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ALGEBRAICALLY IRREDUCIBLE REPRESENTATIONS OF $L_1(G)$

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Let G be a locally compact, noncompact group and π a weakly continuous, uniformly bounded representation of G on a Hilbert space H. Suppose there exists a non-zero ξ in H such that the function $x \to \langle \pi(x)\xi, \xi\rangle$ vanishes at infinity. Then π is not algebraically irreducible when lifted to a representation of $L_1(G)$. This implies that the left regular representation of $L_1(G)$, for G noncompact, contains no algebraically irreducible subrepresentations.

We investigate irreducible representations of locally compact, noncompact groups which lift to algebraically irreducible representations of $L_1(G)$. Algebraically irreducible representations lie somewhere between the irreducible finite dimensional ones and the topologically irreducible ones, not necessarly coinciding with either. A theorem of R. Kadison [6] shows that the topologically irreducible *-representations of a C^*-algebra are all algebraically irreducible. Although (by a result of L. T. Gardner [4]) $L_1(G)$ is never a C^*-algebra unless G is finite, algebraically irreducible representations occur quite naturally in several classes of Banach *-algebras. Also given their nice properities (see the paper by B. Barnes [1]) it would be interesting to know if $L_1(G)$ has any non-finite dimensional ones, and where in the representation theory of G they are located.

A. Weil [10, pp. 69–70] has shown that noncompact groups have no finite dimensional square integrable representations and a result of M. Rieffel [9, Corollary 5.12] shows that an infinite discrete group has no irreducible square integrable representations. Our main result (Theorem 5) is that for locally compact, non-compact groups, representations of $L_1(G)$ which belong to a class containing the square integrable ones are never algebraically irreducible.

Notation and Preliminaries. Let G be a locally compact topological group with left Haar measure μ. Let $L_1(G)$ denote the equivalence classes of integrable functions on G with repect to μ, $L_2(G)$ the equivalence classes of square-integrable functions on G with respect to μ and $L_\infty(G)$ the equivalence classes of essentially bounded functions on G with respect to μ. Let $C_0(G)$ denote the set of continuous functions on G which vanish at infinity and $C_{00}(G)$ the set of continuous functions on G with compact support.

The $L_1(G)$ norm is denoted by $\|\cdot\|_1$, the $L_2(G)$ norm is denoted by $\|\cdot\|_2$ and the uniform norm on $C_0(G)$ is denoted by $\|\cdot\|_u$.

If f is a function on G and x is in G then the function xf is defined by $xf(y)=f(xy)$ for all y in G.

Let H be a Hilbert space and let $B(H)$ denote the set of bounded operators on H. If T is a bounded operator on H then T^* denotes its adjoint. By a representation of G on H we mean a homomorphism of G into the group of invertible operators in $B(H)$. We call π weakly continuous if π is continuous into $B(H)$ with the weak-operator topology. We say that π is uniformly bounded if

$$\sup\{\|\pi(x)\|: x\in G\}<\infty$$

and denote this number by $\|\pi\|$.

Let π be a weakly continuous uniformly bounded representation of G on a Hilbert space H. Then π may be lifted to a continuous representation of $L_1(G)$ by the following formula

$$\langle \pi(f)\xi,\eta\rangle=\int_G f(x)\langle\pi(x)\xi,\eta\rangle d\mu(x)$$

for all f in $L_1(G)$ and ξ,η in H. If in addition π is a unitary representation of G then π lifts to a $*$-representation of $L_1(G)$. If K is a subset of H, the closure of K is denoted by cl K and the linear span of K is denoted by sp K. Let ξ be in H, M a subset of G and S a subset of $L_1(G)$. Then

$$\pi(M)\xi=\{\pi(x)\xi: x\in M\}$$

and

$$\pi(S)\xi=\{\pi(f)\xi: f\in S\}.$$

We call π topologically irreducible if

$$\operatorname{clsp}\,\pi(G)\xi=H$$

for all nonzero ξ in H. This is equivalent to

$$\operatorname{cl}\,\pi(L_1(G))\xi=H$$

for all nonzero ξ in H. We call π an algebraically irreducible representation of $L_1(G)$ if

$$\pi(L_1(G))\xi=H$$

for all nonzero ξ in H.

The Main Result. Throughout this section, unless otherwise specified, G denotes a locally compact, noncompact group and π denotes a weakly continuous, uniformly bounded representation of G on a Hilbert space H.

LEMMA 1. *Suppose π is irreducible and the function $p(x) = \langle \pi(x)\xi, \gamma\rangle$ belongs to $C_0(G)$ for some nonzero ξ and γ in H. Then the functions*

$$x \to \langle \pi(x)\eta, \psi\rangle$$

belong to $C_0(G)$ for all η and ψ in H.

Proof. We first show that

$$\text{clsp } \pi(G) * \xi = H.$$

Suppose for some ζ in H we have $\langle \zeta, \pi(x) * \xi\rangle = 0$ for all x in G. Then $\langle \pi(x)\zeta, \xi\rangle = 0$ for all x in G and since π is irreducible we must have $\zeta = 0$.

Let $g(x) = \langle \pi(x)\eta, \psi\rangle$ and $\epsilon > 0$. Choose $x_1, \cdots, x_n$ in G and scalars $\lambda_1, \cdots, \lambda_n$ such that

$$\left\| \eta - \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi \right\| < [2\{\|\psi\| \, \|\pi\| + 1\}]^{-1}\epsilon$$

Now choose $y_1, \cdots, y_m$ in G and scalars $\beta_1, \cdots, \beta_m$ such that

$$\left\| \psi - \sum_{j=1}^{m} \beta_j \pi(y_j) * \gamma \right\| < \left[2\left\{ \|\pi\| \left\| \sum_{i=1}^{n} \lambda_i \pi(x_i) \right\| + 1 \right\}\right]^{-1} \epsilon.$$

Let $r(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \bar{\beta}_j p(y_j x x_i)$. Then $r(x)$ belongs to $C_0(G)$ and

$$\begin{aligned} |g(x) - r(x)| &= \left| \langle \pi(x)\eta, \psi\rangle - \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \bar{\beta}_j \langle \pi(y_j x x_i)\xi, \gamma\rangle \right| \\ &= \left| \langle \pi(x)\eta, \psi\rangle - \left\langle \pi(x) \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi, \sum_{j=1}^{m} \beta_j \pi(y_j) * \gamma \right\rangle \right| \\ &\leqq \left| \left\langle \pi(x)\eta - \pi(x) \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi, \psi \right\rangle \right| \\ &\quad + \left| \left\langle \pi(x) \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi, \psi - \sum_{j=1}^{m} \beta_j \pi(y_j) * \gamma \right\rangle \right| \end{aligned}$$

$$\leqq \|\pi\| \left\| \eta - \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi \right\| \|\psi\|$$

$$+ \|\pi\| \left\| \sum_{i=1}^{n} \lambda_i \pi(x_i)\xi \right\| \left\| \psi - \sum_{j=1}^{m} \beta_j \pi(y_j) * \gamma \right\|$$

$$< \epsilon.$$

This completes the proof of the lemma.

LEMMA 2. *Let π be an irreducible representation of G on an infinite dimensional Hilbert space H and let ξ be a nonzero vector in H. Then given any compact subset M of G and elements $x_1, \cdots, x_n$ in M there exists x_{n+1} in $G \setminus M$ such that $\pi(x_{n+1}) * \xi$ is linearly independent from the set $\{\pi(x_1) * \xi, \cdots, \pi(x_n) * \xi\}$.*

Proof. By the first part of the proof of Lemma 1 we have that

$$\operatorname{clsp} \pi(G) * \xi = H.$$

If $\operatorname{clsp} \pi(M) * \xi \neq H$ we are done.

Suppose $\operatorname{clsp} \pi(M) * \xi = H$. We claim that $\operatorname{clsp} \pi(G \setminus M) * \xi = H$. To see this suppose there is a η in H such that

$$\langle \eta, \pi(x) * \xi \rangle = 0$$

for all x in $G \setminus M$. Since G is not compact there exists an x_0 in $B \setminus M^{-1}M$. Then Mx_0 is disjoint from M. So we have

$$\langle \pi(x_0)\eta, \pi(x) * \xi \rangle = \langle \eta, \pi(xx_0) * \xi \rangle = 0$$

for all x in M. But then it follows that $\pi(x_0)\eta = 0$ and so $\eta = \pi(x_0)^{-1}\pi(x_0)\eta = 0$. Therefore we may assume that $\operatorname{clsp}(G \setminus M) * \xi = H$. But now we are done since

$$\operatorname{clsp}\{\pi(x_i) * \xi : i = 1, \cdots, n\}$$

is finite dimensional.

LEMMA 3. *Let $h \in C_0(G)$ and let Y be the closed subspace of $C_0(G)$ generated by the left translates of h by elements of G. Suppose there exists an inner product $\langle \cdot, \cdot \rangle$ on Y such that the norm $\|\cdot\|$ determined by it is equivalent to the uniform norm on Y and the functions*

$$x \to \langle xf, g \rangle$$

belong to $C_0(G)$ for all f and g in Y. Then there exists a compact subset M_0 of G and elements $x_1, \cdots, x_n$ in M_0 such that

$$xh \in \mathrm{sp}\{x_1h, \cdots, x_nh\}$$

for all x in $G \setminus M_0$.

Proof. Suppose the contrary. Then given any compact subset M of G and elements $x_1, \cdots, x_n$ in M, there exists x_{n+1} in $G \setminus M$ such that $x_{n+1}h$ is linearly independent from $\{x_1h, \cdots, x_nh\}$. In particular Y is infinite dimensional.

There exists a constant $K > 1$ such that

$$K^{-1}\|f\| \leqq \|f\|_u \leqq K\|f\|$$

for all f in Y.

Let $\gamma_1 = h$. Having chosen $x_1, \cdots, x_n$ in G and $\gamma_1, \cdots, \gamma_n$ in $\mathrm{sp}\{x_1h, \cdots, x_nh\}$ such that

(1) the set $\{\gamma_1, \cdots, \gamma_n\}$ is orthogonal

(2) $\|\gamma_1 + \cdots + \gamma_n\|_u \leqq (1 + 2^{-1} + \cdots + 2^{-n+1})\|h\|_u$

and

(3) $\|\gamma_k\| \geqq (K^{-2} - 2^{-k})\|h\|$, for $k = 1, \cdots, n$ we choose γ_{n+1}.

Let $\phi_k(x) = \langle x\gamma_1, \gamma_k\rangle$ for $k = 1, \cdots, n$. Then ϕ_k belongs to $C_0(G)$. Let

$$M_1 = \left\{x \in G\colon \sum_{k=1}^{n} \|\gamma_k\|^{-2}\|\gamma_k\|_u\, |\phi_k(x)| \geqq 2^{-n-1}\|h\|_u\right\}$$

$$M_2 = \left\{x \in G\colon \sum_{k=1}^{n} |\gamma_k(x)| \geqq 2^{-n-1}\|h\|_u\right\}$$

$$M_3 = \{x \in G\colon |h(x)| \geqq 2^{-n-1}\|h\|_u\}$$

and

$$M_4 = \left\{x \in G\colon \sum_{k=1}^{n} \|\gamma_k\|^{-1}|\phi_k(x)| \geqq 2^{-n-1}\|h\|\right\}.$$

Then since all functions concerned are in $C_0(G)$, the M_i are compact. Let $M_0 = \bigcup_{i=1}^4 M_i$ and $M = M_0 \cup M_0^{-1}$. Then M and hence M^2 are compact. So there exists x_{n+1} in $G \setminus (M^2 \cup M)$ such that $x_{n+1}h$ is linearly independent from $\{\gamma_1, \cdots, \gamma_n\}$. Note that $x_{n+1}^{-1}M$ is disjoint from M. Let

$$\gamma_{n+1} = x_{n+1}h - \sum_{k=1}^{n} \phi_k(x_{n+1})\|\gamma_k\|^{-2}\gamma_k.$$

Since $\phi_k(x) = \langle xh, \gamma_k \rangle$, the $\gamma_1, \cdots, \gamma_{n+1}$ are orthogonal by the Gram-Schmidt process.

Next we verify (2) of the inductive hypothesis. We claim that

$$\left\| x_{n+1} h + \sum_{k=1}^{n} \gamma_k \right\|_u \leqq (1 + 2^{-1} + \cdots + 2^{-n+1} + 2^{-n-1}) \| h \|_u .$$

To see this suppose that for some x in G

$$\left| h(x_{n+1}x) + \sum_{k=1}^{n} \gamma_k(x) \right| > (1 + 2^{-1} + \cdots + 2^{-n+1} + 2^{-n-1}) \| h \|_u .$$

Then either

(i) $|h(x_{n+1}x)| > 2^{-1} \| h \|_u$

or

(ii) $|\sum_{k=1}^{n} \gamma_k(x)| > 2^{-1} \| h \|_u$.

Suppose (i) holds. Then $x_{n+1}x \in M_3$ and so $x \in x_{n+1}^{-1} M$. Therefore $x \notin M_2$ and so

$$\left| \sum_{k=1}^{n} \gamma_k(x) \right| < 2^{-n-1} \| h \|_u .$$

But then

$$\begin{aligned} \left| h(x_{n+1}x) + \sum_{k=1}^{n} \gamma_k(x) \right| &< \| h \|_u + 2^{-n-1} \| h \|_u \\ &\leqq (1 + 2^{-1} + \cdots + 2^{-n+1} + 2^{-n-1}) \| h \|_u . \end{aligned}$$

Next suppose (ii) holds. Then $x \in M_2$ and so $x \notin x_{n+1}^{-1} M$. Therefore $x_{n+1}x \notin M_3$ and so

$$| h(x_{n+1}x) | < 2^{-n-1} \| h \|_u .$$

But then

$$\begin{aligned} \left| h(x_{n+1}x) + \sum_{k=1}^{n} \gamma_k(x) \right| &< 2^{-n-1} \| h \|_u + \| \gamma_1 + \cdots + \gamma_n \|_u \\ &\leqq 2^{-n-1} \| h \|_u + (1 + 2^{-1} + \cdots + 2^{-n+1}) \| h \|_u \\ &= (1 + 2^{-1} + \cdots + 2^{-n+1} + 2^{-n-1}) \| h \|_u . \end{aligned}$$

Therefore

$$
\begin{aligned}
\|\gamma_1+\cdots+\gamma_{n+1}\|_u &= \left\| x_{n+1}h - \sum_{k=1}^{n} \phi_k(x_{n+1}) \|\gamma_k\|^{-2}\gamma_k + \sum_{k=1}^{n}\gamma_k \right\|_u \\
&\leqq \left\| x_{n+1}h + \sum_{k=1}^{n}\gamma_k \right\|_u + \sum_{k=1}^{n} |\phi_k(x_{n+1})| \|\gamma_k\|^{-2}\|\gamma_k\|_u \\
(x_{n+1}\notin M_1) &\leqq (1+2^{-1}+\cdots+2^{-n+1}+2^{-n-1})\|h\|_u + 2^{-n-1}\|h\|_u \\
&= (1+2^{-1}+\cdots+2^{-n})\|h\|_u .
\end{aligned}
$$

which verifies (2).

Now we check (3). First note that for any $x\in G$ we have

$$\|h\| \leqq K\|h\|_u = K\|xh\|_u \leqq K^2\|xh\| .$$

So

$$
\begin{aligned}
\|\gamma_{n+1}\| &= \left\| x_{n+1}h - \sum_{k=1}^{n} \phi_k(x_{n+1})\|\gamma_k\|^{-2}\gamma_k \right\| \\
&\geqq \|x_{n+1}h\| - \sum_{k=1}^{n} |\phi_k(x_{n+1})| \|\gamma_k\|^{-1} \\
(x_{n+1}\notin M_4) &\geqq K^{-2}\|h\| - 2^{-n-1}\|h\| \\
&= (K^{-2}-2^{-n-1})\|h\| .
\end{aligned}
$$

This verifies (3).

Choose N such that $2^{-N-1} < K^{-2}$. Then for $n>N$ we have

$$
\begin{aligned}
\|\gamma_{n+1}\|^2 &\geqq (K^{-2}-2^{-n-1})^2\|h\|^2 \\
&\geqq (K^{-2}-2^{-n})K^{-2}\|h\|^2 .
\end{aligned}
$$

Let $\Psi_n^n = \Sigma_{k=N+1}\gamma_k$. Then

$$
\begin{aligned}
\|\Psi_n\| &= \left\| \sum_{k=1}^{n}\gamma_k - \sum_{k=1}^{N}\gamma_k \right\|_u \\
&\leqq \left\| \sum_{k=1}^{n}\gamma_k \right\|_u + \left\| \sum_{k=1}^{N}\gamma_k \right\|_u \\
&\leqq 4\|h\|_u \qquad \text{by (2).}
\end{aligned}
$$

But since the γ_k are orthogonal,

$$\begin{aligned}\|\Psi_n\|^2 &= \sum_{k=N+1}^{n} \|\gamma_k\|^2 \\ &\geqq \sum_{k=N+1}^{n} (K^{-2}-2^{-k})K^{-2}\|h\|^2 \\ &\geqq [K^{-2}(n-N)-1]K^{-2}\|h\|^2.\end{aligned}$$

Therefore for $n > K^2 + N$,

$$\|\Psi_n\| \geqq (K^{-2}(n-N)-1)^{1/2} K^{-1}\|h\|$$

which is impossible if $\|\cdot\|_u$ and $\|\cdot\|$ are equivalent on Y. This contradiction proves the lemma.

LEMMA 4. *Suppose π is irreducible and there exists a nonzero ξ in H such that the function $p(x) = \langle \pi(x)\xi, \xi\rangle$ belongs to $C_0(G)$. Then H is infinite dimensional.*

Proof. Suppose H is finite dimensional. Let Γ be the closure of $\pi(G)$ in $B(H)$. We show that Γ is a compact group. Since π is uniformly bounded, Γ is compact.

Now let S and T be in Γ. Choose sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in G such that

$$\pi(x_n) \to S \text{ and } \pi(y_n) \to T.$$

Since $\|\pi(x_n)^{-1}\| = \|\pi(x_n^{-1})\| \leqq \|\pi\|$ for all n, by Dunford and Schwartz [**3**, VII 8.1 and VII 6.1] S is invertible and $\pi(x_n^{-1}) = \pi(x_n)^{-1} \to S^{-1}$. So $\pi(x_n^{-1} y_n) \to S^{-1}T$ and therefore $S^{-1}T \in \Gamma$.

It follows from Dixmier [**2**, 16.1.1 and 16.2.1] that we must have $\{xp: x \in G\}$ relatively compact in the set of bounded continuous functions on G. But this is impossible for $p \neq 0$ and $p \in C_0(G)$. Because we can choose $p_0 \in C_{00}(G)$ such that $\|p - p_0\|_u < 4^{-1}\|p\|_u$. Let K be the support of p_0 and $x_1 = e$. Having chosen $x_1, \cdots, x_n$ in G such that $x_1K, \cdots, x_nK$ are pairwise disjoint, choose x_{n+1} in $G \setminus (\bigcup_{j=1}^{n} x_jKK^{-1})$. This can be done since G is not compact. It follows by the choice of x_{n+1} that the sets $x_1K, \cdots, x_{n+1}K$ are pairwise disjoint. Now let x be in K such that $|p_0(x)| = \|p_0\|_u$. Then for $i \neq j$ we have $x_i^{-1}x_jx \notin K$ and so

$$\begin{aligned}\|x_j^{-1}p_0 - x_i^{-1}p_0\|_u &\geqq |x_j^{-1}p_0(x_jx) - x_i^{-1}p_0(x_jx)| \\ &= |p_0(x)| \\ &= \|p_0\|_u.\end{aligned}$$

Therefore

$$\|x_j^{-1}p - x_i^{-1}p\|_u \geqq \|x_j^{-1}p_0 - x_i^{-1}p_0\|_u - \|x_j^{-1}p - x_j^{-1}p_0\|_u - \|x_i^{-1}p - x_i^{-1}p_0\|_u$$
$$\geqq 4^{-1}\|p\|.$$

This contradiction proves the lemma.

We are now ready to prove the main result.

THEOREM 5. *Let π be a weakly continuous uniformly bounded representation of a locally compact, noncompact group G on a Hilbert space H. Suppose there exists a nonzero vector ξ in H such that the function $p(x) = \langle \pi(x)\xi, \xi\rangle$ belongs to $C_0(G)$. Then π is not algebraically irreducible when lifted to a representation of $L_1(G)$.*

Proof. Suppose π is algebraically irreducible on $L_1(G)$. For any η in H and f in $L_1(G)$ we have

$$\pi(f)\eta = \int_G f(x)\pi(x)\eta\, d\mu(x).$$

Let $\mathcal{J} = \{f \in L_1(G)\colon \pi(f)\xi = 0\}$. Then $\mathcal{J}$ is a closed left ideal of $L_1(G)$. Since $\pi(L_1(G))\xi = H$ the map

$$0\colon L_1(G)/\mathcal{J} \to H$$

defined by

$$\theta(f + \mathcal{J}) = \pi(f)\xi$$

is one-to-one and onto. We claim that θ is also continuous. To see this let f be in $L_1(G)$ and g in $\mathcal{J}$. Then

$$\begin{aligned}\|\pi(f)\xi\| &= \|\pi(f-g)\xi\|\\ &= \left\|\int_G (f(x)-g(x))\pi(x)d\mu(x)\right\|\\ &\leqq \|\pi\| \int_G |f(x)-g(x)|\,d\mu(x)\\ &= \|\pi\|\,\|f-g\|_1.\end{aligned}$$

and so

$$\|\pi(f)\xi\| \leqq \|\pi\| \inf_{g\in\mathcal{J}} \|f-g\|_1 = \|\pi\|\,\|f+\mathcal{J}\|_1.$$

By the open mapping theorem there exists a constant $K > 0$ such that

$$K^{-1}\|\pi(f)\xi\| \leqq \|f + \mathscr{J}\|_1 \leqq K\|\pi(f)\xi\|$$

for all f in $L_1(G)$.

By the above inequality it follows that the adjoint map

$$'\theta\colon H^* \to (L_1(G)/\mathscr{J})^*$$

is a bicontinuous isomorphism. Now $(L_1(G)/\mathscr{J})^*$ may be naturally identified with $\mathscr{J}^\perp$, the annihilator of $\mathscr{J}$ in $L_\infty(G)$, see Dunford and Schwartz [**3**, II4.18b].

$$\left(\text{i.e. } \mathscr{J}^\perp = \left\{h \in L_\infty(G)\colon \int_G f(x)\overline{h(x)}d\mu(x) = 0 \text{ for all } f \text{ in } \mathscr{J}\right\}\right).$$

Therefore $\mathscr{J}^\perp$ is equivalent to a Hilbert space in the norm induced from the inner product.

$$\langle f, g\rangle = \langle '\theta^{-1}(f), '\theta^{-1}(g)\rangle$$

for f and g in $\mathscr{J}^\perp$.

For η in H^* we determine $'\theta(\eta)$ explicitly: Let f be in $L_1(G)$. Then

$$\begin{aligned}\int_G f(x)\overline{'\theta(\eta)}(x)d\mu(x) &= \langle\theta(f + \mathscr{J}), \eta\rangle\\ &= \langle\pi(f)\xi, \eta\rangle\\ &= \int_G f(x)\langle\pi(x)\xi, \eta\rangle d\mu(x)\\ &= \int_G f(x)\overline{\langle\eta, \pi(x)\xi\rangle}d\mu(x).\end{aligned}$$

Therefore

$$'\theta(\eta)(x) = \langle\eta, \pi(x)\xi\rangle \qquad \text{a.e.}$$

In particular $'\theta(\xi) = \bar{p}$. It follows from Lemma 1 that

$$\mathscr{J}^\perp \subset C_0(G).$$

Also if y is in G, then

$$\begin{aligned} y\,'\theta(\eta)(x) &= '\theta(\eta)(yx) \\ &= \langle \eta, \pi(yx)\xi \rangle \\ &= \langle \pi(y)*\eta, \pi(x)\xi \rangle \\ &= '\theta(\pi(y)*\eta)(x). \end{aligned}$$

So $\mathscr{J}^{\perp}$ is closed under left translates.

Let f be in $\mathscr{J}^{\perp}$ and x in G. Then

$$\begin{aligned} '\theta(\pi(x)*'\theta^{-1}(f)) &= x\,'\theta('\theta^{-1}(f)) \\ &= xf \\ &= '\theta('\theta^{-1}(xf)) \end{aligned}$$

and so

$$'\theta^{-1}(xf) = \pi(x)*'\theta^{-1}(f).$$

Then for f and g in $\mathscr{J}^{\perp}$ and x in G

$$\begin{aligned} \langle xf, g \rangle &= \langle\!\langle '\theta^{-1}(xf), '\theta^{-1}(g) \rangle\!\rangle \\ &= \langle\!\langle \pi(x)*'\theta^{-1}(f), '\theta^{-1}(g) \rangle\!\rangle \\ &= \langle\!\langle '\theta^{-1}(f), \pi(x)'\theta^{-1}(g) \rangle\!\rangle. \end{aligned}$$

This implies by Lemma 1 that the functions

$$x \to \langle xf, g \rangle$$

belong to $C_0(G)$ for all f and g in $\mathscr{J}^{\perp}$.

Let Y be the closed subspace of $\mathscr{J}^{\perp}$ generated by the left translates of $\bar{p}$. Then Y and $\bar{p}$ satisfy the hypothesis of Lemma 3 with $h = \bar{p}$. We show that the conclusion of Lemma 3 is not satisfied. By Lemma 4, H is infinite dimensional. So the contradiction follows from Lemma 2 since

$$'\theta(\pi(x)*\xi) = x\bar{p}$$

for all x in G and $'\theta$ is one to one. This proves the theorem.

COROLLARY 6. *The left regular representation of $L_1(G)$, for G noncompact, contains no nontrivial algebraically irreducible sub-representations.*

Proof. Let $\lambda : G \to B(L_2(G))$ denote the left regular representation of G. By Hewitt and Ross [**6**, 32.43(e)] the functions

$$x \to \langle \lambda(x)f, f \rangle$$

belong to $C_0(G)$ for all f in $L_2(G)$. Therefore Theorem 5 applies.

The next lemma, when G is unimodular and π is a continuous unitary representation of G, is a special case of a result due to R. A. Kunze [**8**, Theorem 1]. His proof also works in the more general case below.

LEMMA 7. *Let G be a locally compact group and π a weakly continuous, uniformly bounded representation of G on a Hilbert space H. Suppose the functions*

$$x \to \langle \pi(x)\xi, \eta \rangle$$

belong to $L_2(G)$ for all ξ and η in H. Then there exists a constant $K > 0$ such that

$$\|\pi(f)\| \leqq K \|f\|_2$$

for all f in $C_{00}(G)$.

Before proving the next corollary we will need the following elementary fact from measure theory:

LEMMA 8. *Let f be in $L_1(G) \cap L_2(G)$ and $\epsilon > 0$. Then there exists g in $C_{00}(G)$ such that $\|f - g\|_1 < \epsilon$ and $\|f - g\|_2 < \epsilon$.*

Proof. By Hewitt and Ross [**6**, 32.30 and 32.33(b)] there exists h in $C_{00}(G)$ such that $\|f - f * h\|_1 < 3^{-1}\epsilon$ and $\|f - f * h\|_2 < 3^{-1}\epsilon$. Choose a compact subset K of G such that $\|(f * h)|_{G \setminus K}\|_1 < 3^{-1}\epsilon$ and $\|(f * h)|_{G \setminus K}\|_2 < 3^{-1}\epsilon$.

Let U be open such that $K \subset U$ and $[\mu(U \setminus K)]^{p^{-1}} < [6\{\|f * h\|_u + 1\}]^{-1}\epsilon$ for $p = 1$ and 2. Pick k in $C_{00}(G)$ such that $k \equiv 1 = \|k\|_u$ on K and $k \equiv 0$ on $G \setminus U$. Then for $p = 1$ and 2 we have

$$\begin{aligned} \|f * h - (f * h)k\|_p &\leqq \|(f * h)|_{G \setminus U}\|_p + \|(f * h - (f * h)k)|_{U \setminus K}\|_p \\ &< 3^{-1}\epsilon + 2\|f * h\|_u [\mu(U \setminus K)]^{p^{-1}} \\ &< 3^{-1}\epsilon + 3^{-1}\epsilon. \end{aligned}$$

So if $g = (f * h)k$ we have $g \in C_{00}(G)$ and $\|f - g\|_1 < \epsilon$ and $\|f - g\|_2 < \epsilon$.

COROLLARY 9. *Let G be a locally compact, noncompact group and π a weakly continuous uniformly bounded representation of G on a Hilbert space H. Suppose the functions*

$$x \to \langle \pi(x)\xi, \eta \rangle$$

belong to $L_2(G)$ for all ξ and η in H. If π lifts to an algebraically irreducible represetntation of $L_1(G)$, then $\pi = 0$.

Proof. Suppose π is algebraically irreducible on $L_1(G)$ and $H \neq \{0\}$. Let f be in $L_1(G)$ and g in $L_1(G) \cap L_2(G)$. Then since $\|f * g\|_2 \leqq \|f\|_1 \|g\|_2$ we have that $f * g$ is in $L_1(G) \cap L_2(G)$.

Let ξ be in H with $\|\xi\| = 1$. Then $\pi(L_1(G) \cap L_2(G))\xi$ is an invariant subspace for $\pi(L_1(G))$. Since $L_1(G) \cap L_2(G)$ is dense in $L_1(G)$ and π is algebraically irreducible we must have that

$$\pi(L_1(G) \cap L_2(G))\xi = H. \tag{1}$$

Let K be the constant in Lemma 7. So $\|\pi(f)\| \leqq K \|\|f\|_2$ for all f in $C_{00}(G)$. By the density of $C_{00}(G)$ in $L_2(G)$ we may extend π to a continuous map $\tilde{\pi}$ of $L_2(G)$ into $B(H)$. Let $f \in L_1(G) \cap L_2(G)$. We show that $\tilde{\pi}(f) = \pi(f)$. By Lemma 8 there exists a sequence $\{g_n\}_{n=1}^{\infty} \subseteq C_{00}(G)$ such that $\|f - g_n\|_1 \to 0$ and $\|f - g_n\|_2 \to 0$. So $\|\pi(f) - \pi(g_n)\| \to 0$. Therefore

$$\begin{aligned} \|\tilde{\pi}(f) - \pi(f)\| &= \lim_n \|\tilde{\pi}(f) - \pi(g_n)\| \\ &= \lim_n \|\tilde{\pi}(f) - \tilde{\pi}(g_n)\| \\ &\leqq \lim_n K \|f - g_n\|_2 \\ &= 0. \end{aligned}$$

By (1) we must have that

$$\tilde{\pi}(L_2(G))\xi = H.$$

Let $M = \{f \in L_2(G) : \tilde{\pi}(f)\xi = 0\}$. Then $\tilde{\pi}(M^{\perp})\xi = H$. Since the subspace M is closed in $L_2(G)$, the continuous map

$$f \to \tilde{\pi}(f)\xi$$

of $M^{\perp}$ onto H is one to one. So by the open mapping theorem there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_2 \leqq \|\tilde{\pi}(f)\xi\| \leqq C\|f\|_2$$

for all f in M.

Let $\lambda : L_1^+(G) \to B(L_2(G))$ denote the left regular representation of $L_1(G)$.

Let f be in $L_1(G)$ and g in $M^\perp$. Choose $\{g_n\}_{n=1}^\infty \subseteq C_{00}(G)$ such that $\|g - g_n\|_2 \to 0$. Then

$$\begin{aligned}
\|\pi(f)\tilde{\pi}(g) - \tilde{\pi}(f * g)\| &= \lim_n \|\pi(f)\pi(g_n) - \tilde{\pi}(f * g)\| \\
&= \lim_n \|\pi(f * g_n) - \tilde{\pi}(f * g)\| \\
&= \lim_n \|\tilde{\pi}(f * g_n) - \tilde{\pi}(f * g)\| \\
&\leqq \lim_n K\|f * g_n - f * g\|_2 \\
&\leqq \lim_n K\|f\|_1\|g_n - g\|_2 \\
&= 0.
\end{aligned}$$

And so we have

$$\begin{aligned}
\|\pi(f)\tilde{\pi}(g)\xi\| &= \|\tilde{\pi}(f * g)\xi\| \\
&\leqq \|\tilde{\pi}(f * g)\| \\
&\leqq K\|f * g\|_2 \\
&\leqq K\|\lambda(f)\|\,\|g\|_2 \\
&\leqq KC\|\lambda(f)\|\,\|\tilde{\pi}(g)\xi\|.
\end{aligned}$$

Hence

$$\|\pi(f)\| \leqq CB\|\lambda(f)\|$$

for all f in $L_1(G)$.

Let $C_L^*(G)$ denote the C^* enveloping algebra of $\lambda(L_1(G))$ in $B(L_2(G))$. Then by the above inequality we may extend π from $L_1(G)$ to a representation of $C_L^*(G)$ on H. Moreover, π is algebraically irreducible on $C_L^*(G)$ since it is on $L_1(G)$. A result of Barnes [**1**, Theorem 4.1] implies that π is similar to a *-representation of $C_L^*(G)$ on H. So there exists a positive invertible operator V in $B(H)$ such that the map

$$a \to V^{-1}\pi(a)V$$

is a *-representation of $C_L^*(G)$ on H. Therefore the map

$$x \to V^{-1}\pi(x)V$$

is a continuous unitary representation of G on H. Let

$$p(x) = \langle V^{-1}\pi(x)V\xi, \xi\rangle = \langle \pi(x)V\xi, V^{-1}\xi\rangle.$$

Then p is a continuous positive definite function on G and p belongs to $L_2(G)$. So by Godement's Theorem [**2**, p. 269, 13.8.6], $p = q * q = q * \tilde{q}$ where $q \in L_2(G)$ and $\tilde{q}(x) = q(x^{-1})$. But then by [**5**, Theorem 20.16] p belongs to $C_0(G)$. This is a contradiction by Lemma 1 and Theorem 5.

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