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ON RIGHT UNIPOTENT SEMIGROUPS

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We investigate the implications of certain conditions on right unipotent semigroups. We describe the greatest idempotent-separating congruence β on a right unipotent semigroup S. Necessary and sufficient conditions for (i) S to be a union of groups, (ii) S to be an inverse semigroup, (iii) the idempotents of S to be in the centre of S and (iv) the quotient semigroup S/β to be isomorphic with the subsemigroup of idempotents of S are also obtained.

It is known that any regular semigroup has the greatest idempotent-separating congruence [5], [6]. Such a congruence on an inverse semigroup was obtained by Howie [4]. For the general terminology and notation the reader is referred to [1], [2].

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1. Preliminary matters.

An orthodox semigroup S is a regular semigroup in which the idempotents form a subsemigroup. An inverse of an idempotent of S is an idempotent, and if a', b' are inverses of the elements a, b in S then $b'a'$ is an inverse of ab [7].

A semigroup S is called *a right* (*left*) *unipotent semigroup* if every principal right (left) ideal of S has a unique idempotent generator. Such semigroups are called left (right) inverse by the author [9], [10]. Lemma 1 below is a part of the left-righ dual of Theorem 1 in [10].

LEMMA 1. *Let S be a regular semigroup. Then the following statements are equivalent.*

(A) *$fef = fe$ for any two idempotents e, f in S.*

(B) *If a' and a'' are inverses of the element a in S then $aa' = aa''$.*

(C) *S is a right unipotent semigroup.*

LEMMA 2. *Let S be a right unipotent semigroup and e be an idempotent of S. Let $x \in S$ and x', x'' be inverses of x. Then xex' is an idempotent and $xex' = xex''$.*

Proof. By Lemma 1 we have $xe = xx'xe = x(x'xex'x) = xex'x$. So xex' is an idempotent. Also $xex' = (xex')xx' = (xex')xx'' = (xex'x)x'' = xex''$, using Lemma 1.

2. The statements (Px), (Qx) and (Rx). Let S be a right unipotent semigroup and $x \in S$. Throught $E = E(S)$ denotes the subsemigroup of idempotents of S and $V(x)$ denotes the set of inverses of the element x. The symbols (Px), (Qx) and (Rx) stand for the statements indicated below.

(Px) $exe = ex$ and $ex'e = ex'$ for all $e \in E$ and for at least one $x' \in V(x)$.

(Qx) $xex' = xx'e$ for all $e \in E$ and $x' \in V(x)$.

(Rx) $xex' = exx'$ for all $e \in E$ and $x' \in V(x)$.

REMARK. Let S be a left unipotent semigroup. Then the left-right dual of (Px), (Qx) and (Rx) are obtained by replacing respectively the equations in them by $xe = xe$ and $ex'e = x'e$, $x'ex = ex'x$ and $x'ex = x'xe$.

THEOREM 1. *Let S be a right unipotent semigroup and $E = E(S)$. Then*

(1) $(Rx) \Rightarrow (Qx) \Rightarrow (Px)$ *for any* $x \in S$.

(2) *E is contained in the center of S if and only if (Rx) is satisfied for all $x \in S$.*

Proof. (1) Let $x \in S$ and $x' \in V(x)$.

Assume (Rx). Then for any $e \in E$ we have $xex' = exx'$ and hence $ex = (exx')x = xex'x = x(x'xex'x) = x(x'xe) = xe$ by Lemma 1. So $xx'e = xx'(ex)x' = xx'(xe)x' = xex'$, giving (Qx).

Assume (Qx). Then $xex' = xx'e$ for any $e \in E$. Therofore, by Lemma 1, we get $exe = ex(x'xe) = ex(x'xex'x) = e(xex')x = e(xx'e)x = exx'x = ex$ and $ex'e = ex'(xx'e) = ex'(xex') = ex'xx' = ex'$, giving (Px).

(2) The only if part is trivial. The if part follows since, for any $x \in S$ and $e \in E$, as shown above, (Rx) implies $ex = xe$.

Let S be a right unipotent semigroup. Then the statements (1) S is union of groups, (2) each $\mathscr{L}$-class of S is a left group and (3) each $\mathscr{R}$-class of S is a group are equivalent [8]. An alternate characterization for S to be a union of groups is obtained in the following

THEOREM 2. *Let S be a right unipotent semigroup and $E = E(S)$. Then S is a union of groups if and only if (Px) is satisfied for all x in S.*

Proof. Let S be a union of groups. Let $x \in S$ and $e \in E$. Let x^{-1} be the inverse of x in the group H_x. Then $x^{-1}x = xx^{-1}$. Let a and b respectively be the group inverses of ex and ex^{-1}. As $x^{-1}e$ is an inverse of ex, and xe is an inverse of ex^{-1}, by Lemma 1 we have $exa = exx^{-1}e$ and $ex^{-1}b = ex^{-1}xe$. But $exx^{-1}e = exx^{-1}$ and $ex^{-1}xe = exx^{-1}e = exx^{-1}$ by Lemma 1. So both ex and ex^{-1} and hence their product $exex^{-1}$ belong to the group with identity element exx^{-1}. As $exex^{-1}$ is an idempotent we conclude that $exex^{-1} = exx^{-1}$. Therefore $exe = ex(x^{-1}xe) = ex(x^{-1}xex^{-1}x) = (exex^{-1})x = exx^{-1}x = ex$ by Lemma 1. Further since ex^{-1} belongs to the group with identity element exx^{-1}, we have $ex^{-1} = ex^{-1}(exx^{-1}) = ex^{-1}(xx^{-1}exx^{-1}) = ex^{-1}(xx^{-1}e) = ex^{-1}e$, by Lemma 1. So we get (Px).

Conversely let (Px) be satisfied for all $x \in S$. (This part of the proof holds for any regular semigroup S). Let $x \in S$ and $x' \in V(x)$. Taking $e = xx'$ in $ex = exe$ we have $x = x^2x' \in x^2S$. So S is a right regular semigroup and hence a union of group [1], [3].

Let S be a right unipotent semigroup. Then S is an inverse semigroup if and only it S satisfies the left-right dual of any of the statements of Lemma 1. We now obtain a necessary and sufficient condition in terms of (Px) and (Rx) for S to be an inverse semigroup.

THEOREM 3. *Let S be a right unipotent semigroup and $E = E(S)$. Then S is an inverse semigroup if and only if (Px) implies (Rx) for all x in S.*

Proof. Let S be an inverse semigroup. Let $x \in S$. Assume (Px). Then for any $e \in E$ we have $exe = ex$ and $ex^{-1}e = ex^{-1}$. As the idempotents in S commute we get $exx^{-1} = (exe)x^{-1} = e(xex^{-1}) = (xex^{-1})e = x(ex^{-1}e) = xex^{-1}$, giving (Rx).

Conversely let (Px) imply (Rx) for all $x \in S$. Let $g, h \in E$. Then for any $e \in E$, by Lemma 1, we have $e(gh)e = egh$. As $gh \in V(gh)$, by hypothesis we conclude $ghegh = eghgh$. So, by Lemma 1, we get $ghe = egh$. Taking $e = h$, by Lemma 1, we have $gh = hg$. Thus S is an inverse semigroup.

COROLLARY. *Let S be a right unipotent semigroup and $E = E(S)$. Then S is an inverse semigroup if and only if (Px), (Qx) and (Rx) are equivalent for all x in S.*

REMARK. The left-right dual of Theorems 1, 2 and 3 hold for a left unipotent semigroup.

3. The congruences α and β. In this section we construct the greatest idempotent-separating congruence on a right (left) unipotent semigroup.

Theorems 4 and 7 below generalize known results for inverse semigroups [4]. In [6] Munn relates the greatest idempotent-separating congruence on an inverse semigroup to a certain full inverse semigroup. We need the following

LEMMA 3. *Let S be an orthodox semigroup and σ be an idempotent-separating congruence on S. If $(x, y) \in \sigma$ then these exist $u \in V(x)$ and $v \in V(y)$ such that $(u, v) \in \sigma$.*

Proof. Let $(x, y) \in \sigma$, $x' \in V(x)$ and $y' \in V(y)$. Since σ is a congruence we get $(x'x, x'y) \in \sigma$ and hence $(x'xy'y, x'y) \in \sigma$. By transitivity of σ we conclude $(x'xy'y, x'x) \in \sigma$. This, since σ is idempotent-separating, implies $x'xy'y = x'x$. So $xy'y = x$. Similarly we get $xx'yy' = yy'$ and $xx'y = y$

Set $u = y'yx'$ and $v = y'xx'$. Then $u \in V(x)$ and $v \in V(y)$. Now from $(x, y) \in \sigma$ we have $(y'xx', y'yx') \in \sigma$, that is $(v, u) \in \sigma$ and thus $(u, v) \in \sigma$. Hence the lemma.

Let S be a regular semigroup and E be the set of idempotents of S. Define the binary relations α and β on S thus:

$$\alpha = \{(x, y) \in S \times S : x'ex = y'ey \text{ for all } e \in E,\ x' \in V(x) \text{ and } y' \in V(y)\}\ .$$
$$\beta = \{(x, y) \in S \times S : xex' = yey' \text{ for all } e \in E,\ x' \in V(x) \text{ and } y' \in V(y)\}\ .$$

THEOREM 4. *Let S be a right* (left) *unipotent semigroup and $E = E(S)$. Then $\beta(\alpha)$ is an idempotent-separating congruence on S. Further, if σ is any idempotent-separating congruence on S then $\sigma \subseteqq \beta(\sigma \subseteqq \alpha)$.*

Proof. We prove the theorem for the right unipotent semigroup S. Clearly β is an equivalence relation on S. Let $(x, y) \in \beta$. Let $c \in S$ and $c' \in V(c)$ and $x' \in V(x)$. Then $x'c' \in V(cx)$ and $y'c' \in V(cy)$. As $c(xex')c' = c(yey')c'$, by Lemma 2, we get $(cx, cy) \in \beta$ and β is a left congruence. Further, since cec' is an idempotent for any $e \in E$, $c'x' \in V(xc)$ and $c'y' \in V(yc)$ we have $x(cec')x' = y(cec')y'$. So by Lemma 2, $(xc, yc) \in \beta$. Therefore β is a right congruence and hence a congruence relation on S.

Now let $g, h \in E$ and suppose that $(g, h) \in \beta$. Then by Lemma 2, for any $e \in E$ we have $geg = heh$. Taking $e = g$ and $e = h$ in turn we obtain $g = hgh = hg$ and $h = ghg = gh$ using Lemma 1. Therefore $g = h(gh) = hh = h$ proving that β is idempotent-separating.

Now let σ be any idempotent-separating congruence on S. Let $(x, y) \in \sigma$. Then by Lemma 3 there exist $x' \in V(x)$ and $y' \in V(y)$ such that $(x', y') \in \sigma$. As σ is a congruence, for any $e \in E$ we have $(xe, ye) \in \sigma$ and hence $(xex', yey') \in \sigma$. But xex' and yey' are idempotents and σ is idempotent-separating. Therefore $xex' = yey'$. This, by

Lemma 2, implies $(x, y) \in \beta$ and thus $\sigma \subseteqq \beta$. Hence the theorem.

COROLLARY [4]. *Let S be an inverse semigroup. Then $\alpha(=\beta)$ is the greatest idempotent-separating congruence on S.*

THEOREM 5. *Let S be a right unipotent semigroup and $E = E(S)$. For each $x \in S$ let $\theta_x \colon E \rightarrow E$ be the mapping defined by $\theta_x(e) = xex'$ where $x' \in V(x)$. Then*

(1) *θ_x is an endomorphism, and*

(2) *the following statements are equivalent.*

(A) *θ_x is an idempotent.*

(B) *The $\mathscr{H}$-class H_x is a subgroup of S and $xx^{-1}e = xex^{-1} = x^{-1}ex$ for all $e \in E$ where x^{-1} is the group inverse of x in H_x.*

(C) *$\theta_x = \theta_g$ where $g = xx'$.*

Proof. Let $x \in S$ and $x' \in V(x)$.

(1) For any $e, f \in E$, by Lemma 1, we have $(xex')(xfx') = (xex'x)fx' = xefx'$, proving (1).

(2) Assume (A). Then $xex' = xxex'x'$ for all $e \in E$. Taking $x'x$ for e we have $xx' = xxx'x'$ and theorefore $x'x = x'(xx')x = x'xxx'x'x = x'xxx'$ using Lemma 1. So $x = xx'x = x^2x'$ and $x\mathscr{R}x^2$.

Now taking $x'x'xx$ for e in $xex' = xxex'x'$, and using $x = x^2x'$ and Lemma 1, we get $xx'x'x = xx'$ and hence $x'x'x = x'$. Therefore $x = xx'x = xx'x'x^2$ and $x\mathscr{L}x^2$. Thus $x\mathscr{H}x^2$ and H_x is a subgroup of S.

By hypothesis and Lemma 2, for all $e \in E$ we have $xex^{-1} = x^2ex^{-2}$ and therefore $x^{-1}xe = x^{-1}xex^{-1}x = x^{-1}(x^2ex^{-2})x = xex^{-1}$ since $x^{-1}x = xx^{-1}$. Again taking $x^{-2}ex^2$ for e in $xex^{-1} = x^2ex^{-2}$ we get $x^{-1}ex = xx^{-1}exx^{-1} = xx^{-1}e$ using Lemma 1. So we get (B).

Assume (B). Then by Lemma 1, for all $e \in E$ we have $xex^{-1} = xx^{-1}e = xx^{-1}exx^{-1}$, giving (C). Clearly (C) implies (A).

THEOREM 6. *Let S be a right unipotent semigroup and $E = E(S)$. Then*

(1) *$T = \{\theta_x \colon x \in S\}$ is a right unipotent semigroup.*

(2) *The mapping $\theta \colon S \rightarrow T$ defined by $\theta(x) = \theta_x$ is an onto homomorphism and $\theta \cdot \theta^{-1} = \beta$.*

(3) *Set $\gamma = \theta | E$ (θ restricted to E). Then γ is an isomorphism of E upon $\theta(E)$.*

Proof. As $\theta_x\theta_y = \theta_{xy}$ it follows that T is a regular semigroup. We now show directly that T is right unipotent. Let θ_x and θ_y be idempotents of T. Then, for all $e \in E$, using (B) of Theorem 5 repeatedly we have $x(yxex^{-1}y^{-1})x^{-1} = xx^{-1}y(xex^{-1})y^{-1} = xx^{-1}yy^{-1}(xex^{-1}) = xx^{-1}(yy^{-1})xex^{-1} = xx^{-1}x(yy^{-1})ex^{-1} = x(yy^{-1}e)x^{-1} = xyey^{-1}x^{-1}$, and hence

$\theta_{xyx} = \theta_{xy}$ by Lemma 2. So, by Lemma 1, T is a right unipotent semigroup, proving (1).

(2) follows directly. As for (3) we need only to show that γ is one-to-one. Let $\gamma(g) = \gamma(h)$ for $g, h \in E$. Then by Lemma 2, we have $(g, h) \in \beta$. This, by Theorem 4, implies $g = h$ and so γ is an isomorphism.

We now consider the quotient semigroup S/β. The following theorem gives a necessary and sufficient condition for S/β to be an idempotent semigroup.

THEOREM 7. *Let S be a right unipotent semigroup and $E = E(S)$. Then the quotient semigroup S/β is isomorphic with E if and only if the statement (Qx) is satisfied for all x in S.* (The left-right dual holds for a left unipotent semigroup).

Proof. Let S/β be isomorphic with E. As S/β is a homomorphic image of S, each idempotent of S/β is the image an idempotent of S [10]. So each β-class of S contains at least one and hence exactly one idempotent of S. Let $x \in S$. Then there exists $h \in E$ such that $(x, h) \in \beta$. So for any $e \in E$ and $x' \in V(x)$ we have $xex' = heh$. In particular taking $e = x'x$ and $e = h$ in turn we get $xx' = hx'xh$ and $xhx' = h$. The first equation gives $xx'h = xx'$ and the second $xx'h = h$. So $xx' = h$. Hence for any $e \in E$, by Lemma 1, we have $xex' = heh = he = xx'e$ giving (Qx).

Conversely let (Qx) be satisfied for all $x \in S$. Let $x \in S$. Then for any $e \in E$ and $x' \in V(x)$, by Lemma 1, we have $xex' = xx'e = xx'exx'$. Therefore $(x, xx') \in \beta$ and hence each β-class contains a unique idempotent. Let β^* be the natural homomorphism of S upon S/β. Then the mapping β^* restricted to E is an isomorphism of E upon S/β. This completes the proof of the theorem.

REMARK. One may appeal to Theorems 5 and 6 to prove Theorem 7. Clearly, for any $x \in S$, the statements (Qx), and (C) of Theorem 5 are equivalent. Therefore (Qx) is satisfied for all $x \in S$ if and only if $T = \theta(E)$ and hence if and only if $S|\beta$ is isomorphic with E.

From Theorems 1, 2, 7 and the corollary of Theorem 3 we have the following.

COROLLARY [4]. *Let S be an inverse semigroup. Then the following statements are equivalent.*

(A) *The quotient semigroup S/β is isomorphic with E.*

(B) *S is a union groups.*

(C) *The idempotents of S are contained in the centre of S.*

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