PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

COLLECTIVELY COMPACT SETS AND THE ERGODIC THEORY OF SEMI-GROUPS

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Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group of linear operators on a Banach space X such that 1 is an eigenvalue of each $T(t)$ and $T(a)$ is compact for some $a > 0$. Then the ergodic limit $A(t) = \lim_{n\to\infty} (1/n)\{T(t) + T^2(t) + \cdots + T^n(t)\}$ exists for each t. In this paper it is proved that if each $T(t)$, $t > 0$, is compact and 1 is, in a certain sense, an isolated eigenvalue of all $T(t)$, then for $t > 0$, the dimension of the null space of $T(t) - I$ is independent of t. Sufficient conditions are also obtained for the $\lim_{t\to\infty} T(t)$ to exist.

Suppose X is a real or complex Banach space. Let B denote the unit ball in X and $[X]$ the space of bounded linear operators on X into X. A set $\mathscr{K} \subset [X]$ is said to be collectively compact if $\mathscr{K}B = \{Kx: K \in \mathscr{K}, x \in B\}$ is relatively compact. Basic properties of such sets were obtained by Anselone and Palmer [1, 2]. Some of their results will be applied to semi-groups in the following sections.

2. Ergodic family associated with a semi-group. Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group of linear operators on a B-space X such that $T(a)$ is compact for some $a > 0$ and 1 is an eigenvalue of each $T(t)$. Then [4, VIII. 8.4] yields that

$$\begin{aligned}\lim_{n\to\infty} \cdot (1/n)\{T(t) + T^2(t) + \cdots + T^n(t)\} &= \lim_{n\to\infty} \cdot (1/n)\{T(t) + \cdots + T(nt)\} \\ &= A(t)\end{aligned}$$

say, exists in the uniform operator topology and defines a projection operator satisfying $A(t) = T(t)A(t) = A(t)T(t)$. Also, $A(t)$ is the residue operator in the Laurent expansion of $(\lambda - T(t))^{-1}$ in the neighbourhood of 1 and is represented by the Dunford integral

$$A(t) = (1/2\pi i)\int_C (\lambda - T(t))^{-1} d\lambda$$

where C is a sufficiently small circle with centre at 1 and will, in general, depend upon t. For $t = 0$, $T(0) = A(0) = I$ the identity operator. $\{A(t): t \geqq 0\}$ will be called the ergodic family associated with $\{T(t): t \geqq 0\}$.

When $T(a)$ is compact for some $a > 0$, the family $\{T(t): t \geqq a\}$ is collectively compact and totally bounded in the uniform operator topology [3]. But nothing can be said about $T(t)$ for $t < a$. However, the following proposition shows that the situation is more satisfactory in respect of the family $\{A(t)\}$.

PROPOSITION 2.1. *Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group on a B-space X such that 1 is an eigenvalue of each $T(t)$ and $T(a)$ is compact for some $a > 0$. Then for each $\varepsilon > 0$, the ergodic family $\{A(t): t \geqq \varepsilon\}$ is collectively compact.*

Proof. Since $A(t) = T(t)A(t)$, by repeated application, we have $A(t) = T(nt)A(t)$ for all positive integers n. For $t > 0$ $T(nt)$ is compact for sufficiently large n. Hence, $A(t)$ will also be compact. Now for $\varepsilon > 0$, choose a positive integer k so that $k\varepsilon \geqq a$. Then $kt \geqq a$ for $t \geqq \varepsilon$. Therefore, $\mathscr{K} = \{T(kt): t \geqq \varepsilon\}$ is collectively compact. Now, $\| T(t) \| \leqq M$ implies $\| A(t) \| \leqq M$. Hence, $\mathscr{M} = \{A(t): t \geqq \varepsilon\}$ is uniformly bounded. We can now conclude that

$$\mathscr{K}\mathscr{M} = \{T(kt) \circ A(s): t, s \geqq \varepsilon\}$$

is collectively compact, because, by [1, Prop. 2.3], if $\mathscr{K}$ is collectively compact and $\mathscr{M}$ is uniformly bounded, then $\mathscr{K}\mathscr{M}$ is collectively compact. The result now follows from the fact that $\{A(t): t \geqq \varepsilon\}$ is a subset of $\mathscr{K}\mathscr{M}$.

COROLLARY 2.2. *The family $\{A(t): t \geqq \varepsilon\}$ is totally bounded in the uniform operator topology.*

Proof. The above arguments in respect of $\{T(t)\}$ also hold for the dual semi-group $\{T^*(t)\}$. Hence, $\{A(t): t \geqq \varepsilon\}$ and $\{A^*(t): t \geqq \varepsilon\}$ are both collectively compact. The desired conclusion follows from the fact that if $\mathscr{K}, \mathscr{K}^*$ are both collectively compact, then $\mathscr{K}$ is totally bounded in the uniform operator topology [6, Thm. 3.1].

3. Spectral properties. In Section 2 strong continuity of $T(t)$ was not necessary. But, for the spectral properties to be discussed now, we shall require strong continuity. Let, as usual, $\sigma(T(t))$, $\tilde{\rho}(T(t))$ denote the spectrum and the extended resolvent set of $T(t)$ and $R(\lambda, T(t))$ the inverse $(\lambda - T(t))^{-1}$. The following proposition proved in [3] leads to the strong continuity of $A(t)$ which is crucial for the main result.

PROPOSITION 3.1. *Let $\{T(t): t \geqq 0\}$ be a uniformly bounded, strongly continuous semi-group on a B-space X such that $T(t)$ is compact for each $t > 0$. For $a > 0$, let Ω be a neighbourhood of $\sigma(T(a))$, and $\Lambda = \tilde{\rho}(T(a)) - \Omega$. Then*

(i) *there exists a* $\partial > 0$ *such that, for* $|t - a| < \partial$, $\sigma(T(t)) \subset \Omega$ *and* $\Lambda \subset \tilde{\rho}(T(t))$.

(ii) *As* $t \to a$, $(\lambda - T(t))^{-1}x \to (\lambda - T(a))^{-1}x$ *uniformly on* Λ.

PROPOSITION 3.2. *Let* $\{T(t): t \geqq 0\}$ *be a uniformly bounded, strongly continuous semi-group on a B-space* X *such that* 1 *is an eigenvalue of each* $T(t)$ *and* $T(t)$ *is compact for each* $t > 0$. *Suppose further, that for each* $a > 0$, *there exists a circle* C *with centre at* 1 *in the complex plane and a real number* ∂ *such that, for* $|t - a| < \partial$, $T(t)$ *has no eigenvalue in* C *except* 1. *Then for* $t > 0$, $A(t)$ *is strongly continuous in* t.

Proof. We know that

$$A(t)x = (1/2\pi i)\int_{\Gamma} (\lambda - T(t))^{-1}x d\lambda$$

where, in general, Γ depends upon t. But, under the hypothesis on C, we may assume that

$$A(t)x = (1/2\pi i)\int_{C} (\lambda - T(t))^{-1}x d\lambda$$

for all t with $|t - a| < \delta$. Again, by proposition 3.1, $(\lambda - T(t))^{-1}x \to (\lambda - T(a))^{-1}x$ uniformly on C as $t \to a$. Hence, for any $a > 0$, we have $A(t)x \to A(a)x$ as $t \to a$. This completes the proof.

Let $R(T)$, $N(T)$ denote the range and null space of an operator T and $\dim R(T)$, $\dim N(T)$ their dimensions. When $T_n \to T$ pointwise, $\dim R(T_n) > \dim R(T)$ eventually. But, when we are dealing with projections, the following result [2, 4.2] gives a more precise estimate.

THEOREM 3.3. *Let* E *and* E_n, $n \geqq 1$, *be projections in* $[X]$ *such that* $E_n \to E$ *pointwise and* $\{E_n - E\}$ *is collectively compact. Then, eventually* $\dim E_n X = \dim EX$.

THEOREM 3.4. *Let* $\{T(t)\}$ *be the semi-group of Proposition* 3.2. *Then for each* $a > 0$, *there exists* $\partial > 0$ *such that* $\dim N(T(t) - I) = \dim N(T(a) - I)$ *whenever* $|t - a| < \partial$.

Proof. Observe first that $R(A(t)) = N(T(t) - I)$ since $A(t)$ is the ergodic limit of $T(t)$. Suppose that there does not exist any ∂ with the required property. Then, by using Proposition 2.1, it is possible to construct a sequence of projections for which the Theorem 3.3 will not be true.

THEOREM 3.5. *If $\{T(t)\}$ is the semi-group of Proposition 3.4, then for $t > 0$, $\dim N(T(t) - I)$ is independent of t.*

Proof. By the last theorem the function $t \to \dim N(T(t) - I)$ is continuous and since it takes only positive integral values, it must be a constant function.

4. The nature of $T(t)$ at $t = \infty$. The first result of this section will be a lemma proved in [5] by using collective compactness. However, the proof given here throws some light on the nature of $T(t)$ at $t = \infty$.

LEMMA 4.1. *Let $T \in [X]$ be such that $\{T^n : n \geqq 1\}$ is uniformly bounded and T^k is compact for some $k \geqq 1$. Then $\{T^n : n \geqq 1\}$ is totally bounded in the uniform operator topology.*

Proof. The conditions on T ensure that $\sigma(T)$ is contained in the closed unit disc with centre 0 in the complex plane and that there are only a finite number of simple poles of $R(\lambda, T)$, say $\lambda_1, \cdots, \lambda_p$ on the circumference of the disc [4, VIII. 8.1]. If $A_1, \cdots, A_p$ are the residue operators in the Laurent expansion of $R(\lambda, T)$ in the neighbourhoods of $\lambda_1, \cdots, \lambda_p$ respectively, then by using the standard technics of operational calculus, it can be easily proved that

$$(\mathrm{I}) \qquad T^n x = (1/2\pi i)\int_{C_0} \lambda^n R(\lambda, T)x d\lambda + \lambda_1^n A_1(x) + \cdots + \lambda_p^n A_p(x)$$

where C_0 is a circle with centre at 0 and radius less than 1. Since $|\lambda| < 1$ on C_0, the first term on the right tends to zero uniformly on C_0 and on bounded sets of X. Again $|\lambda_\kappa| = 1$ for $k = 1, \cdots, p$. Hence $\{\lambda_\kappa^n : n \geqq 1, k = 1, \cdots, p\}$ is totally bounded. It is easy to extract a sequence $\{n_i\}$ such that $\lambda_k^{n_i}$ converges to μ_k say, for $k = 1, \cdots, p$. It then follows that

$$T^{n_i} x \longrightarrow \mu_1 A_1(x) + \cdots + \mu_p A_p(x) .$$

The convergence is obviously uniform on bounded sets of X.

PROPOSITION 4.2. *Let $\{T(t) : t \geqq 0\}$ be a uniformly bounded semi-group such that $T(a)$ is compact for some $a > 0$. For any $t > 0$, let $A_1 = A_1(t), \cdots, A_p = A_p(t)$ be the residue operators in the Laurent expansion of $R(\lambda, T(t))$ in the neighbourhoods of its poles $\lambda_1, \cdots, \lambda_p$ say, lying on the circumference of the unit disc. Then, there exists a sequence $n_i = n_i(t)$ such that, as $n_i \to \infty$, $T(n_i t)x$ converges to a linear combination of $A_1 x, \cdots, A_p x$ for each x.*

Proof. Follows immediately when the previous lemma is applied to $T(t)$.

PROPOSITION 4.3. *Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group on a B-space X such that $T(a)$ is compact for some $a > 0$. Suppose 1 is the only eigenvalue of $T(t)$ on the unit circle in the complex plane for each $t > 0$. Then $\lim_{n\to\infty} T(nt) = A$ exists in the uniform operator topology for each $t > 0$ and A is independent of t.*

Proof. When the conditions of the proposition are satisfied equation (I) of this section applied to $T(t)$ gives

$$\text{(II)} \qquad T(nt)x = T^n(t)x = (1/2\pi i)\int_{C_0} \lambda^n(\lambda - T(t))^{-1}xd\lambda + A(t)x$$

where $A = A(t)$ is the residue operator in the Laurent expansion of $(\lambda - T(t))^{-1}$ around 1 and C_0 is a circle with centre at 0 and radius less than 1. Now $|\lambda| < 1$ on C_0 and therefore $\lambda^n \to 0$ as $n \to \infty$. Hence taking limits in the above equation as $n \to \infty$ we get $\lim_{n\to\infty} T(nt)x = A(t)x$, the limit being uniform on bounded sets of X. Now, to prove that $A(t) = A$ is independent of t note that $A(t)A(s) = \lim_{n\to\infty} T(nt)T(ns) = \lim_{n\to\infty} T(n(s+t)) = A(s+t) = A(s)A(t)$. Hence $\{A(t)\}$ is a semi-group. Again, for $t \neq s$, let $t < s$. Then by the semi-group property $R(A(t)) \supset R(A(s))$ where R denotes the range. Also $A(t)$ is a projection. Therefore $A(t) = A^n(t) = A(nt)$ for each positive integer n. Choose n so that $nt > s$. Then we have $R(A(t)) = R(A(nt)) \subset R(A(s))$. Hence $R(A(t)) = R((A(s))$. Now, it is easy to prove that if P, Q are projections such that $PQ = QP$ and $R(P) = R(Q)$ then $P = Q$. Therefore we must have $A(t) = A(s)$. This completes the proof.

COROLLARY 4.4. *$N(T(t) - I)$ is independent of t for $t > 0$.*

Proof. An application of corollary, Theorem 2 in [7, VIII, § 3] leads to $N(T(t) - I) = R(A(t)) = R(A)$.

REMARK 4.5. In the proof of the Proposition 4.3 $A(t)$ is proved to be independent of t by using the fact that $A(t)A(s) = A(t+s) = A(s)A(t)$ and that $A(t)$ is a projection. This means that the only semi-group of projections is the semi-group $T(t) = P$ a projection for all t.

In the next proposition it will be seen that the pointwise convergence of $T(nt)$ in t proved in Proposition 4.3 can be strengthened so as to be locally uniform in t. This essential for the final ergodic theorem.

PROPOSITION 4.6. *Let $\{T(t): t \geqq 0\}$ be a uniformly bounded, strongly continuous semi-group on a B-space X such that $T(t)$ is compact for each $t > 0$. Suppose there exists a circle in the complex plane with centre at 0 and radius less than 1 such that 1 is the only eigenvalue of each $T(t)(t > 0)$ outside this circle. Then for each $t > 0$, $\lim_{n\to\infty} T(nt) = A$ exists in the uniform operator topology, the convergence being locally uniform on t,* i.e., *for each $a > 0$ there exists a positive ∂ depending upon a such that the convergence is uniform on the interval $|t - a| < \partial$.*

Proof. The equation (II) of this section is now

$$T(nt)x = (1/2\pi i)\int_{C_0} \lambda^n(\lambda - T(t))^{-1}x d\lambda + Ax \tag{III}$$

where in general C_0 depends upon t. But, under the hypothesis made, C_0 may be assumed to be independent of t. Then by Proposition 3.1 the set $\{(\lambda - T(t))^{-1}: \lambda \in C_0, |t - a| < \partial\}$ is uniformly bounded for some $\partial > 0$, i.e., $\|(\lambda - T(t))^{-1}\| < M$ say, for all $\lambda \in C_0$ and $|t - a| < \partial$. Hence by equation (III) we have

$$\begin{aligned} \| T(nt)x - Ax \| &\leqslant |(1/2\pi i)| \int_{C_0} \| \lambda^n(\lambda - T(t))^{-1}x \| \, |d\lambda| \\ &< (M/2\pi)\int_{C_0} |\lambda|^n \, |d\lambda| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \text{ since } |\lambda| < 1 \text{ on } C_0 . \end{aligned}$$

It is obvious that the convergence is of the required type.

LEMMA 4.7. *Suppose $\{T(t): t \geqq 0\}$ is a family of bounded linear operators on a B-space X such that $\lim_{n\to\infty} T(nt) = A$ exists in the uniform operator topology where A is independent of t and the convergence is uniform on some open interval. Then $\lim_{t\to\infty} T(t) = A$ in the uniform operator topology.*

Proof. We may assume that $T(nt) \rightarrow A$ uniformly on $|t - a| < \partial$. Then for $\varepsilon > 0$, there exists $\partial > 0$ and a positive integer N such that $\| T(nt) - A \| < \varepsilon$ for all $n \geqq N$ and all t in $|t - a| < \partial$ or equivalently $\| T(t) - A \| < \varepsilon$ for $n \geqq N$ and $|t - na| < n\partial$. Let I_n be the interval $(na - n\partial, na + n\partial)$. Then I_n has a nonempty intersection with I_{n+1} if and only if $na + n\partial > (n + 1)a - (n + 1)\partial$ or equivalently $(2n + 1)\partial > a$. Choose M so that $n \geqq M$ implies $n \geqq N$ and $(2n + 1)\partial > a$. Then, for each $n \geqq M$, I_n and I_{n+1} have a nonempty intersection. Hence, any $t > Ma - M\partial$ falls in some I_n with $n \geqq M$. Therefore we must have $\| T(t) - A \| < \varepsilon$ for $t > Ma - M\partial$. This completes the proof.

In the light of this lemma, the Proposition 4.6 and the Corollary 4.4, the following ergodic theorem now becomes obvious.

THEOREM 4.8. *If* $\{T(t): t \geqq 0\}$ *is the semi-group of Proposition 4.6 then* $\lim_{t\to\infty} T(t)$ *exists in the uniform operator topology, and* $N(T(t) - I)$ *is constant for* $t > 0$.

REMARK. In §§ 3 and 4 the continuity of $R(\lambda, T(t))$ in t plays a crucial role. If it is assumed that $T(t)$ is compact for each $t > 0$, then the strong continuity of $T(t)$ in t implies uniform continuity and this in turn implies the uniform continuity of $R(\lambda, T(t))$ for $\lambda \in \rho(T(t))$. But if instead of assuming the compactness of $T(t)$ for each $t > 0$, it is assumed that for each $t > 0$ there exists $\partial > 0$ such that $\{T(t) - T(a): |t - a| < \partial\}$ is collectively compact then the strong continuity of $T(t)$ would not imply uniform continuity but the theory of collectively compact operators can be invoked to ensure the strong continuity of $R(\lambda, T(t))$.

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Received February 27, 1974.

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