

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF RETARDED DIFFERENTIAL DIFFERENCE EQUATIONS

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The asymptotic behavior of the solutions of nonautonomous n^{th} order linear retarded differential difference equations is studied in this paper. It is shown that if the coefficients satisfy certain restrictions, then for any real K there exists a finite dimensional subspace $F(K)$ of the solution space having the following property. For any solution x of the equation one has for all $t > 0$ that $x(t) = x_x(t) + x_r(t)$ where x_x belongs to $F(K)$ and $x_r(t) = 0$ ($\exp(-Kt)$) as $t \rightarrow \infty$. As in the author's earlier papers, considering the periodic and almost periodic cases, the spaces $F(K)$ are obtained by treating the nonautonomous equation as a perturbation of an n^{th} order autonomous equation.

1. Introduction and notation. We consider perturbations of the autonomous n^{th} order equation

$$(1.1) \quad L_0(x(t)) = x^{(n)}(t) + \sum_{k=0}^m \sum_{l=0}^{l(k)} c_{lk} x^{(l)}(t - \Delta_k) = 0$$

where $0 = \Delta_0 < \Delta_1 < \Delta_m$ and the c_{lk} , for all pairs (l, k) occurring in (1.1), are real numbers. We assume $m \geq 1$. We also assume that $l(0) < n$ and that $l(k) < n$, $c_{l(k)k} \neq 0$ for $k = 1, \dots, m$. The perturbed equations will be of the form

$$(1.2) \quad L_0(x(t)) = D(x(t))$$

where

$$(1.3) \quad D(x(t)) = \sum_{h=0}^V \sum_{g=0}^{g(h)} q_{gh}(t) x^{(g)}(t - \sigma_h).$$

Here it is assumed that the q_{gh} belong to $C^{2n}(-\infty, \infty)$ and there exists an $M_1 > 0$ such that

$$(1.4) \quad |q_{gh}^{(j)}(t)| \leq M_1$$

for $t \in (-\infty, \infty)$, $j \leq 2n$, and $(g, h) \in B$, where B denotes the set of all pairs (g, h) in (1.3) for which $q_{gh}(t) \neq 0$.

In earlier papers, the author has established, in the cases where the coefficients q_{gh} are periodic [6] or almost periodic [7], that for $K > 0$ and sufficiently large there exist finite dimensional solution spaces $F(K)$ of (1.2) possessing the following properties. Any so-

lution of (1.2) has a representation of the form

$$(1.5) \quad x(t) = x_K(t) + x_r(t).$$

Here $x_K \in F(K)$ and there exists an $M(x)$ such that

$$(1.6) \quad |x_r(t)| < M(x) \exp(-Kt) \quad \text{for all } t \geq 0.$$

If $H(K)$ denotes the corresponding subspace for (1.1) then there exists a sequence a_r , $\lim a_r = -\infty$ such that

$$(1.7) \quad n(a_r) = \dim H(-a_r) = \dim F(-a_r).$$

In § 3 we shall extend these results to systems of the form (1.2) under certain additional restrictions on the perturbations (1.3). In order to state these restrictions we next introduce the notion of a distribution diagram for (1.1) and (1.3).

Associated with (1.1) we have the characteristic polynomials

$$(1.8) \quad P(z) = \sum_{k,l} \sum c_{lk} z^l \exp(-\Delta_k z) + z^n$$

and $G(z) = P(z) \exp(\Delta_m z)$ where $\sum_{k,l} \sum$ denotes the double sum occurring in (1.1). For $G(z)$ we define its distribution diagram [6] as follows. Let $S(G)$ denote the set of points, in a Cartesian plane, consisting of the point (Δ_m, n) and the points $(\Delta_m - \Delta_k, l)$ where (l, k) is any pair of subscripts in (1.1) for which $c_{lk} \neq 0$. We denote by $p_j = (\beta_j, m_j)$, $j = 0, \dots, k$, those points in $S(G)$ which possess the following properties:

- (a) $p_0 = (0, l(m))$, $p_k = (\Delta_m, n)$
- (b) $\beta_j < \beta_{j+1}$ and $m_j < m_{j+1}$ for $j = 0, 1, \dots, k-1$
- (c) if l_j denotes the line segment connecting p_{j-1} and p_j then every point of $S(G)$ lies on or below at least one of the l_j , $j = 1, \dots, k$,
- (d) if u_j denotes the slope of l_j then $u_1 > u_2, \dots, > u_k$.

The graph consisting of the line segments l_1, \dots, l_k is referred to as the distribution diagram of G . For $j = 1, \dots, k$ we denote by $G_j(z)$ those terms in $G(z)$ which correspond to points in $S(G)$ on the line segment l_j . Each of the polynomials $G_j(z)$ may be factored

$$(1.9) \quad G_j(z) = b_j z^{m_j-1} \exp(\beta_{j-1} z) \prod_{h=1}^{\alpha(j)} (z \exp(z/u_j) - \gamma_{jh})^{\beta(j,h)}$$

where $\gamma_{jh} \neq \gamma_{ji}$ for $i \neq h$ and b_j denotes the coefficient in $G(z)$ of the term $z^{m_j} \exp(\beta_j z)$. Since our estimates for $|G(z)|$ depend directly on the numbers $\beta(j, h)$ in (1.9) we associate with $P(z)$ the following constant

$$(1.10) \quad \sigma(P) = \max \beta(j, h) \text{ for } j = 1, \dots, k \text{ and } h \in [1, \dots, \alpha(j)] .$$

DEFINITION. Equation (1.1) is said to satisfy condition $I\alpha$ if $\sigma(P) = \alpha$ for its associated polynomial $P(z)$ as defined in (2.1).

As mentioned earlier it is necessary to impose a restriction, condition $II\alpha$, on the perturbation term (1.3). In order to define this condition we associate with (1.3) a distribution diagram. Let

$$(1.11) \quad F(z) = \exp(\Delta_m z) \sum_{hg} \sum_{g^h} r_{gh} z^g \exp(-\sigma_h z)$$

where $r_{gh} = 1$ if $(g, h) \in B$, $r_{gh} = 0$ if $(g, h) \notin B$, and $\sum_{hg} \sum_{g^h}$ denotes the double sum in (1.3). As in the case of $G(z)$ one now defines the set of points $S(G)$ and then the distribution diagram of F which is also referred to as the distribution diagram associated with (1.3).

Let $R(\sigma)$ denote the closed region bounded by the lines $x = \sigma/u_k$, $y = 0$, $x = \Delta_m$, and the line segments $l_1(\sigma), \dots, l_k(\sigma)$. Here $l_j(\sigma)$ is the line segment joining the points $p_j(\sigma) = (\beta_j + \sigma/u_k, m_j)$ and $p_{j-1}(\sigma) = (\beta_{j-1} + \sigma/u_k, m_{j-1})$ where p_j and p_{j-1} are the endpoints of l_j in the distribution diagram of $G(z)$.

DEFINITION. We say that (1.3) satisfies conditions II_α for any real α if its distribution diagram is contained in $R(\sigma)$ for some $\sigma > \alpha$.

For an intuitive discussion of condition II_α the reader is referred to the author's earlier paper [6]. Before stating the estimates that we shall need in § 2 we introduce the following notation.

For any complex number z_1 we have

$$(1.12) \quad k(z_1) = |\exp(-z_1/u_k)| \quad \text{and}$$

$$(1.13) \quad l(z_1) = \{z: \text{Re}(z) = \text{Re}(z_1)\} .$$

For any pair $(g, h) \in B$ we define

$$(1.14) \quad H(g, h, w, z) = \exp[-\Delta_h(z - iw)](z - iw)^j/P(z - iw) .$$

Let Z denote the set of zeros of $P(z)$. In an earlier work [6] the author has shown that there exists a decreasing sequence of real numbers $\{a_r\}$, $\lim a_r = -\infty$, $a_0 - 2$ lies to the right of Z and 1, and an M_4 such that the strips $S_r = \{z: |\text{Re } z - a_r| < M_4 k^{-1}(a_r)\}$ do not contain any points of Z . Using the estimates established in [6] one has the following results. If (1.1) satisfies I_α and (1.3) satisfies $II_{\alpha+\gamma}$, $\gamma \geq 0$, then there exists $\beta > 0$, $H_1 > 0$ such that for any pair $(g, h) \in B$, $r \geq H_1$ and $\delta_r = M_4 k^{-1}(a_r)/4$ one has the estimates

$$(1.15) \quad |H(g, h, 0, z)| \leq k^{-(\beta+\gamma)}(a_r) \text{ for } \text{Re } z \in (a_r - \delta_r, a_r + \delta_r) = \alpha_r$$

$$(1.16) \quad \int |H(g, h, 0, z)|^2 |dz| \leq k^{-2(\beta+\gamma)+1}(a_r)$$

$$(1.17) \quad \int |H(g, h, 0, z)| |dz| \leq k^{-(\beta+\gamma)+1}(a_r)$$

where the integrals in (1.16) and (1.17) are along any line $l(z_1)$ for $\text{Re } z_1 \in \alpha_r$. We also have that there exists an $M \geq 0$ such that for any $\sigma \geq 0$

$$(1.18) \quad |H(g, h, 0, z)| \leq M/|z|^2 \quad \text{for } z \in l(a_0 + \sigma)$$

$$(1.19) \quad \int |H(g, h, 0, z)| |dz| \leq M/(a_0 + \sigma)$$

$$(1.20) \quad \int |H(g, h, 0, z)|^2 |dz| \leq M/(a_0 + \sigma)^3$$

where the integrals in (1.19) and (1.20) are along the line $l(a_0 + \sigma)$.

2. The Greens function. In this section we shall obtain in Theorem 2.1 a representation result for the Greens function associated with equation (1.2). The proof of Theorem 2.1 consists of treating equation (1.2) as a perturbation of (1.1) and solving by successive approximations. To facilitate this discussion we introduce the following notation. Let $G(t, s)$, called the Greens function, denote the function which vanishes for $t < s$, satisfies (1.2) as a function of t for $t > s$, $G^{(n-1)}(s^+, s) - G^{(n-1)}(s^-, s) = 1$ and $G^{(j)}(t, s)$ is continuous in t for all $j \leq n - 2$. Here $f^{(j)}(t, s)$ denotes the j^{th} partial derivative of $f(t, s)$ with respect to t . The j^{th} partial derivative of $f(t, s)$ with respect to s will be denoted by $f^{(j)}(t, \underline{s})$. We shall denote by $G(0, t, s)$ the Greens function for equation (1.1). Associated with $G(0, t, s)$ we have

$$(2.1) \quad G(0, t, s, r, 1) = \int_r \exp [z(t - s)] \Delta(z) dz \quad \text{for all values of } t,$$

$$(2.2) \quad G(0, t, s, r, 2) = \int_{a_r} \exp [z(t - s)] \Delta(z) dz \quad \text{for } t > s,$$

where $\Delta(z) = 1/P(z)$ and the integrals are defined as follows. The symbol \int_{a_r} denotes the line integral $1/(2\pi i) \int_{a_r - i\infty}^{a_r + i\infty}$ and $\int_{a_{\bar{r}}}$ denotes the negative of this integral. The symbol \int_r denotes the sum of the two integrals $\int_{a_{\bar{r}}}$ and \int_{a_0} where the a_j are defined as in § 1. Since Z contains only a finite set of points to the right of $l(a_r)$ it follows easily [1] that

$$(2.3) \quad G(0, t, s) = G(0, t, s, r, 1) + G(0, t, s, r, 2)$$

for all values of t . Here $G(0, t, s, r, 1)$ satisfies equation (1.1) for all values of t and $G(0, t, s, r, 2)$ satisfies equation (1.1) for all $t > s$. It also follows [1] that for any $k \geq 0$ and r sufficiently large that there exist constants $M(k, r)$ such that for $j \leq n - 1$

$$\begin{aligned}
 |G^{(j)}(0, \underline{t}, s, r, 1)| &\leq M(k, r) \exp [b(t - s)] \\
 &\quad \text{for } t \geq s \text{ and } b \in [\alpha + k, \alpha + k + 1] \\
 |G^{(j)}(0, \underline{t}, s, r, 1)| &\leq M(k, r) \exp [\gamma(t - s)] \\
 (2.4) \qquad \qquad \qquad &\quad \text{for } t \leq s \text{ and } \gamma \in \alpha_r \\
 |G^{(j)}(0, \underline{t}, s, r, 2)| &\leq M(k, r) \exp [\alpha(t - s)] \\
 &\quad \text{for } t \geq s \text{ and } \alpha \in \alpha_r .
 \end{aligned}$$

Since $G(0, t, s, r, 1)$ and $G(0, t, s, r, 2)$ satisfy equation (1.1) for t and s in appropriate domains it follows that except for a finite set of points in $[0, (n + 1)\Delta_m]$ the derivatives of order $j \leq 2n$ will exist and the constants $M(k, r)$ may be chosen so that one has for these higher derivatives the estimates (2.4) for the indicated values of t and s .

We now establish, in Theorem 2.1, a similar representation result for $G(t, 0)$. For $i = 1, 2$ we define $G(0, t, r, i) \equiv G(0, t, 0, r, i)$. For $j \geq 1$ we define for all values of t

$$\begin{aligned}
 G(j, t, r, 1) &= \int_{-\infty}^t D(G(j - 1, w, r, 1))G(0, t - w, r, 2)dw \\
 (2.5) \qquad &+ \int_0^t D(G(j - 1, w, r, 1))G(0, t - w, r, 1)dw \\
 &+ \int_0^\infty D(G(j - 1, w, r, 2))G(0, t - w, r, 1)dw
 \end{aligned}$$

and for $t > 0$

$$\begin{aligned}
 G(j, t, r, 2) &= \int_\infty^t D(G(j - 1, w, r, 2))G(0, t - w, r, 1)dw \\
 (2.6) \qquad &+ \int_0^{-\infty} D(G(j - 1, w, r, 1))G(0, t - w, r, 2)dw \\
 &+ \int_0^t D(G(j - 1, w, r, 2))G(0, t - w, r, 2)dw .
 \end{aligned}$$

For $t \leq 0, j \geq 1$ we set $G(j, t, r, 2) = -G(j, t, r, 1)$. We then set

$$\begin{aligned}
 G(t, r, 1) &= \sum_{j=1}^\infty G(j, t, r, 1) \\
 (2.7) \qquad G(t, r, 2) &= \sum_{j=0}^\infty G(j, t, r, 2) .
 \end{aligned}$$

THEOREM 2.1. *If equation (1.1) satisfies I_α and Equation (1.3)*

satisfies $II_{\alpha+2}$ then there exists an $H_1 > 0$ such that for any $r \geq H_1$ the function $G(t, 0)$ can be written for all values of t

$$(2.8) \quad G(t, 0) = G(t, r, 1) + G(t, r, 2) .$$

The function $G(t, r, 1)$ satisfies equation (1.2) for all values of t and $G(t, r, 2)$ satisfies (1.2) for all $t > 0$. Furthermore there exists a constant k and constants $N(k, r)$ for $r \geq H_1$ such that

$$(2.9) \quad \begin{aligned} |G^{(j)}(t, r, 1)| &\leq N(k, r) \exp(bt) && \text{for } t \geq 0, b \in [a_0 + k, a_0 + k + 1] \\ |G^{(j)}(t, r, 1)| &\leq N(k, r) \exp(\gamma t) && \text{for } t \leq 0, \gamma \in [a_r - \delta_r/2, a_r + \delta_r/2] = \beta_r \\ |G^{(j)}(t, r, 2)| &\leq N(k, r) \exp(\alpha t) && \text{for } t \geq 0, \alpha \in \beta_r . \end{aligned}$$

Proof. For any function f we denote by $[f]^+$ the function which equals f for $t \geq 0$ and is zero for $t < 0$. We denote by $[f]^-$ the function $f - [f]^+$. We denote by $\|f\|_1$ and $\|f\|_2$ the $L^1(-\infty, \infty)$ and $L^2(-\infty, \infty)$ norms of f . We also introduce the functions

$$f(\gamma, m, h, g, r, j, t) \equiv \exp(-\gamma t)G^{(j)}(m, t - \sigma_h, r, j) .$$

Then using (2.4), (2.9), and (1.17) it follows for $\gamma \in \beta_r$, r sufficiently large, and any pair $(g, h) \in B$

$$(2.10) \quad \begin{aligned} \|[f(\gamma, 0, h, g, r, 1, -)]^-\|_1 &< k^{-\beta/2}(a_r) \\ \|[f(\gamma, 0, h, g, r, 2, -)]^+\|_1 &< k^{-\beta/2}(a_r) \end{aligned}$$

and the $L^2[-\infty, \infty]$ norms of the above functions are bounded by $k^{-\beta/2-3/2}(a_r)$. Similarly using (1.19) in place of (1.17) for any $k > 0$, $(g, h) \in B$ and $b \in [a_0 + k, a_0 + k + 1]$ it follows that

$$(2.11) \quad \|[f(b, 0, h, g, 1, -)]^+\|_1 < M/[a_0 + k]$$

and its $L^2[-\infty, \infty]$ norm is bounded by $M^{1/2}/(a_0 + k)^{3/2}$. Next we note that if $f \in L^1[-\infty, \infty]$, $g \in L^2[-\infty, \infty]$ then

$$h(y) = \int_{-\infty}^{\infty} f(y + x)g(x)dx$$

belongs to $L^2[-\infty, \infty]$ and

$$(2.12) \quad |h(y)| \leq \|f\|_1 \|g\|_2$$

$$(2.13) \quad \|h\|_2 \leq \|f\|_1 \|g\|_2 .$$

For $k = 1, 2$ let $C(k, m, \gamma, b)$ denote $\max(\|[f(\gamma, m, h, g, r, 2, \cdot)]^+\|_k, \|[f(\gamma, m, h, g, r, 1, \cdot)]^-\|_k, \|[f(b, m, h, g, r, 1, \cdot)]^+\|_k)$ where the max is over all pairs $(g, h) \in B$, and $\gamma \in \beta_r$.

Then from (2.5), (2.6), (2.10), (2.13) we have for all $(g, h) \in B$, $m \geq 1$ that

$$\begin{aligned}
 & \| [f(b, m, h, g, r, 1, \cdot)]^+ \|_2 \leq 2M_1 v(n-3)C(2, m-1, \gamma, b) \\
 & \quad [k^{-\beta/2}(a_r) + M/[a_0 + k]] \\
 (2.14) \quad & \| [f(\gamma, m, h, g, r, 1, \cdot)]^- \|_2 \\
 & \leq 3M_1 v(n-3)C(2, m-1, r, b)k^{-\beta/2}(a_r) \\
 & \| [f(\gamma, m, h, g, r, 2, \cdot)]^+ \|_2 \\
 & \leq 3M_1 v(n-3)C(2, m-1, \gamma, b)k^{-\beta/2}(a_r).
 \end{aligned}$$

Thus for $m \geq 1$ one has

$$\begin{aligned}
 (2.15) \quad C(2, m, \gamma, b) & \leq M_1 v(n-3)[3k^{-\beta/2}(a_r) \\
 & + 2M/[a_0 + k]]C(2, m-1, \gamma, b).
 \end{aligned}$$

Similarly from (2.12) we obtain the estimates for all values of t

$$\begin{aligned}
 & |f(b, m, h, g, r, 1, t)|^+ \leq M_1 v(n-3)C(2, m-1, \gamma, b) \\
 & \quad \times [3k^{-\beta/2-3/2}(a_r) + 2M^{1/2}/[a_0 + b]^{3/2}] \\
 (2.16) \quad & | [f(\gamma, m, h, g, r, 1, t)]^- | \\
 & \leq 3M_1 v(n-3)C(2, m-1, \gamma, b)k^{-\beta/2-3/2}(a_r) \\
 & | [f(\gamma, m, h, g, r, 2, t)]^+ | \\
 & \leq 3M_1 v(n-3)C(2, n-1, \gamma, b)k^{-\beta/2-3/2}(a_r).
 \end{aligned}$$

Thus it follows that if k and r are chosen so that

$$(2.17) \quad M_1 v(n-3)[3k^{-\beta/2}(a_r) + 2M/[a_0 + k]] < \frac{1}{2}$$

then the series in (3.7) will converge uniformly and absolutely in every finite interval and for $j = 0$ and $\gamma = \alpha$ one has the estimates (2.9). For $m = 0$, $j = 1, 2$ and $r > H_1$ we note that the functions $G(m, t, r, j)$ are independent of the choice of $\gamma \in \beta_r$. By induction this then holds for $m \geq 1$. Thus we have the estimates in (2.9) for the case $j = 0$. It also follows that $G(t, r, 2)$ and $G(t, r, 1)$ satisfy the equations

$$\begin{aligned}
 G(t, r, 1) & = \int_{-\infty}^t D(G(w, r, 1))G(0, t-w, r, 2)dw \\
 & \quad + \int_0^t D(G(w, r, 1))G(0, t-w, r, 1)dw + \int_0^\infty D(G(w, r, 2)) \\
 & \quad \times G(0, t-w, r, 1)dw \quad \text{for all values of } t \\
 (2.18) \quad G(t, r, 2) & = \int_{-\infty}^t D(G(w, r, 2))G(0, t-w, r, 2)dw \\
 & \quad + \int_0^{-\infty} D(G(w, r, 1))G(0, t-w, r, 2)dw \\
 & \quad + \int_0^\infty D(G(w, r, 2))G(0, t-w, r, 1)dw \quad \text{for } t \geq 0.
 \end{aligned}$$

Now using the estimates (2.4) it follows from (2.18) that $G(t, r, 1)$ satisfies equation (1.2) for all values of t and $G(t, r, 2)$ satisfies equation (1.2) for $t > 0$. It also follows from (2.18), the estimates for the terms $D(G(w, r, i))$ above, and (2.17) that the estimates given in (2.9) hold for $j \leq n - 1$ with $N(k, r) = 2M(k, r)$. This completes the proof of Theorem 2.1.

Associated with equations (1.1) and (1.2) we have the adjoint equations (2.19) and (2.20) given below. These equations are adjoint relative to the inner product given by (3.1) in the next section.

$$(2.19) \quad L_0^+(y(s)) = y^{(n)}(s) + \sum_{kl} \sum (-1)^{l+n} c_{lk} y^{(l)}(s + \Delta_k)$$

$$(2.20) \quad L_0^+(y(s)) = D^+(y(s)) = \sum_{hg} \sum (-1)^{g+n} [q_{gh}(s + \sigma_k) y(s + \sigma_k)]^{(g)} .$$

Now in place of the equations (3.5) and (3.6) we have for $j \geq 1$ and all value of t

$$(2.21) \quad \begin{aligned} G(j, t, s, r, 1) = & \int_{-\infty}^t D(G(j - 1, w, s, r, 1))G(0, t, w, r, 2)dw \\ & + \int_s^t D(G(j - 1, w, s, r, 1))G(0, t, w, r, 1)dw \\ & + \int_s^\infty D(G(j - 1, w, s, r, 2))G(0, t, w, r, 1)dw \end{aligned}$$

and for $t \geq s$

$$(2.22) \quad \begin{aligned} G(j, t, s, r, 2) = & \int_\infty^t D(G(j - 1, w, s, r, 2))G(0, t, w, r, 1)dw \\ & + \int_s^{-\infty} D(G(j - 1, w, s, r, 1))G(0, t, w, r, 2)dw \\ & + \int_s^t D(G(j - 1, w, s, r, 2))G(0, t, w, r, 2)dw . \end{aligned}$$

For $t < s$ and $j \geq 1$ we set $G(j, t, s, r, 2) = -G(j, t, s, r, 1)$. Then defining

$$(2.23) \quad \begin{aligned} G(t, s, r, 1) &= \sum_{j=0}^\infty G(j, t, s, r, 1) \\ G(t, s, r, 2) &= \sum_{j=0}^\infty (G(j, t, s, r, 2)) \end{aligned}$$

we have the following theorem.

THEOREM 2.2. *If equation (1.1) satisfies I_α and equation (1.3) satisfies $II_{\alpha+2}$ for some α then there exists $H_2 > 0$ such that for any $r \geq H_2$ the function $G(t, s)$ can be written for all value of t, s in the form*

$$(2.24) \quad G(t, s) = G(t, s, r, 1) + G(t, s, r, 2) .$$

The function $G(t, s, r, 1)$ satisfies equation (1.2) as a function of t and equation (2.20) as a function of s for all values of s and t . The function $G(t, s, r, 2)$ satisfies (1.2) as a function of t and (2.20) as a function of s for all $t > s$. The mixed partials $\partial^{i+j}G(t, s, r, 1)/\partial t^i \partial s^j$ are continuous and independent of order for all $i \leq n - 1, j \leq n - 1$, and all values of t and s . Furthermore these partials satisfy estimates of the form (2.9) with appropriate constants $N^*(k, r)$ and t replaced by $t - s$ on the right side of the inequalities.

Proof. We first note that the change of variable $t_1 = t - s$ converts the equations (2.21), (2.22) into equations of the form (2.18) where the coefficients used in defining D have been shifted. Since the estimates on the coefficients were uniform in t we have from Theorem 2.1 the assertions that $G(t, s, r, 1)$ and $G(t, s, r, 2)$ are solutions as functions of t , of equation (1.2) and satisfy equations (2.25), (2.26) for the appropriate values of t .

$$(2.25) \quad \begin{aligned} G(t, s, r, 1) = & \int_{-\infty}^t D(G(w, s, r, 1))G(0, t, w, r, 2)dw \\ & + \int_s^t D(G(w, s, r, 1))G(0, t, w, r, 1)dw \\ & + \int_s^\infty D(G(w, s, r, 2))G(0, t, w, r, 1)dw \end{aligned}$$

$$(2.26) \quad \begin{aligned} G(t, s, r, 2) = & \int_\infty^t D(G(w, s, r, 2))G(0, t, w, r, 1)dw \\ & + \int_s^{-\infty} D(G(w, s, r, 1))G(0, t, w, r, 2)dw \\ & + \int_s^t D(G(w, s, r, 2))G(0, t, w, r, 2)dw . \end{aligned}$$

The results of Theorem 2.1 also justify integrating the equations (2.21) and (2.22) by parts, obtaining

$$(2.27) \quad \begin{aligned} & G(j, t, s, r, 1) \\ = & \int_{-\infty}^t \sum_{gh} G(j - 1, w - \sigma_h, s, r, 1)[q_{gh}(w)G(0, t, w, r, 2)]^{(g)}dw \\ & + \int_s^t \sum_{gh} G(j - 1, w - \sigma_h, s, r, 1)[q_{gh}(w)G(0, t, w, r, 1)]^{(g)}dw \\ & + \int_s^\infty \sum_{gh} G(j - 1, w - \sigma_h, s, r, 2)[q_{gh}(w)G(0, t, w, r, 1)]^{(g)}dw \end{aligned}$$

and

$$\begin{aligned}
 &G(j, t, s, r, 2) \\
 &= \int_{-\infty}^t \sum_{gh} G(j-1, w - \sigma_h, s, r, 2) [q_{gh}(w)G(0, t, w, r, 1)]^{(g)} dw \\
 (2.28) \quad &+ \int_s^{-\infty} \sum_{gh} G(j-1, w - \sigma_h, s, r, 1) [q_{gh}(w)G(0, t, w, r, 2)]^{(g)} dw \\
 &+ \int_s^t \sum_{gh} G(j-1, w - \sigma_h, s, r, 2) [q_{gh}(w)G(0, t, w, r, 2)]^{(g)} dw
 \end{aligned}$$

where $[\]^{(g)}$ denotes the g^{th} derivative with respect to w . But then it follows from (2.27) and (2.28) for $i = 1$ that

$$(2.29) \quad L_0^+ G(i, t, s, r, 1,) = D^+(G(i - 1, t, s, r, 1))$$

for all values of t and s , and for $t \geq s$ that

$$(2.30) \quad L_0^+ G(i, t, s, r, 2) = D^+(G(i - 1, t, s, r, 2)) .$$

Then assuming the relations (2.29) and (2.30) hold for $i = n$ one extends them to $i = n + 1$. This is done by noting that due to the smoothness properties of $G(n, t, s, r, k)$, $k = 1, 2$, and its derivatives for $n \geq 1$ and the identity $G(n, t, s, r, 2) = -G(n, t, s, r, 1)$ for $t < s$ one may commute the operator L_0^+ with the integral and summation signs in equations (2.27) and (2.28). Then by the induction hypothesis one obtains $D^+G(n - 2, w - \sigma_h, s, r, k)$ inside the integral as $k = 1$ or 2 in the given integral. But then, for the reasons mentioned above, one is able to commute D^+ with the integral and summation signs to obtain $D^+G(n - 1, w - \sigma_h, s, r, 1)$ in (2.27) and $D^+G(n - 1, w - \sigma_h, s, r, 2)$ in (2.28). This completes the desired induction. The uniform convergence results of Theorem 2.1 now establish the assertions that $G(t, s, r, 1)$ and $G(t, s, r, 2)$ satisfy equation (2.20) for t and s in the appropriate domains. The assertions concerning the mixed partial derivatives of $G(t, s, r, 1)$ are obtained from the identity

$$\begin{aligned}
 &G(t, s, r, 1) \\
 &= \int_{-\infty}^t \sum_{gh} G(w - \sigma_h, s, r, 1) [q_{gh}(w)G(0, t, w, r, 2)]^{(g)} dw \\
 (2.31) \quad &+ \int_s^t \sum_{gh} G(w - \sigma_h, s, r, 1) [q_{gh}(w)G(0, t, w, r, 1)]^{(g)} dw \\
 &+ \int_s^{\infty} \sum_{gh} G(w - \sigma_h, s, r, 2) [q_{gh}(w)G(t, w, r, 1)]^{(g)} dw .
 \end{aligned}$$

In light of the existence of the partials of $G(t, s, r, k)$ and $G(0, t, s, r, k)$ with respect to s and t and the fact that they satisfy bounds of the form (2.4), (2.9), the mixed partials $\partial^{i+j}/\partial t^i \partial s^j$ are obtained by taking these partials inside the integrals along with the additional terms due to the simple discontinuities of $G^{(n-1+k)}(0, \underline{t}, w, r, 2)$ at the finite set

of isolated points t_{kj} in the interval $[0, (n + 1)A_m]$. These terms are of the form $G^{(l)}(t_{kj} - \sigma_h, \underline{s}, r, 1)$ multiplied by appropriate constants where $l = g + i + j - (n - 1 + k)$. Thus the mixed partials possess the stated properties. This completes the proof of Theorem 2.2.

We now denote by K the $\max g(h)$ for $h = 0, \dots, v$. We then have the following result.

THEOREM 2.3. *The conclusions of Theorem 2.2 remain valid if the hypothesis that (1.3) satisfies $II_{\alpha+2}$ is replaced by the hypothesis that (1.3) satisfies $II_{\alpha+1/2}$ and the assumption that for all $j \leq k$ and pairs $(g, h) \in B$ one has that $q_{gh}^{(j)}$ are in $L^1[-\infty, \infty]$.*

Proof. Referring to the proof of Theorem 2.1 we note that in obtaining the estimates (2.14) we used (2.13). The restriction $II_{\alpha+2}$ was used in obtaining a bound for the L^1 norm of the terms playing the role of f . But now using the fact that the $q_{gh} \in L^1[-\infty, \infty]$ one can obtain for the terms playing the role of g , in Theorem 2.1, an estimate on their L^1 norm. Thus one needs only an estimate on the L^2 norm of the terms playing the role of f in the proof of Theorem 2.1. But for this it is sufficient to have (1.3) satisfy the condition $II_{\alpha+1/2}$. Thus the results of Theorem 2.1 remain valid if the $q_{gh} \in L^1[-\infty, \infty]$ for all pairs $(g, h) \in B$. The additional hypothesis that the $q_{gh}^{(j)} \in L^1[-\infty, \infty]$ for $j \leq k$ are needed in the proof of Theorem 2.2 where the adjoint equation is considered. In particular, they are required for the representation given by equation (2.30). This completes the proof of Theorem 2.3.

3. Representation result. In this section we shall establish the representation result given by equation (1.5) and described in §1. This is done by showing that the function $G(t, s, r, 1)$, occurring in Theorem 2.2 defines the finite dimensional subspace $F(-a_r)$ described in §1. This result is then used to establish Theorem 3.2.

Let C denote the space of functions having $n - 1$ continuous derivatives on $[-A_m, 0]$ with the uniform norm $\|\cdot\|$ and C^+ denote the space of functions having $n - 1$ continuous derivatives on $[0, A_m]$ along with the uniform norm. Then associated with equation (1.2) and its adjoint equation (2.20) we have for every real number σ and $f \in C, d \in C^+$ the following inner product

$$[f, d, \sigma] = \sum_{i=1}^n (-1)^{i-1} f^{(n-i)}(0) d^{(i-1)}(0) - \sum_{kl} \left\{ \int_{-A_k}^0 f^{(l)}(\xi) c_{lk} d(\xi + A_k) \right.$$

$$\begin{aligned}
 (3.1) \quad & + \sum_{i=1}^l (-1)^i f^{(l-i)}(0) c_{lk} d^{(i-1)}(\Delta_k) \\
 & + \sum_{hg} \sum \left\{ \int_{-\sigma_h}^0 f^{(g)}(\xi) q_{gh}(\sigma + \xi + \sigma_h) d(\xi + \sigma_h) d\xi \right. \\
 & \left. + \sum_{i=1}^g (-1)^i f^{(l-i)}(0) [q_{gh}(\sigma + \sigma_h) d(\sigma_h)]^{(i-1)} \right\}.
 \end{aligned}$$

Here we have written

$$[q_{gh}(\sigma + \sigma_h) d(\sigma_h)]^{(j)} \quad \text{for} \quad \frac{d^j [q_{gh}(\sigma + \sigma_h + \xi) d(\xi + \sigma_h)]}{d\xi^j}$$

evaluated at $\xi = 0$. If y denotes a solution of the adjoint equation (2.20) for $s \leq s_0$, then for any $\sigma \leq s_0$ we define $y_\sigma \in C^+$ by the equation $y_\sigma(\xi) = y(\sigma + \xi)$ for $\xi \in [0, \Delta_m]$. Similarly if x is a solution of (1.2) for $t \geq t_0$, we denote by x_σ , $\sigma \geq t_0$, the function $x_\sigma \in C$ defined by $x_\sigma(\xi) = x(\sigma + \xi)$ for $\xi \in [-\Delta_m, 0]$. For any r , $r \geq H_2$ as given in Theorem 2.2, we define the linear operator $E(r)$ on C by setting

$$(3.2) \quad E(r, f)(t) = [f, G_0(t, \cdot, r, 1), 0] \quad \text{for} \quad t \in [-\Delta_m, 0].$$

We then have the following result.

THEOREM 3.1. *For any $r \geq H_2$ the linear operator $E(r)$ defines a projection of C into C . The range of $E(r)$ has the dimension $n(a_r)$ as defined in § 1.*

Proof. We first establish the identity

$$(3.3) \quad G(t, s, r, 1) = [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 1), 0]$$

for $r \geq H_2$ and all values of t and s . Since the mixed partials of $G(t, s, r, 1)$ of order up to $2n$ are continuous and since $G(t, s, r, 1)$ is a solution of (1.2) and (2.20) it follows that the right side of (3.3) is a solution of (1.2), as a function of t , and a solution of (2.20), as a function of s for all values of s and t . We consider (3.3) for the case in which $t > \Delta_m$. For these values of s and t [4] one has the identity

$$\begin{aligned}
 (3.4) \quad G(t, s, r, 1) &= [G_0(\cdot, s, r, 1), G_0(t, \cdot), 0] \\
 &= [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 1), 0] \\
 &\quad + [G_0(\cdot, s, r, 1), G_0(t, \cdot, r, 2), 0].
 \end{aligned}$$

Now it is known [4] that for all values of $\sigma \leq 0$

$$(3.5) \quad [G_\sigma(\cdot, s, r, 1), G_\sigma(t, \cdot, r, 2), \sigma]$$

is a constant which is independent of σ . From Theorem 2.2 it follows

that there exists an $M > 0$ such that (3.5) is bounded by

$$M \exp [(a_r - \delta_{r/2})(t - \sigma)] \exp [(a_r + \delta_{r/2})(\sigma - s)].$$

Letting $\sigma \rightarrow -\infty$ we have that the expression in (3.5) is zero and so we obtain (3.3) for all $t > \Delta_m$. Using the fact that solutions of (1.2) have unique backward continuations [5], [4], the identity (3.3) follows for all values of t . Using the continuity of the mixed partials of $G(t, s, r, 1)$ as given in Theorem 2.2 and Fubini's theorem one may carry out the interchanges in the orders of integrations and differentiations [6] to obtain the identity

$$(3.6) \quad \begin{aligned} & [[f, G_0(t, \cdot, r, 1), 0], G_0(\alpha, \cdot, r, 1), 0] \\ & = [f, [G_0(\cdot, s, r, 1), G_0(\alpha, \cdot, r, 1), 0], 0] \end{aligned}$$

where t varies in $[-\Delta_m, 0]$ and s in $[0, \Delta_m]$. Thus we have that $E(r, E(r, f)) = E(r, f)$ and it follows that $E(r)$ defines a projection on C . In order to prove that the dimension of the range of $E(r)$ is $n(a_r)$ one introduces the one parameter family of equations

$$(3.7) \quad L_0(x(t)) = uD(x(t)) \quad \text{for } 0 \leq u \leq 1.$$

Then from Theorem 2.2 one again obtains a function $G(t, s, r, 1, u)$ which for $u = 1$ reduces to $G(t, s, r, 1)$. As in [6], the inner product (3.1) is modified to $[f, d, \sigma, u]$ by replacing the functions q_{gh} by the functions uq_{gh} . One again shows that

$$(3.8) \quad E(r, f, u)(t) = [f, G_0(t, \cdot, r, 1, u), 0, u]$$

defines a projection. It then follows from the proof of Theorems 2.1 and 2.2 that the mixed partials $G^j(t, s, r, 1, u)$ are continuous functions of u for $|t| \leq 2\Delta_m$, $|s| \leq 2\Delta_m$, and $j \leq 2n$. Thus it follows that the norm of $E(r, u)$ is a continuous function of u for $0 \leq u \leq 1$. Since the dimension of the range of $E(r, 0)$ is $n(a_r)$ it then follows by the usual arguments [6] that the dimension of the range of $E(r, 1)$ is also $n(a_r)$. This completes the proof of Theorem 3.1.

Let $C(r)$ denote the range of $E(r)$. Then since $C(r)$ has the dimension $n(a_r)$ we may select a basis ζ_{rj} , $j = 1, \dots, n(a_r)$ for $C(r)$. Thus for every $f \in C(r)$ we have a unique representation of the form

$$(3.9) \quad E(r, f) = \sum_{j=1}^{n(a_r)} a(f, j) \zeta_{rj}.$$

Now for $t > 0$ we have [4] for any solution x of (1.2)

$$(3.10) \quad \begin{aligned} x(t, f, 0) &= [f, G_0(t, \cdot), 0] = [f, G_0(t, \cdot, r, 1), 0] \\ &+ [f, G_0(t, \cdot, r, 2), 0]. \end{aligned}$$

Since $G(t, s, r, 1)$ satisfies equation (1.2) for all values of t it follows that $[f, G_0(t, \cdot, r, 1), 0]$ has a unique backward extension to $[-A_m, 0]$. Thus it follows from (3.9) and (3.10) that for all $t > 0$

$$(3.11) \quad x(t, f, 0) = \sum_{j=1}^{n(a_r)} a(f, j)x(t, \zeta_{r,j}, 0) + x_r(t).$$

From the results of Theorem 2.2 we have that there exists a constant $k(r)$, depending only on r , such that for all $t > 0$

$$(3.12) \quad \|x_{r,t}\| \leq k(r) \|f\| \exp(a_r t).$$

Summarizing these results we have the following theorem.

THEOREM 3.2. *Under the hypothesis of Theorem 2.2 or Theorem 2.3 one has for every $r > H_2$ and every solution $x(t, f, 0)$ of (1.2) a unique representation of the form (3.11) where the remainder term x_r satisfies the estimate (3.12).*

We note finally that the hypothesis of Theorem 2.3 are more restrictive than those of Hale [3] and Cooke [2]. Thus in this case their results are applicable. These results assure us that each of the solutions $x(t, \zeta_{r,j}, 0)$ is asymptotic as $t \rightarrow \infty$ to a solution of (1.1).

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