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A CHARACTERIZATION OF COMPLETELY REGULAR FIELDS

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Dedicated to H. B. Mann on the occasion of his Seventieth Birthday

We prove a theorem on tamely ramified extensions and apply this theorem to obtain a characterization of completely regular fields.

Let Q_p denote the p-adic completion of Q, F a finite extension of Q_p, $e(F \mid Q_p) = a$, the ramification degree of F over Q_p, and ζ_p a primitive p^{th} root of unity.

We say that F is regular if $\zeta_p \notin F$.

When first studying the extension $F(\zeta_p)$ over F, it is a common error to assume that the extension $F(\zeta_p)$ over F is ramified. Or, to put it another way, that if K is an unramified extension of F, then K must also be regular.

On this question, Borevič, [1], has made the following definition: Let $K \supset F$, $e(K \mid F) = 1$. If $\zeta_p \notin K$, for all such K, then we say that F is completely regular.

Using class field theoretic techniques, Borevič, [1], has given a characterization of completely regular fields. This characterization is a corollary to the following theorem.

THEOREM. *Let $L \cap F = L'$, where $e(L' \mid Q_p) = l$, degree $(L \mid L') = e(L \mid L') = f$, $(f, p) = 1$ and $d = (a/l, f)$. Then $L = L'(\pi^{1/f})$, where π is some prime element in L' and $e(F(\pi^{1/f}) \mid F) = f/d$. Furthermore, if K is an unramified extension of F, then $e(K(\pi^{1/f}) \mid K) = f/d$.*

Proof. Given $(a/l, f) = d$, we find an x such that $(x, f) = 1$ and $(a/l)x - yf = d$. To find x, set $d = d_1 d_2$, where d_2 is the largest divisor of d such that $(f_1, d_2) = 1$, where $f_1 = f/d$. Then $(f_1 d_1, d_2) = 1$. Let $(a/ld) \cdot x_1 \equiv 1 (\bmod f_1)$. Since $(f_1 d_1, d_2) = 1$, we can solve the system of congruences $x \equiv x_1 (\bmod f_1 d_1)$, $x \equiv 1 (\bmod d_2)$.

Since $(f, p) = 1$, we have that $L = L'(\pi^{1/f})$ (Weiss, page 89), where π is some prime element in L'. Then $\pi = \alpha \pi_1^{a/l}$, where π_1 is some prime element in F and α is a unit. Since $(f, x) = 1$, we have that

$$F(\pi^{1/f}) = F((\pi^x)^{1/f}) = F((\alpha^x \pi_1^d)^{1/f}) = F((\pi_1 (\alpha^x)^{1/d})^{1/f_1}) .$$

But $F((\alpha^x)^{1/d})$ is unramified over F and $F(\pi^{1/f})$ over $F((\alpha^x)^{1/d})$ is defined by the polynomial $x^{f_1} - (\alpha^x)^{1/d} \pi_1$, which is an Eisenstein polynomial.

Hence $e(F(\pi^{1/f})|F) = f/d$.

Let $e(K|F) = 1$. Then $e(K(\pi^{1/f})\,|F(\pi^{1/f})) = 1$. So we have that $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})|K) \cdot e(K|F) = e(K(\pi^{1/f})|K)$. But also $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})\,|F(\pi^{1/f})) \cdot e(F(\pi^{1/f})|F) = e(F(\pi^{1/f})|F)$. Hence $e(K(\pi^{1/f})|F) = e(K(\pi^{1/f})|K) = e(F(\pi^{1/f})\,|F) = f/d$.

COROLLARY. *The field F is completely regular iff $p - 1 \nmid e(F|Q_p)$.*

Proof. Let degree $(F \cap Q_p(\zeta_p)|Q_p) = l < p - 1$, with $fl = p - 1$, and

$$\text{degree}\,(Q_p(\zeta_p)|F \cap Q_p(\zeta_p)) = e(Q_p(\zeta_p)|F \cap Q_p(\zeta_p)) = f\ .$$

Then $(f, p) = 1$. Let $d = (a/l, f)$. Then $d = f$ iff $p - 1|a$. If K is an unramified extension of F, then $e(K(\zeta_p)|K) > 1$ iff $p - 1 \nmid a$.

COROLLARY. *Let K be an unramified extension of F and $p - 1|a$, then $\zeta_p \in K$ iff $(p - 1)/l\,|$degree$\,(K|F)$, where $l =$ degree$\,(F \cap Q_p(\zeta_p)|Q_p)$.*

Proof. If $p - 1|a$, then $d = f = (a/l, f)$. So $e(F(\zeta_p)|F) = 1$ and degree $(F(\zeta_p)|F) = (p - 1)/l$.

If $\zeta_p \in K$, then $K \supset F(\zeta_p) \supset F$. But degree$\,(F(\zeta_p)|F) = (p - 1)/l$, so $(p - 1)/l\,|$degree$\,(K/F)$.

Conversely, if $(p-1)/l\,|$degree$\,(K|F)$, then since K over F has cyclic galois group, we have a field F_1, $K \supset F_1 \supset F$, where degree$\,(F_1|F) = (p - 1)/l$ and $e(F_1|F) = 1$. But we have shown that $e(F(\zeta_p)|F) = 1$, and degree $(F(\zeta_p)|F) = (p - 1)/l$. Hence $F_1 = F(\zeta_p)$, since there is exactly one unramified extension for each degree. So $\zeta_p \in K$.

REFERENCES

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