi)^*)$*

*Proof.* Let $D^*$ denote the pseudo-metric on $(X^*, \tau^*)$ which extends the pseudo-metric $D$ on $(X, \tau(\theta \vee \Psi))$. Since $\theta \vee \Psi: X \times X \to \mathscr{D}$ is a uniformly continuous map [8] it can be extended (Cor. 6.2, Ch. 14 of [3]) to a continuous map $(\theta \vee \Psi)^*: X^* \times X^* \to \mathscr{D}$. The work of Sherwood [11, 12] implies that $(\theta \vee \Psi)^* \in \mathscr{M}(X)$ and $\tau((\theta \vee \Psi)^*) = \tau^*$. Since $T$ is uniformly continuous on $(X, D)$ there exists an extension $T^*$ which is a self-homeomorphism of $(X^*, \tau^*)$. Now since $\tau^*$ is compact, $(\theta \vee \Psi)^* \in \mathscr{M}(X^*, T^*)$.

**4. Entropy.** We begin by investigating the relation among $h(T, \theta \vee \Psi)$, $h(T, \theta)$ and $h(T, \Psi)$. Several lemmas are required.

LEMMA 2. *Let $(X, \rho)$ be a compact pseudo-metric space and $T: X \to X$ be a homeomorphism. Let $\{\mathscr{U}_n: n \in \boldsymbol{Z}^+\}$ be a sequence of open covers of $X$ satisfying $\rho$-diam $\mathscr{U}_n \to 0$ as $n \to \infty$. Then $h(T, \mathscr{U}_n) \to h(T)$.*

*Proof.* Let $\{\mathscr{U}_{n_j}: j \in \boldsymbol{Z}^+\}$ be a subsequence of the $\{\mathscr{U}_n\}$. Using the Lebesgue covering lemma one can select a subsequence $\{m_j\}$ of the $\{n_j\}$ such that $\mathscr{U}_{m_j} \prec \mathscr{U}_{m_{j+1}}$ for $j \geqq 1$. Now applying the Corollary on page 314 of [1] the desired result is obtained.

LEMMA 3. *Suppose $D$ and $d$ are pseudo-metrics on $X$ satisfying $D \geqq d$, $\tau(d)$ is compact and $\tau(D)$ is totally bounded. Assume $T$ is a self-homeomorphism of $(X, d)$ and of $(X, D)$. Let $\{\mathscr{V}_n: n \in \boldsymbol{Z}^+\}$ be a sequence of $\tau(D)$-open covers of $X$ such that $D$-diam $\mathscr{V}_n \to 0$ as $n \to \infty$ and each $\mathscr{V}_n$ possesses a finite subcover of $X$. Then sup $\{h(T, \mathscr{U}): \mathscr{U} \subset \tau(d)\} \leqq \overline{\lim}_{n\to\infty} h(T, \mathscr{V}_n)$.*

*Proof.* Let $\mathscr{W}_n$ be a sequence of $\tau(d)$-open covers of $X$ such that $d$-diam $\mathscr{W}_n \to 0$. Then for each $n > 0$ there exists an $m \geqq n$ such that $\mathscr{W}_n \prec \mathscr{V}_m$. Thus $\lim_n h(T, \mathscr{W}_n) \leqq \overline{\lim}_n h(T, \mathscr{V}_n)$. Now $\sup \{h(T, \mathscr{U}): \mathscr{U} \subset \tau(d)\} = \lim_n h(T, \mathscr{W}_n)$.

LEMMA 4. *Let $\theta, \Psi \in \mathscr{M}(X, T)$ and let $D$ be a pseudo-metric on $X$ for which $\tau(D) = \tau(\theta \vee \Psi)$. For each $\varepsilon > 0$ let $\mathscr{U}_\varepsilon = \{B(X, p, D, \varepsilon): p \in X\}$. Then $h(T^*, (\theta \vee \Psi)^*) = \sup_\varepsilon h(T, \mathscr{U}_\varepsilon)$.*

*Proof.* Let $\mathscr{U}_\varepsilon^* = \{B(X^*, p, D^*, \varepsilon): p \in X\}$. Then $\mathscr{U}_\varepsilon = \{A \cap X: A \in \mathscr{U}_\varepsilon^*\}$ and $h(T^*, (\theta \vee \Psi)^*) = \sup_\varepsilon h(T^*, \mathscr{U}_\varepsilon^*)$. It is easy to verify that $N(\bigvee_0^K T^{*i}\mathscr{U}_\varepsilon^*) = N(\bigvee_0^K T^i\mathscr{U}_\varepsilon)$ for all $K \geqq 0$. Consequently $h(T^*, \mathscr{U}_\varepsilon^*) = h(T, \mathscr{U}_\varepsilon)$ and the lemma is proven.

THEOREM 2. *Let $\theta, \Psi \in \mathscr{M}(X, T)$. Then:*

$$\operatorname{Max}(h(T, \theta), h(T, \Psi)) \leqq h(T^*, (\theta \vee \Psi)^*) \leqq h(T, \theta) + h(T, \Psi) .$$

*Proof.* Let $d_1$ and $d_2$ be pseudo-metrics on $X$ which generate $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then $D = d_1 + d_2$ is a pseudo-metric for $\tau(\theta \vee \Psi)$. Let $\varepsilon_n$ be a sequence of positive numbers such that $\varepsilon_n \to 0$. Let $\mathscr{V}_n = \{B(X, p, D, \varepsilon_n): p \in X\}$ and $\mathscr{V}_n^* = \{B(X^*, p, D^*, \varepsilon_n): p \in X\}$. Applying lemma 3, we have $h(T^*, (\theta \vee \Psi)^*) = \lim_{n\to\infty} h(T^*, \mathscr{V}_n^*) \geqq \overline{\lim}_{n\to\infty} h(T, \mathscr{V}_n) \geqq \sup\{h(T, \mathscr{U}): \mathscr{U} \subset \tau(d_1)\} = h(T, \theta)$.

Let $\mathscr{P}_n = \{B(X, p, d_1, 1/n): p \in X\}$, $\mathscr{Q}_n = \{B(X, p, d_2, 1/n): p \in X\}$ and $\mathscr{R}_n = \{B(X, p, D, 1/n): p \in X\}$. Since $\mathscr{R}_n \prec \mathscr{P}_{4n} \vee \mathscr{Q}_{4n}$ we have $h(T, \mathscr{R}_n) \leqq h(T, \mathscr{P}_{4n} \vee \mathscr{Q}_{4n}) \leqq h(T, \mathscr{P}_{4n}) + h(T, \mathscr{Q}_{4n})$. Lemma 4 yields $h(T^*, (\theta \vee \Psi)^*) = \lim_{n\to\infty} h(T, \mathscr{R}_n) \leqq \lim_{n\to\infty} h(T, \mathscr{P}_{4n}) + \lim_{n\to\infty} h(T, \mathscr{Q}_{4n}) = h(T, \theta) + h(T, \Psi)$.

EXAMPLE 1. Let $Y = \{0, 1, 2\}$ and $X = Y^Z$. Define the shift $T: X \to X$ by $T(\{y_i\}) = \{y_{i+1}\}$. Let $d_1$ and $d_2$ be pseudo-metrics on $X$ given by:

$$d_1(\{u_i\}, \{z_i\}) = \sum_{i=-\infty}^{\infty} \frac{|\omega(u_i) - \omega(z_i)|}{2^{|i|}}$$

where

$$\omega(a) = \begin{cases} 0 & \text{if} \quad a = 2 \\ 1 & \text{if} \quad a = 0, 1 \end{cases}$$

and

$$d_2(\{u_i\}, \{z_i\}) = \sum_{i=-\infty}^{\infty} \frac{|\alpha(u_i) - \alpha(z_i)|}{2^{|i|}}$$

where

$$\alpha(a) = \begin{cases} 0 & \text{if} \quad a = 0 \\ 1 & \text{if} \quad a = 1, 2 \end{cases}$$

for all $\{u_i\}$ and $\{z_i\} \in X$.

Define $\theta, \Psi \in \mathscr{M}(X, T)$ by:

$$\theta_{uz}(\varepsilon) = H(\varepsilon - d_1(u, z))$$

and

$$\Psi_{uz}(\varepsilon) = H(\varepsilon - d_2(u, z)) \qquad \text{for all } \varepsilon > 0 \text{ and all } u, z \in X\,.$$

Then it follows that $h(T^*, (\theta \vee \Psi)^*) = ln\ 3$, $\mathrm{Max}\,(h(T, \theta), h(T, \Psi)) = ln\ 2$, and $h(T, \theta) + h(T, \Psi) = ln\ 4$.

DEFINITION. If $\theta, \Psi \in \mathscr{M}_F(X, T)$ let $h_T(\theta \mid \Psi) = h(T^*, (\theta \vee \Psi)^*) - h(T, \Psi)$

PROPOSITION 3. *Assume* $\theta, \Psi, \Gamma \in \mathscr{M}_F(X, T)$. *Then*:

(a) $0 \leqq h_T(\theta \mid \Psi) \leqq h(T, \theta)$

(b) $h_T(\theta \mid \theta) = 0$

(c) $h_T(\theta \vee \Psi \mid \Gamma) = h_T(\theta \mid \Psi \vee \Gamma) + h_T(\Psi \mid \Gamma)$ *provided* $\theta \vee \Psi$, $\Psi \vee \Gamma \in \mathscr{M}_F(X, T)$

(d) $h_T(\theta \mid \Xi) = h(T, \theta)$

*Proof.* Statement (*a*) is a corollary of Theorem 2. Statements (b), (c) and (*d*) follow quickly from the definitions.

PROPOSITION 4. *Let* $\theta, \Gamma \in \mathscr{M}_F(X, T)$. *Suppose* $\theta \vee \Gamma \in \mathscr{M}_F(X, T)$ *and that* $(X, \Gamma)$ *is a Menger space. Then* $h_T(\theta \mid \Gamma) = 0$.

*Proof.* This follows from the fact that any two compact metrizable topologies on $X$, each of which renders $T$ a homeomorphism, yield the same topological entropy for $T$.

PROPOSITION 5. *Let* $\theta, \Psi \in \mathscr{M}_F(X, T)$. *Assume that* $\theta \vee \Psi \in \mathscr{M}_F(X, T)$ *and that* $\theta_{pq} = H$ *if and only if* $\Psi_{pq} = H$. *Then* $h_T(\theta|\Psi) = 0$.

*Proof.* For $x, y \in X$, define $x \sim y$ if and only if $\theta_{xy} = H$. This equivalence relation on $X$ induces a self-homeomorphism $\tilde{T}$ of $X/\sim$. It is easy to verify that $h(T) = h(\tilde{T})$. One can then apply Proposition 4.

PROPOSITION 6. *Let* $\theta, \Psi, \Gamma \in \mathscr{M}_F(X, T)$ *and suppose* $\Psi \leqq \Gamma$. *Then* $h_T(\Psi \mid \theta) \geqq h_T(\Gamma \mid \theta)$.

*Proof.* Lemma 4 of [7] yields $\tau(\Gamma) \subset \tau(\Psi \vee \theta)$. Since $\tau(\theta) \subset \tau(\Psi \vee \theta)$ we have $\tau(\Gamma \vee \theta) \subset \tau(\Psi \vee \theta)$. Let $(X_1^*, \tau_1^*)$ and $(X_2^*, \tau_2^*)$ denote the completions of $(X, \tau(\theta \vee \Gamma))$ and $(X, \tau(\theta \vee \Psi))$ respectively, and let $T_1^*$ and $T_2^*$ denote the extensions of $T$ to $X_1^*$ and $X_2^*$ respectively. One may assume that $X_1^* \subset X_2^*$ and that $T_2^*$ extends $T_1^*$. Then the relative topology on $X_1^*$ induced by $\tau_2^*$ contains $\tau_1^*$. Let $D_1$ and $D_2$ denote pseudo-metrics on $X_1^*$ which generate $\tau_1^*$ and $\tau_2^* |_{X_1^*}$ (the topology induced on $X_1^*$ by $\tau_2^*$) respectively. By replacing $D_2$ with $D_1 + D_2$, if necessary, we may assume that $D_1 \leqq D_2$. Let $D_2^*$ denote the extension of $D_2$ to $(X_2^*, \tau_2^*)$. Let $\mathscr{V}_n^* = \{B(X_2^*, p, D_2^*, 1/n): p \in X_1^*\}$ and $\mathscr{V}_n = \{A \cap X_1^*: A \in \mathscr{V}_n^*\}$. Applying Lemma 3 together with the fact that $h(T_1^*, \mathscr{V}_n) = h(T_2^*, \mathscr{V}_n^*)$, we have:

$$\begin{aligned} h(T_1^*, (\theta \vee \Gamma)^*) &\leqq \overline{\lim_{n\to\infty}}\, h(T_1^*, \mathscr{V}_n) \\ &\leqq \sup_n h(T_2^*, \mathscr{V}_n^*) \\ &= h(T_2^*, (\theta \vee \Psi)^*)\,. \end{aligned}$$

EXAMPLE 2. Two examples are given to show that, in general, if $\theta, \Psi, \Gamma \in \mathscr{M}_F(X, T)$ and $\Psi \leqq \Gamma$ then $h_T(\theta \mid \Psi)$ may be less than or greater than $h_T(\theta \mid \Gamma)$.

First, note that $\Psi \leqq \Xi$ and $h_T(\theta \mid \Xi) = h(T, \theta)$. Thus, if $h(\mathrm{T}, \theta) > 0$, then $0 = h_T(\theta \mid \theta) < h_T(\theta \mid \Xi)$.

We now provide an example in which $\Psi \leqq \Gamma$ and $h_T(\theta \mid \Psi) > h_T(\theta \mid \Gamma)$. Let $Y = \{1, 2, 3, 4, 5\}$ and $X = Y^Z$. Define three pseudo-metrics on $X$ as follows:

$$D(\{y_i\}, \{z_i\}) = \sum_{-\infty}^{\infty} \frac{|c(y_i) - c(z_i)|}{2^{|i|}}$$

where

$$c(a) = \begin{cases} 1 & \text{if} \quad a = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$d(\{y_i\}, \{z_i\}) = \sum_{-\infty}^{\infty} \frac{|s(y_i) - s(z_i)|}{2^{|i|}}$$

where

$$s(a) = \begin{cases} 2 & \text{if} \quad a = 1 \quad \text{or} \quad 3 \\ 1 & \text{if} \quad a = 2 \quad \text{or} \quad 4 \\ 0 & \text{if} \quad a = 5 \end{cases}$$

$$\rho(\{y_i\}, \{z_i\}) = \sum_{-\infty}^{\infty} \frac{|t(y_i) - t(z_i)|}{2^{|i|}}$$

where

$$t(a) = \begin{cases} 1 & \text{if} \quad a = 1 \quad \text{or} \quad 2 \\ 0 & \text{otherwise} \end{cases}$$

for all $\{y_i\}, \{z_i\} \in X$.

Define $\theta, \Psi, \Gamma \in \mathscr{M}(X)$ as follows:

$$\theta_{pq}(\varepsilon) = H(\varepsilon - \rho(p, q))$$
$$\Gamma_{pq}(\varepsilon) = H(\varepsilon - D(p, q))$$
$$\Psi_{pq}(\varepsilon) = H(\varepsilon - d(p, q))$$

for all $p, q \in X$.

Let $T: X \to X$ be the shift given by $T(\{y_i\}) = \{y_{i+1}\}$. Then $\theta, \Gamma, \Psi \in \mathscr{M}(X, T)$, $\Psi \leqq \Gamma$, $h_T(\theta \mid \Psi) = h(T, \theta \vee \Psi) - h(T, \Psi) = ln\ 5 - ln\ 3 = ln\ 5/3$, and $h_T(\theta \mid \Gamma) = h(T, \theta \vee \Gamma) - h(T, \Gamma) = ln\ 3 - ln\ 2 = ln\ 3/2$.

THEOREM 3. *Let $\theta_n, \Psi_n, \theta, \Psi \in \mathscr{M}_F(X, T)$. Suppose $\theta_n \geqq \theta$, $\Psi_n \geqq \Psi$, $\theta_n \xrightarrow{\mathscr{D}} \theta$, $\Psi_n \xrightarrow{\mathscr{D}} \Psi$ and $\theta \vee \Psi \in \mathscr{M}(X, T)$. Then $h_T(\theta_n \mid \Psi_n) \to h_T(\theta \mid \Psi)$ as $n \to \infty$.*

The proof of this theorem is based upon the following lemma.

LEMMA 5. *Assume $\Sigma \in \mathscr{M}(X, T)$, $\Omega \in \mathscr{M}(X)$, $\tau(\Omega)$ is totally bounded, $\Omega \geqq \Sigma$ and $T$ is a self-homeomorphism of $(X, \tau(\Omega))$. Then $\Omega \in \mathscr{M}(X, T)$.*

*Proof.* We must show $\tau(\Omega)$ is compact. Let $X^*$ be the completion of $(X, \tau(\Omega))$ and $T^*$ be the extension of $T$ to $X^*$. So $\Omega^* \in \mathscr{M}(X^*, T^*)$. Since $X^*$ is compact, Lemma 3 of [7] is applicable. Thus, given $q \in X^*$, $\{N(q, \varepsilon, \lambda, \Omega^*): \varepsilon, \lambda > 0\}$ is a local basis for $\tau(\Omega^*)$ at $q$. If $p \in X$ observe that $N(p, \varepsilon, \lambda, \Omega^*) \cap X = N(p, \varepsilon, \lambda, \Omega)$. Hence, for $p \in X$, $\{N(p, \varepsilon, \lambda, \Omega): \varepsilon, \lambda > 0\}$ is a local basis for $\tau(\Omega)$ at $p$. Now an argument analogous to that used in lemma 4 of [7] yields $\tau(\Omega) \subset \tau(\Sigma)$.

*Proof of Theorem* 3. Since $\theta_n \vee \Psi_n \geqq \theta \vee \Psi \in \mathscr{M}_F(X, T)$ and $\theta_n \vee \Psi_n \in \mathscr{M}(X)$ the preceding lemma yields $\theta_n \vee \Psi_n \in \mathscr{M}(X, T)$. Now, applying the main theorem of [7], $h(T, \theta_n \vee \Psi_n) \to h(T, \theta \vee \Psi)$ and $h(T, \Psi_n) \to h(T, \Psi)$. The desired result now follows.

We conclude with a brief consideration of a special class of pseudo-Menger maps.

DEFINITION. The map $\theta \in \mathscr{M}(X, T)$ is $(X, T)$-deterministic if $h(T, \theta) = 0$.

PROPOSITION 7. *Let $\theta, \Gamma \in \mathscr{M}(X, T)$ and suppose that $\theta \geqq \Gamma$. If $\Gamma$ is $(X, T)$-deterministic then so is $\theta$.*

*Proof.* This follows at once from Lemma 4 of [7] where it is shown that $\tau(\theta) \subset \tau(\Gamma)$.

The following two propositions are consequences of Theorem 2.

PROPOSITION 8. *Let $\theta, \Gamma \in \mathscr{M}(X, T)$. Then $\theta$ and $\Gamma$ are $(X, T)$-deterministic if and only if $\theta \vee \Gamma$ is $(X^*, T^*)$-deterministic.*

PROPOSITION 9. *If $\Gamma \in \mathscr{M}(X, T)$ is $(X, T)$-deterministic and $\theta \in \mathscr{M}_F(X, T)$ then $h_T(\theta \mid \Gamma) = h(T, \theta)$.*

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