PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 2, 1976

EXACT FUNCTORS AND MEASURABLE CARDINALS

ANDREAS BLASS

The purpose of this paper is to prove that all exact functors from the category $\mathscr{S}$ of sets to itself are naturally isomorphic to the identity if and only if there are no measurable cardinals. The first step in the proof is to approximate arbitrary left-exact endofunctors F of $\mathscr{S}$ with endofunctors of a special sort, reduced powers, and to characterize reduced powers in terms of category-theoretic properties. The next step is to determine the effect, on the approximating reduced powers, of the additional assumption that F preserves coproducts or coequalizers. It turns out that preservation of coequalizers is an extremely strong condition implying preservation of many infinite coproducts. From this fact, the main theorem follows easily.

Left-exact functors, by definition, preserve equalizers and finite products. It follows that they preserve pullbacks (including intersections as a special case) and monomorphisms. Note that any functor on $\mathscr{S}$ preserves all epimorphisms, because they split. Also note that, if $F: \mathscr{S} \to \mathscr{S}$ is left exact and $F(\phi) = \phi$, then, for F to preserve a coproduct $\coprod_{\alpha \in I} A_\alpha$ with injections i_α, it is necessary and sufficient that the maps $F(i_\alpha)$ be jointly epic; indeed, left-exactness guarantees that these maps are monic and that the ranges of any two of them have intersection $F(\phi)$ which was assumed to be empty.

To avoid annoying special cases later, observe that there is only one (up to natural isomorphism) product-preserving $F: \mathscr{S} \to \mathscr{S}$ for which $F(\phi) \neq \phi$, namely the functor sending every set to a singleton (i.e. the functor represented by ϕ). To see this, simply note that the second projection $F(X) \times F(\phi) \to F(\phi)$ is an isomorphism (because $X \times \phi \to \phi$ is an isomorphism and F preserves products). This functor will be called the *improper* left-exact endofunctor of $\mathscr{S}$; all others are *proper*.

For the sake of notational simplicity, when F is a product-preserving functor, the natural isomorphism $F(A \times B) \cong F(A) \times F(B)$ (induced by F of the projections) will not be explicitly mentioned. Thus, if $a \in F(A)$ and $b \in F(B)$, then (a, b) will be considered an element of $F(A \times B)$.

For similar reasons, the distinction between sets, classes, and things of even higher type will be suppressed (except in § 5). For example, the category of left-exact endofunctors of $\mathscr{S}$ and natural transformations between them will be treated as though it were a set. Scrupulous readers are invited to assume the existence of a

Grothendieck universe and refer to it when necessary.

1. Reduced powers. Among left-exact set-valued functors, the simplest are the representable functors. Arbitrary set-valued functors are colimits of representable ones [11, Ch. 10], and left-exact set-valued functors are filtered colimits of representable ones. Unfortunately, the diagrams and index categories involved in these colimits can be extremely complicated. It is more useful for some purposes to express left-exact endofunctors of $\mathscr{S}$ as colimits of a very simple sort, directed unions. The cost of this simplification is that one must allow, as terms in the union, functors more complicated than representable ones; one needs reduced powers [3], which we now describe.

Let D be a filter on a set A. (D may be the improper filter, the set of all subsets of A.) For any set x, consider functions f into X whose domains belong to D. Two such functions, f and g, are *equal modulo* D ($f = g \bmod D$) if the set of $a \in A$ such that $f(a)$ and $g(a)$ are both defined and are equal belongs to D. This clearly defines an equivalence relation. Its equivalence classes $[f]_D$, called *germs* (modulo D), are the elements of the *reduced power* D-prod X. Notice that usually a germ will contain a function whose domain is all of A; the only exception is if $A \neq \phi = X$ and D is the improper filter. This exceptional case will often be left for the reader to handle, and so all germs under consideration will be assumed to contain total functions.

Any function $X \to Y$ induces, by composition, a function D-prod $X \to D$-prod Y. The reader can easily verify that D-prod is a left-exact endofunctor of $\mathscr{S}$. (Readers familiar with [1] will recognize D-prod as the composite of the "principal filter" functor $P: \mathscr{S} \to \mathscr{G}$ and the functor $\mathscr{G} \to \mathscr{S}$ represented by D; they can prove its left-exactness by citing Theorems 2 and 9 of [1]. Other readers will note that D-prod is a filtered colimit of representable functors and is therefore left-exact by Theorem 9. 5. 2 of [11].) Notice that, if D is an improper filter, then D-prod is the improper left-exact functor.

A set-valued $F: \mathscr{C} \to \mathscr{S}$ on an arbitrary category is called *weakly reprensentable* if there exist an object A in $\mathscr{C}$ and an element $a \in F(A)$ such that, for every object B of $\mathscr{C}$ and every $b \in F(B)$, there is an $f: A \to B$ in $\mathscr{C}$ such that $F(f)(a) = b$. (If f were required to be unique, then A would represent F and a would be a universal element. Without the uniqueness condition, F is merely a quotient of the representable functor $\mathscr{S}(A, -)$, and a is sometimes called a "versal" element for F.) Note that any reduced power functor D-prod is weakly representable; one can take A to be the set on which D is a filter (or $A = \phi$ if D is improper) and $a \in D$-prod A to be the germ of the identity map of A. This remark establishes half of the following theorem.

THEOREM 1. *The left-exact weakly representable endofunctors of $\mathscr{S}$ are precisely the reduced powers* (up to natural isomorphism).

Proof. It remains to prove that, if $F: \mathscr{S} \to \mathscr{S}$ is left exact and weakly represented by (A, a), then F is naturally isomorphic to a reduced power. This is trivial if F is improper, so assume that $F(\phi) = \phi$.

Let D be the collection of all subsets X of A such that a is in the range of $F(i_X): F(X) \to F(A)$, where $i_X: X \to A$ is the inclusion map. If $X \in D$ and $X \subseteq Y \subseteq A$, then $Y \in D$, because i_X factors through i_Y. If X and Y are in D, then so is $X \cap Y$ because F preserves intersections. Thus, D is a filter on A, proper because F is proper. For any set B and any $f: A \to B$, set $\gamma(f) = F(f)(a)$. Because (A, a) weakly represents F, γ maps $\mathscr{S}(A, B)$ onto $F(B)$. Furthermore, for any two maps $f, g: A \to B$,

$$\begin{aligned}
\gamma(f) = \gamma(g) \quad &\text{iff} \quad F(f)(a) = F(g)(a) \\
&\text{iff} \quad a \in \text{Equalizer of } F(f) \text{ and } F(g) \\
&\text{iff} \quad a \in \text{Image of } F(\text{Equalizer of } f \text{ and } g) \\
&\text{iff} \quad (\text{Equalizer of } f \text{ and } g) \in D \\
&\text{iff} \quad f = g \bmod D\,.
\end{aligned}$$

(The third equivalence used the left-exactness of F.) Thus, γ induces a bijection from D-prod B to $F(B)$. The verification that this bijection is natural is left to the reader.

2. Left-exact functors. Let $F: \mathscr{S} \to \mathscr{S}$ be left-exact. The goal of this section is to express F as the union of a directed family of weakly representable left-exact subfunctors of F. This is trivial if F is the improper left-exact functor, so assume that F is proper.

For any set A and any $a \in F(A)$, define a subfunctor $F_{A,a}$ of F by

$$F_{A,a}(X) = \{F(f)(a) \mid f: A \to X\}\,,$$

and

$$F_{A,a}(f) = F(f) \mid F_{A,a}(\text{Domain of } f)\,.$$

Clearly, $F_{A,a}$, is weakly representable by (A, a). Equally clearly, F is the pointwise union (over all A and all $a \in F(A)$) of the functors $F_{A,a}$,

$$F(X) = \bigcup_{A,a} F_{A,a}(X)\,,$$

because any $x \in F(X)$ belongs to $F_{X,x}(X)$. Since F preserves products, the family of subfunctors $F_{A,a}$ is directed by inclusion. Indeed, if $B = A_1 \times A_2$ and $b = (a_1, a_2) \in F(B)$, then both F_{A_1,a_1} and F_{A_2,a_2} are subfunctors of $F_{B,b}$. Thus, F is a directed union of weakly representable subfunctors $F_{A,a}$. (These subfunctors are just the images in F of the representable functors in the usual representation of F as a colimit of representables.)

It remains to prove that the subfunctors $F_{A,a}$ are left-exact (and therefore isomorphic to reduced powers). The verification that they preserve products is easy and therefore omitted. Consider any equalizer diagram

$$E \overset{i}{\rightarrowtail} B \underset{l}{\overset{k}{\rightrightarrows}} C$$

in $\mathscr{S}$. As $F_{A,a}$ is a subfunctor of F, which preserves monomorphisms, $F_{A,a}(i)$ is monic, and of course its composites with $F_{A,a}(k)$ and $F_{A,a}(l)$ are equal. It remains to check that any element $F(f)(a)$ in $F_{A,a}(B)$ whose images under $F(k)$ and $F(l)$ are equal is in the range of $F_{A,a}(i)$; this range is of course just $F(i)\ (F_{A,a}(E))$. Consider the diagram

$$\begin{tikzcd}
Q \arrow[r, tail, "j"] \arrow[d, "g"'] & A \arrow[d, "f"] \arrow[dl, dashed, "h"] & \\
E \arrow[r, tail, "i"'] & B \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & C
\end{tikzcd}$$

where $Q \overset{j}{\rightarrowtail} A$ is the equalizer of kf and lf, and where g is the unique map making the square commute. As F preserves equalizers and $F(kf)(a) = F(lf)(a)$, there exists $q \in F(Q)$ such that $a = F(j)(q)$. As F is proper, it follows that $Q \neq \phi$; the existence of g then implies $E \neq \phi$. Therefore, there is a map h making the upper triangle in the diagram commute. Then $fj = ig = ihj$, so

$$F(f)(a) = F(fj)(q) = F(ihj)(q) = F(i)F(h)(a) \in F(i)\ (F_{A,a}(E))\ ,$$

as required. This completes the proof of the following theorem.

THEOREM 2. *Every left-exact endofunctor F of $\mathscr{S}$ is the directed union of subfunctors naturally isomorphic to reduced powers. If F is proper, the subfunctors may be taken to be the functors $F_{A,a}$ defined above.*

(Notice that, if F is improper, then $F_{A,a}$ is F when A is empty but fails to be left-exact when A is nonempty.)

As an application of Theorem 2, I sketch an analysis of the collection $\mathrm{Nat}(F, G)$ of natural transformations from one left-exact endofunctor of $\mathscr{S}$ to another. By Theorem 2, these functors may be written as $F = \varinjlim_j (D_j\text{-prod})$ and $G = \varinjlim_i (E_i\text{-prod})$, where i and j range over directed classes and where the transition maps $D_j\text{-prod} \to D_{j'}$, -prod are monic natural transformations. It is clear, from the definition of colimit, that

$$\mathrm{Nat}(F, G) = \varprojlim_j \mathrm{Nat}(D_j\text{-prod}, \varinjlim_i (E_i\text{-prod}))\,.$$

Now any natural transformation from D_j-prod into $\varinjlim_i (E_i\text{-prod})$ must in fact map into a single E_i-prod (because D_j-prod is weakly representable), so

$$\mathrm{Nat}(F, G) = \varprojlim_j \varinjlim_i \mathrm{Nat}(D_j\text{-prod}, E_i\text{-prod})\,.$$

The problem is thus reduced to the special case that both F and G are reduced powers.

Omitting the subscripts i and j, consider a natural transformation $\alpha\colon D\text{-prod} \to E\text{-prod}$, where D is a filter on A and E is a filter on B. The versal element $[id_A]_D$ for D-prod is sent by α_A to some $[f]_E \in E\text{-prod}\, A$. I claim that $[f]_E$ completely determines α; indeed, if $[g]_D \in D\text{-prod}\, X$, then one easily sees, using the naturality of α, that $\alpha_X[g]_D = [g\circ f]_E$ (which is independent of the choice of f in $[f]_E$). I also claim that the filter

$$f(E) = \{X \subseteq A \mid f^{-1}(X) \in E\}$$

is an extension of D. To see this, consider any $X \in D$, let B be the pushout $A \cup_X A$, and let $i_0, i_1\colon A \to B$ be the injections. Then $[i_0]_D = [i_1]_D$ because $X \in D$; applying α_B yields $[i_0 f]_E = [i_1 f]_E$, which means $f^{-1}(X) \in E$.

Conversely, if D and E are filters on A and B respectively, and if $f\colon B \to A$ satisfies $D \subseteq f(E)$, then the formula $\alpha_X[g]_D = [g\circ f]_E$ defines a natural transformation $\alpha\colon D\text{-prod} \to E\text{-prod}$. Let $\mathscr{G}$ be the category of filters defined in [1]; then the preceding discussion may be summarized as follows.

THEOREM 3. *The category of weakly representable left-exact endofunctors of $\mathscr{S}$ is dual to $\mathscr{G}$. The category of all left-exact endofunctors of $\mathscr{S}$ is dual to a full subcategory of the category of pro-objects in $\mathscr{G}$.*

Pro-objects are defined in [5], but for the purposes of this theorem one must relax the definition to allow the inverse systems to be proper classes. The need for large indexing classes will be considered further in § 5.

In the duality considered in Theorem 3, the identity functor on $\mathscr{S}$ corresponds to the principal filter on a singleton. Since this filter is terminal in $\mathscr{G}$, (and in the category of pro-objects) the identity functor is initial in the category of left-exact endofunctors of $\mathscr{S}$.

3. Preservation of coproducts.

THEOREM 4. *Let I be any index set, and let $F: \mathscr{S} \to \mathscr{S}$ be left-exact. Then F preserves all I-indexed coproducts if and only if all the subfunctors $F_{A,a}$ preserve all I-indexed coproducts.*

Proof. Ignoring the trivial case of improper F, assume that F, and therefore also every $F_{A,a}$, are proper as well as left-exact. Suppose B is the coproduct of the family $\{B_\alpha \mid \alpha \in I\}$ with injections $i_\alpha: B_\alpha \to B$. As noted in § 0, this coproduct is preserved by F if and only if the maps $F(i_\alpha)$ are jointly epic, and similarly for $F_{A,a}$.

Suppose all $F_{A,a}$ preserve I-indexed coproducts, and let $x \in F(B)$. Then

$$\begin{aligned} &x \in F_{B,x}(B) \quad \text{so, for some} \quad \alpha \in I , \\ &x \in \text{Range of } F_{B,x}(i_\alpha) \subseteqq \text{Range of } F(i_\alpha) . \end{aligned}$$

Conversely, suppose F preserves I-indexed coproducts and $x \in F_{A,a}(B)$; I must show $x \in \text{Range of } F_{A,a}(i_\alpha) = F(i_\alpha)F_{A,a}(B_\alpha)$, for some $\alpha \in I$. By definition of $F_{A,a}(B)$, there exists $f: A \to B$ such that $x = F(f)(a)$. For each $\alpha \in I$, let $A_\alpha = f^{-1}(B_\alpha) \subseteqq A$, so A is the coproduct of the A_α, with injections j_α. Also, let $f_\alpha: A_\alpha \to B_\alpha$ be the restriction of f. Thus, the diagram

$$\begin{tikzcd} A_\alpha \arrow[r, tail, shift left, "j_\alpha"] \arrow[d, "f_\alpha"'] & A \arrow[l, dotted, shift left, "g"] \arrow[d, "f"] \\ B_\alpha \arrow[r, tail, "i_\alpha"] & B , \end{tikzcd}$$

without the dotted arrow, commutes. As F preserves the coproduct A of $\{A_\alpha \mid \alpha \in I\}$, a must be in the range of some $F(j_\alpha)$. Fix this $\alpha \in I$, and let $q \in F(A_\alpha)$ satisfy $a = F(j_\alpha)(q)$. As F is proper, the existence of q implies that $A_\alpha \neq \phi$, so there exists $g: A \to A_\alpha$ with $g \circ j_\alpha = \text{identity of } A_\alpha$. Then $fj_\alpha = i_\alpha f_\alpha g j_\alpha$, so

$$\begin{aligned} x &= F(f)(a) = F(fj_\alpha)(q) = F(i_\alpha f_\alpha g j_\alpha)(q) \\ &= F(i_\alpha)F(f_\alpha g)(a) \in F(i_\alpha)(F_{A,a}(B_\alpha)) , \end{aligned}$$

as required.

Thus, the study of coproduct-preserving left-exact functors reduces to the study of coproduct-preserving reduced powers.

THEOREM 5. *The reduced power D-*prod *preserves finite coproducts if and only if D is an ultrafilter. It also preserves κ-indexed coproducts if and only if the intersection of any κ sets in the ultrafilter D is itself in D.*

Proof. D-prod preserves the empty coproduct if and only if D is proper (by inspection).

Suppose D is a filter on A and D-prod preserves finite coproducts. If A is partitioned into two sets, $A = B \cup C$, then $[id_A]_D$ must be in D-prod B or in D-prod C. Without loss of generality, assume $[id_A]_D = [f]_D$ where $f: A \to B$. Then the subset $\{x \in A \mid f(x) = x\}$ of B is in D, so $B \in D$. Thus, D is an ultrafilter.

Conversely, suppose D is an ultrafilter and $X = Y \cup Z$. It suffices to show that every member $[f]_D$ of D-prod X is in either D-prod Y or D-prod Z. But $f^{-1}(Y)$ and $f^{-1}(Z)$ form a partition of a set in D, so one of them, say $f^{-1}(Y)$, is in D. By altering the definition of f on the complement of this set, one can clearly get a map into Y that is equal to f modulo D.

The argument for κ-indexed coproducts is the same.

Theorems 2, 4, and 5 immediately imply the following corollary, which is also obtainable from Theorem IV 1.4 of [10]; using Theorem 3 also, one obtains the result of Joyal [6] (modulo set-class difficulties to be considered in § 5). I thank G. Reyes for bringing these references to my attention.

COROLLARY. *The left-exact endofunctors of $\mathscr{S}$ that preserve finite coproducts are* (up to natural isomorphism) *the same as the directed unions of ultrapowers.*

Directed unions of ultrapowers are the same as limit ultrapowers, as defined in [7].

4. Preservation of coequalizers.

THEOREM 6. *A proper left-exact endofunctor of $\mathscr{S}$ preserves coequalizers if and only if it preserves countably-indexed coproducts.*

Proof. Let $F: \mathscr{S} \to \mathscr{S}$ be proper and left-exact. Suppose that

F preserves coequalizers. Consider first the effect of F on $N = \{0, 1, 2, \cdots\}$, the coproduct of countably many copies of the singleton 1, with injections $i_n: 1 \to N$. Let $S: N \to N$ be the successor function, $S(x) = x + 1$. Then $N \underset{S}{\overset{\mathrm{id}}{\rightrightarrows}} N \to 1$ is a coequalizer, so, by hypothesis, $F(N) \underset{F(S)}{\overset{\mathrm{id}}{\rightrightarrows}} F(N) \to 1$ is a coequalizer. (1 is, of course, preserved by F as F is left-exact.) Thus, it is impossible to partition $F(N)$ into two nonempty disjoint subsets both closed under $F(S)$. Using the equations $Si_n = i_{n+1}$, one easily sees that the set

$$\tilde{N} = \bigcup_n \text{Range of } \quad F(i_n) \quad = \{\tilde{0}, \tilde{1}, \tilde{2}, \cdots\}$$

is closed under $F(S)$. (Here $\tilde{n}$ is the unique member of the range of $F(i_n)$. Note that all the $\tilde{n}$ are distinct, because F is left-exact, and that $F(S)(\tilde{n}) = \widetilde{n+1}$.) I claim that $F(N) - \tilde{N}$ is also closed under $F(S)$. For suppose $p \in F(N) - \tilde{N}$ and $F(S)(p) = \tilde{n} \in \tilde{N}$. As i_0 and S have pullback (= intersection) ϕ and F is proper and left-exact, we see that $\tilde{0} \notin$ Range of $F(S)$, so $n = k + 1$ for some k and $\tilde{n} = F(S)(\tilde{k})$. But S is monic, so $F(S)$ is monic, so $p = \tilde{k}$, a contradiction. Thus both $\tilde{N}$ and $F(N) - \tilde{N}$ are closed under $F(S)$, and therefore $\tilde{N} = F(N)$. (For functors that are known to preserve finite coproducts, the preceding argument could be replaced by a reference to Freyd's characterization [4] of natural-number-objects in topoi.)

Consider now an arbitrary countably indexed coproduct $A = \coprod_{n \in N} A_n$ with injections j_n. Let $F: A \to N$ be the map sending all of A_n to n; thus

$$\begin{tikzcd} A_n \arrow[r, tail, "j_n"] \arrow[d] & A \arrow[d, "f"] \\ 1 \arrow[r, tail, "i_n"] & N \end{tikzcd}$$

is a pullback. As F is left-exact, the range of $F(j_n)$ is the preimage of $\tilde{n}$ under the map $F(f)$. But the union of these preimages is all of $F(A)$ because $F(N) = \tilde{N}$. Therefore, the maps $F(j_n)$ are jointly epic, as required.

Conversely assume F preserves countably indexed coproducts (and is proper and left-exact as always). To show that F preserves coequalizers, consider the following detailed description of how the coequalizer of a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is to be found in $\mathscr{S}$. First, let $R \rightarrowtail B \times B$ be the image of $(f, g): A \to B \times B$, and let $\Delta: B \rightarrowtail B \times B$ be the diagonal. Second, define inductively

$$R^1 = R \cup \Delta \cup tR$$
$$R^{n+1} = R^n \circ R^1 .$$

Here $t: B \times B \to B \times B$ is the map (p_2, p_1) that interchanges the two factors, and the composite $S \circ T$ of two relations in defined by forming the pullback

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & B \times T \arrow[d, tail] \\ S \times B \arrow[r, tail] & B \times B \times B \end{tikzcd}$$

and then taking the image of $P \to B \times B \times B \xrightarrow{(p_1, p_3)} B \times B$. Third, define $\bar{R}$ to be the union of all the R^n. Finally, let $B \xrightarrow{h} C$ be the unique epimorphism such that

$$\begin{tikzcd} \bar{R} \arrow[r, tail] \arrow[d, tail] & B \times B \arrow[r, "p_1"] & B \arrow[dd, "h"] \\ B \times B \arrow[d, "p_2"] & & \\ B \arrow[rr, "h"] & & C \end{tikzcd}$$

is a pullback. Then h is the coequalizer of f and g.

By inspecting this description of coequalizers, one sees that F preserves them provided it preserves images and countable unions, for all the other concepts used in the description are preserved by all left-exact functors. But the image of $X \xrightarrow{k} Y$ can be described (up to isomorphism) as the middle object in an epi-mono factorization $X \twoheadrightarrow I \rightarrowtail Y$ of k, and countable unions can be described as images of countable coproducts. But F preserves epimorphisms (see § 0), monomorphisms, and countable coproducts, so it preserves images and countable unions as well.

It is well-known (see, for example [8, Thm. 2.1]) that any ultrafilter closed under countable intersections is necessarily closed under κ-indexed intersections for all cardinals κ smaller than the first measurable cardinal. (By convention, if there is no measurable cardinal, every cardinal is to be considered "smaller than the first measurable cardinal." in this situation, only principal ultrafilters are closed under countable intersections.) Combining this fact with Theorems 4, 5 and 6, one immediately obtains the following corollary.

COROLLARY. *All exact endofunctors of $\mathscr{S}$ preserve all κ-indexed coproducts for all cardinals κ smaller than the first measurable cardinal.*

If there is no measurable cardinal, then exact endofunctors of $\mathscr{S}$ preserve all coproducts; since every set is a coproduct of singletons, it follows that these functors are naturally isomorphic to the identity. On the other hand, if there is a measurable cardinal κ and if D is a κ-complete non-principal ultrafilter on κ, then D-prod is an exact functor (by Theorems 5 and 6) not naturally isomorphic to the identity (because D-prod κ is not isomorphic to κ). These remarks prove the following theorem.

THEOREM 7. *The existence of a measurable cardinal is equivalent to the existence of an exact endofunctor of $\mathscr{S}$ not naturally isomorphic to the identity.*

5. **Do small limits suffice?** In Theorem 2, left-exact functors were expressed as directed unions of reduced powers. The reduced powers occurring there were indexed by the class of pairs (A, a) with $a \in F(A)$, a class which is clearly not a set. In this section, I consider briefly the question of whether Theorem 2 remains true if only a set of indices is allowed.

Call a functor $F: \mathscr{S} \to \mathscr{S}$ κ-bounded if, for every A and every $a \in F(A)$, there is an inclusion $i: X \rightarrowtail A$ such that X has cardinality at most κ and $a \in$ Range of $F(i)$. Call F *bounded* if it is κ-bounded for some cardinal κ (a set). It is easy to see that a left-exact F is the union of a set-indexed directed family of reduced powers if and only if it is bounded. Thus, the problem amounts to asking whether every left-exact endofunctor of $\mathscr{S}$ is bounded.

Suppose F is an unbounded left-exact endofunctor of $\mathscr{S}$. For any cardinal κ, there are A and $a \in F(A)$ such that a doesn't come from any subset of A of cardinality $\leqq \kappa$. In the proof of Theorem 1, the subfunctor $F_{A,a}$ of F is isomorphic to D-prod where the filter D contains no sets of cardinality $\leqq \kappa$. Such a D can be extended to an ultrafilter E that also contains no sets of cardinality $\leqq \kappa$. Since any germ modulo D is included in a unique germ modulo E, there is a natural map from D-prod X onto E-prod X for all X. Therefore,

$$|E\text{-prod } X| \leqq |D\text{-prod } X| = |F_{A,a}(X)| \leqq |F(X)| .$$

Since κ was arbitrary, there are uniform ultrafilters E on arbitrarily large cardinals such that, for all X, $|E\text{-prod } X| \leqq |F(X)|$.

Now consider the following hypothesis, first suggested by Keisler:

(R) Every uniform ultrafilter on an infinite cardinal κ is regular .

For the definition of regular, see [2]. The only fact about regularity that is needed here is that, if E is κ-regular and X is infinite, then $|E\text{-prod}\, X| \geqq |X|^{\kappa}$. This fact, proved in [2, Prop. 4.3.7], together with the preceding remarks, clearly shows that hypothesis (R) implies that all left-exact endofunctors of $\mathscr{S}$ are bounded. It should be noted that (R) (or at least the consistency of (R)) is not entirely implausible. It is known that the case $\kappa = \aleph_0$ of (R) is true [2, Prop. 4.3.4], and, if Gödel's axiom of constructibility holds, then (R) is true for all $\kappa < \aleph_\omega$. (See [9] for the case $\kappa = \aleph_1$; the remaining cases are unpublished work of Jensen.) It seems plausible that (R) might hold in the constructible universe; even if it does not, there is hope for weaker hypotheses, like Conjecture 4 in Appendix B of [2], which still suffice to imply boundedness of all left-exact $F: \mathscr{S} \to \mathscr{S}$.

On the other hand, (R) is false if measurable cardinals exist. However, even in this situation, it may still be the case that all left-exact endofunctors of $\mathscr{S}$ are bounded. Indeed, the only construction of an unbounded left-exact endofunctor of $\mathscr{S}$ that I know is the following, which requires a proper class of measurable cardinals.

Let M be the class of measurable cardinals, assumed to be unbounded. For each $\kappa \in M$, let D_κ be a nonprincipal κ-complete ultrafilter on κ. Let I be the class of finite subsets of M, directed by inclusion. For $i = \{\kappa_1, \cdots, \kappa_n\} \in I$, with $\kappa_1 < \cdots < \kappa_n$, let $F_i = D_{\kappa_1}\text{-prod} \cdots D_{\kappa_n}\text{-prod}$, and let $F = \varinjlim_{i \in I} F_i$. The transition maps $F_i \to F_j (i \subseteq j)$ of the direct system are obtained by composing the unique natural transformations $Id \to D_\lambda\text{-prod}$ with the various D_κ-prod's. For any set X, the D_κ-prod's for $\kappa > |X|$ have no effect on X, by Theorem 5; so $F(X)$ can be computed as the limit of the $F_i(X)$ where $i \in I$ and every $\kappa \in i$ is $\leqq |X|$. This shows that $F(X)$ is a set, so F is well-defined. It is clearly left-exact. But it is not bounded because it has all the D_κ-prod's as subfunctors. Thus, assuming sufficiently strong large cardinal axioms, one can obtain unbounded left-exact endofunctors of $\mathscr{S}$.

References

1. A. Blass, *Two closed categories of filters*, Fund. Math., to appear.

2. C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland (1973).

3. T. Frayne, A. Morel, and D. Scott, *Reduced direct products*, Fund. Math., **51** (1962), 195-228.

4. P. Freyd, *Aspects of Topoi*, Bull. Austral. Math. Soc., **7** (1972), 1-76.

5. A. Grothendieck, *Technique de descente et théorèmes d'existence en geometrie algébrique*, Seminaire Bourbaki, **12** (1959-60), exposé 195.

6. A. Joyal, *Functors which preserve elementary operations*, Notices Amer. Math. Soc., **18** (1971), 967.

7. H. J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc., **107** (1963), 383-408.
8. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math., **53** (1964), 225-308.
9. K. Prikry, *On a problem of Gillman and Keisler*, Ann. Math. Logic, **2** (1971), 179-188.
10. G. Reyes, *From sheaves to logic*, Studies in Algebraic Logic, ed. A. Daigneault, MAA Studies in Mathematics, 9, Mathematical Association of America, 1974.
11. H. Schubert, *Categories*, Springer-Verlag (1972).

Received June 3, 1975 and in revised form December 9, 1975.

THE UNIVERSITY OF MICHIGAN