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ON A CLASS OF CONTRACTIVE PERTURBATIONS OF RESTRICTED SHIFTS

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The Sz.-Nagy-Foiaş model theory uses generalized restricted shifts as canonical models for contractions in Hilbert space. This paper considers a class of contractive and unitary perturbations of a generalized restricted shift acting on a Sz.-Nagy-Foiaş space generated by an analytic operator-valued function $S(z)$ whose values are contractions on a separable Hilbert space. The spectra and characteristic functions of the perturbations are computed and related to the original operator. When the perturbation is unitary, a unitary equivalence to multiplication by $e^{i\theta}$ on $L^2(\mu)$, for an operator-valued measure μ, is given.

In [2], D. N. Clark studied the one-dimensional unitary perturbations of restricted shifts in H^2, i.e. $S(z)$ a scalar inner function, and in [3], he announced results for the case where $S(z)$ is an arbitrary scalar (characteristic) function. The general unitary perturbations are implicit in work of de Branges and Rovnyak [1], though in the context of the de Branges-Rovnyak model theory rather than the Sz.-Nagy-Foiaş. P. A. Fuhrmann [5] considered a class of completely nonunitary and unitary perturbations for the case of $S(z)$ an inner function on a finite-dimensional space. In this case, the maps considered are always compact perturbations. Here we generalize results of [5] and [2]. We will follow the general outline of [5], and we correct a minor error occurring there so our description of the perturbations in the general case is actually as sharp as in the finite-dimensional case. As was pointed out in [5], these perturbations have applications to the theory of stability of linear control systems.

1. **Preliminary results.** For notation, let C and C_* be separable Hilbert spaces, and let $L^2(C)$, $L^2(C_*)$, $H^2(C)$, $H^2(C_*)$ denote the standard vector-valued Lebesgue and Hardy spaces defined on the unit circle. (See [6] or [8] for general references.) We will use "t" to denote the argument of a function (vector or operator valued) defined on the unit circle, and for analytic functions, we will freely identify $h(t)$ on the circle with its extension to the disc, denoted $h(z)$. S will denote a fixed purely contractive analytic operator-valued function from C to C_*, i.e. $S(z)\colon C \to C_*$, $\|S(z)\| \leqq 1$ for all $|z| < 1$ and $\|S(0)c\| < \|c\|$ for all $c \in C$, and let $\Delta(t) = (I - S(t)^*S(t))^{1/2}$. (Note that this denotes the unique positive root of the positive operator.) Let $H =$

$H^2(C_*) \oplus \overline{\Delta L^2(C)}$, where the second summand denotes the $L^2(C)$ closure of $\{\Delta(t)g(t) \mid g \in L^2(C)\}$, and $M = \{(S(z)f(z), \Delta(t)f(t)) \mid f \in H^2(C)\} \subset H$. Then M is invariant under the (unilateral) shift U_+ in H defined by $U_+(f, g) = (zf(z), e^{it}g(t))$, so $K = H \ominus M$ is invariant under U_+^*, where U_+^* is the left-shift defined by $U_+^*(f, g) = (z^{-1}(f(z) - f(0)), e^{-it}g(t))$. We call K the Sz.-Nagy-Foiaş space generated by S. Let T denote the restricted right shift in K, i.e. the compression of U_+ to K. Thus, for $(f, g) \in K$, $T(f, g) = P(zf, e^{it}g)$, where P denotes projection onto K, and $T^* = U_+^*|_K$. Note that if S is inner, then $\Delta(t) = 0$ a.e. and $K = H^2(C) \ominus SH^2(C)$.

Let $\tilde{S}(z)$ be the analytic operator-valued function defined by $\tilde{S}(z) = S(\bar{z})^*$, i.e. $\tilde{S}(t) = S(-t)^*: C_* \to C$. Analogously to above, define $\tilde{\Delta}(t): C_* \to C_*$ by $\tilde{\Delta}(t) = (I - \tilde{S}(t)^*\tilde{S}(t))^{1/2}$, $\tilde{H} = H^2(C) \oplus \overline{\tilde{\Delta}L^2(C_*)}$, $\tilde{M} = \{(\tilde{S}f, \tilde{\Delta}f) \mid f \in H^2(C_*)\}$, $\tilde{K} = \tilde{H} \ominus \tilde{M}$, and $\tilde{T} = \tilde{P}\tilde{U}_+|_{\tilde{K}}$, where $\tilde{U}_+$ is the unilateral shift in $\tilde{H}$ and $\tilde{P}$ is projection onto $\tilde{K}$. Note that $\tilde{S}$ is inner if and only if S is inner. (We use "inner" in the sense of [6], i.e. $S(t): C \to C_*$ is unitary a.e.; in the terminology of [8], this is called "inner from both sides".) The following is an extension of [4, Theorem 2.1].

THEOREM 1.1. *The right shift T on K is unitarily equivalent to the left shift $\tilde{T}^*$ on $\tilde{K}$.*

Proof. Let $L = L^2(C_*) \oplus \overline{\Delta L^2(C)}$, $\tilde{L} = L^2(C) \oplus \overline{\tilde{\Delta}L^2(C_*)}$, and consider $\tau: L \to \tilde{L}$ defined by

$$\tau(f, \Delta g) = e^{-it}(S(-t)^*f(-t) + \Delta^2(-t)g(-t)\,,\quad \tilde{\Delta}(t)(f(-t) - S(-t)g(-t)))\,.$$

Then

$$\begin{aligned}\|\tau(f, \Delta g)\|_{\tilde{L}}^2 &= \|S(-t)^*f(-t) + \Delta^2(-t)g(-t)\|_{L^2(C)}^2 \\ &\quad + \|\tilde{\Delta}(t)(f(-t) - S(-t)g(-t))\|_{L^2(C_*)}^2 \\ &= \|f(-t)\|^2 + (\|g(-t)\|^2 - \|S(-t)^*S(-t)g(-t)\|^2) \\ &= \|f(t)\|_{L^2(C_*)}^2 + \|\Delta(t)g(t)\|_{L^2(C)}^2 = \|(f, \Delta g)\|_L^2\,,\end{aligned}$$

so τ extends to an isometry from L to $\tilde{L}$. For $f \in L^2(C)$, $g \in L^2(C_*)$, $(f, \Delta g) = \tau(\tau^*(f, \Delta g))$, where $\tau^*(f, \Delta g) = e^{-it}(S(t)f(-t) + \tilde{\Delta}^2(-t)g(-t)$, $\Delta(t)(f(-t) - S(t)^*g(-t)))$, so τ is unitary.

We can decompose $L = K \oplus M \oplus K^2(C_*)$, where

$$K^2(C_*) = \{(f, 0) \mid f \in L^2(C_*) \ominus H^2(C_*)\}\,,$$

and similarly $\tilde{L} = \tilde{K} \oplus \tilde{M} \oplus K^2(C)$. It is easy to see that $\tau(M) = K^2(C)$ and $\tau(K^2(C_*)) = \tilde{M}$, so therefore $\tau(K) = \tilde{K}$. Hence, $\tau P = \tilde{P}\tau$ (here we consider the domains of P and $\tilde{P}$ to be L and $\tilde{L}$ respectively), and $\tau U = \tilde{U}^*\tau$, where U and $\tilde{U}$ are the bilateral shifts on L and $\tilde{L}$. Thus, $\tau PU = \tilde{P}\tilde{U}^*\tau$, which implies $\tau T = \tilde{T}^*\tau$ on K. Therefore, $\tau|_K$, which we denote simply by τ, is a unitary map from K to $\tilde{K}$ satisfying the theorem.

It is now easy to derive an explicit formula for T will which be useful later on.

COROLLARY. *For* $(f, \Delta g) \in K$,

$$T(f, \Delta g) = (zf(z) - S(z)Q(0), e^{it}\Delta(t)g(t) - \Delta(t)Q(0))$$

where $Q(z)$ *is the first component of* $\tau(f, \Delta g)$.

Proof. This is obtained by computing

$$(\tau^* \tilde{T}^* \tau)(f, \Delta g) .$$

If $F = (f, g) \in K$ and $\tau(F) = (Q, h)$, denote by $(\tau_1 F)(z)$ the C-valued function $Q(z)$. We derive several technical lemmas needed later on.

LEMMA 1.2. *For* $|w| < 1$, $x \in C_*$, $y \in C$, *let*

$$k_{w,x,y} = \left(\frac{I - S(z)S(w)^*}{1 - z\bar{w}}x, \ -\frac{\Delta(t)S(w)^*}{1 - e^{it}\bar{w}}x\right) + \left(\frac{S(z) - S(\bar{w})}{z - \bar{w}}y, \ \frac{\Delta(t)}{e^{it} - \bar{w}}y\right)$$

Then $k_{w,x,y} \in K$ *and*

$$P((x/(1 - z\bar{w}), 0)) = k_{w,x,0} ,$$

$$P\left(\left(\frac{S(t)}{e^{it} - \bar{w}}y, \ \frac{\Delta(t)}{e^{it} - \bar{w}}y\right)\right) = k_{w,0,y} .$$

Proof. Note, for $(Sf, \Delta f) \in M$,

$$(k_{w,x,0}, (Sf, \Delta f)) = \left(\frac{I - S(z)S(w)^*}{1 - z\bar{w}}x, \ S(z)f(z)\right) - \left(\frac{\Delta(t)^2 S(w)^*}{1 - e^{it}\bar{w}}x, \ f(t)\right) = 0$$

and hence $k_{w,x,0} \in K$.

Similarly

$$(k_{w,0,y}, (Sf, \Delta f)) = \left(\frac{S(z) - S(\bar{w})}{z - \bar{w}}y, \ S(z)f(z)\right) + \left(\frac{\Delta(t)^2}{e^{it} - \bar{w}}y, \ f(t)\right)$$

$$= \left(\frac{S(t)^*S(\bar{w})}{e^{it} - \bar{w}}y,\ f(t)\right) + \left(\frac{1}{e^{it} - \bar{w}}y,\ f(t)\right)$$
$$= 0 + 0 = 0\,,$$

since $f(t) \in H^2(C)$. Hence $k_{w,0,y} \in K$.

Furthermore,

$$(x/(1 - z\bar{w}),\ 0) - k_{w,x,0}$$
$$= \left(S(z)\frac{S(w)^*}{1 - z\bar{w}}x,\ \Delta(t)\frac{S(w)^*}{1 - e^{it}\bar{w}}x\right) \in M$$

and

$$(S(t)y/(e^{it} - \bar{w}),\ \Delta(t)y/(e^{it} - \bar{w})) - k_{w,0,y}$$
$$= \left(\frac{S(\bar{w})}{e^{it} - \bar{w}}y,\ 0\right) \in (K \oplus M)^{\perp}\,.$$

All the assertions of the lemma follow.

LEMMA 1.3. *If* $(f, g) = F \in K$, *then*

(i) $(F, k_{w,x,0})_K = (f(w), x)_{C_*}$

(ii) $(F, k_{w,0,y})_K = ((\tau_1 F)(w), y)_C$.

In particular

(iii) *for* $x, y \in C_*$, ζ *and* η *in* D,

$$(k_{\zeta,x,0},\ k_{\eta,y,0})_K = \left(\frac{I - S(\eta)S(\zeta)^*}{1 - \eta\bar{\zeta}}x,\ y\right)_{C_*}$$

(iv) *for* $x, y \in C$,

$$(k_{\zeta,0,x},\ k_{\eta,0,y})_K = \left(\frac{I - S(\bar{\eta})^*S(\bar{\zeta})}{1 - \eta\bar{\zeta}}x,\ y\right)_C$$

and

(v) *for* $x \in C$, $y \in C_*$,

$$(k_{\zeta,0,x},\ k_{\eta,y,0})_K = \left(\frac{S(\eta) - S(\bar{\zeta})}{\eta - \bar{\zeta}}x,\ y\right)_{C_*}.$$

Proof. For $(f, g) \in K$, $x \in C_*$, we have

$$(f(w), x) = \left((f, g), \left(\frac{1}{1 - z\bar{w}}x,\ 0\right)\right)$$
$$= \left((f, g), P\left(\frac{1}{1 - z\bar{w}}x,\ 0\right)\right)$$
$$= ((f, g), k_{w,x,0})$$

by Lemma 1.2, proving (i).

For (ii) note that

$$(\tau_1 F)(t) = e^{-it}(S(-t)^* f(-t) + \Delta(-t)g(-t))$$

is in $H^2(C)$. Hence

$$\begin{aligned}((\tau_1 F)(w), y)_C &= \left(e^{-it}(S(-t)^* f(-t) + \Delta(-t)g(-t)), \frac{1}{1-e^{it}\bar{w}}y\right) \\ &= \left(S(t)^* f(t) + \Delta(t)g(t), \frac{1}{e^{it}-\bar{w}}y\right) \\ &= \left((f, g), \left(\frac{S(t)}{e^{it}-\bar{w}}y, \frac{\Delta(t)}{e^{it}-\bar{w}}y\right)\right) \\ &= \left((f, g), P\left(\frac{S(t)}{e^{it}-\bar{w}}y, \frac{\Delta(t)}{e^{it}-\bar{w}}\right)\right) \\ &= ((f, g), k_{w,0,y})\end{aligned}$$

by Lemma 1.2, and (ii) follows.

A straight-forward computation gives

$$(\tau_1 k_{w,x,0})(z) = \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} x$$

and

$$(\tau_1 k_{w,0,y})(z) = \frac{I - \tilde{S}(z)\tilde{S}(w)^*}{1 - z\bar{w}} y \, .$$

Hence (iii)-(v) follows from (i) and (ii).

We note that if $F = (f, g) \in K$ is orthogonal to $k_{w,x,y}$ for all $w \in D$, $x \in C_*$ and $y \in C$, then $f = 0$ and $\tau_1 F = 0$. From the formula for τ_1, it follows that also $g = 0$, and hence F is the zero element of K. This implies that $\{k_{w,x,y} \mid w \in D, x \in C_*, y \in C\}$ spans a dense subset of K. This fact will make the formulas (iii)-(v) useful for computations later on.

The next lemma follows from the corollary to Theorem 1.1 and direct computations.

LEMMA 1.4. (i) $Tk_{w,x,0} = \bar{w}^{-1}(k_{w,x,0} - k_{0,x,0})$, $w \neq 0$.
(ii) $Tk_{w,0,y} = \bar{w}k_{w,0,y} - k_{0,S(\bar{w})y,0}$.
(iii) $T^*k_{w,x,0} = \bar{w}k_{w,x,0} - k_{0,0,S(w)^*x}$
(iv) $T^*k_{w,0,y} = \bar{w}^{-1}(k_{w,0,y} - k_{0,0,y})$, $w \neq 0$.

We wish to distinguish two subspaces of K defined by

$$k_0 = \text{the closure of } \{k_{0,x,0} \mid x \in C_*\}$$
$$K_0 = \text{the closure of } \{k_{0,0,y} \mid y \in C\} \, .$$

Let us simplify the notation for this special case by writing d_x for $k_{0,x,0}$ and D_y for $k_{0,0,y}$.

LEMMA 1.5. *Let* $F = (f, g) \in K$. *Then*
(i) $T^*F = (z^{-1}f(z), e^{-it}g(t))$ *if and only if* $F \perp k_0$.
(ii) $TF = (zf(z), e^{it}g(t))$ *if and only if* $F \perp K_0$.

Proof. (i) holds if and only if $f(0) = 0$ which holds if and only if $F \perp k_0$ by Lemma 1.3 (i). By the corollary to Theorem 1.1, (ii) follows similarly, using Lemma 1.3 (ii).

LEMMA 1.6. *Let* P_{k_0} *and* P_{K_0} *denote the orthogonal projection onto* k_0 *and* K_0 *respectively. Then* $P_{k_0}F = d_x$, *where*

$$x = (I - S(0)S(0)^*)^{-1}f(0)$$

and $P_{K_0}F = D_y$, *where* $y = (I - S(0)^*S(0))^{-1}(\tau_1 F)(0)$. (Note since $S(z)$ is a pure contractive function, x and y are well-defined for F in some dense subset of K.)

Proof. The map e_1 densely defined by $e_1\colon x \longrightarrow d_{(I-S(0)S(0)^*)^{-1/2}x}$ is an isometry (using Lemma 1.3iii) of C_* into K with range equal to k_0, and with adjoint given by $e_1^*\colon f \longrightarrow (I - S(0)S(0)^*)^{-1/2}f(0)$ (using Lemma 1.3i). The formula for P_{k_0} follows by computing $e_1e_1^*$. The formula for P_{K_0} follows similarly.

2. The perturbations.

DEFINITION 2.1. Let $A\colon C \longrightarrow C_*$ be a bounded linear map. We define $Z(A)$ to be the unique bounded linear map on K such that

$$Z(A)F = \begin{cases} TF & \text{if} \quad F \perp K_0 \\ d_{Ay} & \text{if} \quad F = D_y \,. \end{cases}$$

REMARK 2.2. It follows that $Z(A)^*$ is given by

$$Z(A)^*F = \begin{cases} T^*F \text{ if } F \perp k_0 \\ D_y \text{ if } F = d_x, \text{ where } y \\ \qquad\qquad = (I - S(0)^*S(0))^{-1}A^*(I - S(0)S(0)^*)x \,. \end{cases}$$

We note that $T = Z(-S(0))$ (by Lemma 1.3), and that $Z(A)^*d_x = D_{A^*x}$ if and only if

$$AS(0)^*S(0) = S(0)S(0)^*A \,. \tag{1}$$

THEOREM 2.3. (i) $Z(A)$ *is a contraction if and only if*

$$A^*(I - S(0)^*S(0))A \leqq (I - S(0)S(0)^*) . \tag{2}$$

(ii) *$Z(A)$ is unitary if and only if $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$ for some unitary V.*

(iii) *If A satisfies condition* (1), *then $Z(A)$ is a contraction if and only if $\|A\| \leqq 1$ and $Z(A)$ is unitary if and only if A is unitary.*

Proof. (i) Since $Z(A)$ maps $K_0^\perp$ isometrically onto $k_0^\perp$ and sends K_0 onto k_0, $Z(A)$ is a contraction if and only if it is contractive on K_0. By Lemma 1.3, this holds precisely when $\|Ay\|^2 - \|S(0)^*Ay\|^2 \leqq \|y\|^2 - \|S(0)y\|^2$ for all $y \in C$, but this is clearly equivalent to (2).

(ii) As above, $Z(A)$ is isometric precisely when equality holds in (2). By [5, Theorem 1.7(i)], this holds if and only if $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$ for some isometry V. By Lemma 1.3, $Z(A)^*$ is isometric if and only if $(I-S(0)S(0)^*)=(I-S(0)S(0)^*)^{1/2}VV^*(I-S(0)S(0)^*)^{1/2}$, which holds if and only if $VV^* = I$, so V must be unitary.

(iii) If (1) holds, then (2) reduces to $(A^*A)(I - S(0)^*S(0)) \leqq (I - S(0)^*S(0))$, which, using (1) again, holds if and only if $A^*A \leqq I$, i.e. $\|A\| \leqq 1$. In the second case, (1) implies that $A = V$.

REMARK 2.4. In [5], it was claimed that (1) was a necessary condition for $Z(A)$ to be a contraction. Clearly, if A is bounded, then $Z(\lambda A)$ will be a contraction for all sufficiently small scalars λ. There is also an error in Theorem 1.7(iii) in [5], which states that if P and Q are unitarily equivalent strictly positive operators and X is a solution of $P \geqq X^*QX$ and $Q \geqq XPX^*$, then X is a contraction such that $XP = QX$. The matrices

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

provide a counterexample.

3. Characteristic functions and spectra. The Sz.-Nagy-Foiaş model theory for contractions assigns to each contraction T on a Hilbert space H the triple $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T(\lambda)\}$ where $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$, $\mathscr{D}_T = \overline{D_T H}$, $\mathscr{D}_{T^*} = \overline{D_{T^*}H}$, and $\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1}D_T]|_{\mathscr{D}_T}$ is an analytic operator-valued function whose values are contractions from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$, the defect spaces of T. (This holds since $TD_T = D_{T^*}T$.) We call this triple the characteristic function of T, and if T is completely nonunitary (c.n.u.), i.e. there is no reducing subspace on which T is unitary, then T is unitarily equivalent to the adjoint of the restricted shift on the Sz.-Nagy-Foiaş space generated by its characteristic function [8, p. 248]. In most cases, one is unable to get any "concrete" information from this representation for a specific operator because of computational difficulties involved in simplifying the form of the characteristic function. However, if A satisfies (1), then we can apply Fuhrmann's proof [5, p. 169–172] verbatim to get the following two theorems.

THEOREM 3.1. *If A is a strict contraction satisfying* (1), *then $Z(A)$ is a c.n.u. contraction on K with characteristic function $\{K_0, k_0, \Theta_{Z(A)}(z)\}$, where $\Theta_{Z(A)}(z)$ is given by $\Theta_{Z(A)}(z)D_y = d_{G(z)y}$ where*

$$(I - S(0)S(0)^*)^{1/2}G(z)(I - S(0)^*S(0))^{-1/2}$$
$$= (I - AA^*)^{1/2}(I - \Gamma(z)A^*)^{-1}(\Gamma(z) - A)(I - A^*A)^{-1/2}$$

and

$$\Gamma(z) = (I - S(0)S(0^*))^{1/2}(I - S(z)S(0)^*)^{-1}(S(z) - S(0))(I - S(0)^*S(0))^{-1/2} .$$

Note that the above are matrix fractional linear transformations.

We call an open arc γ of the unit circle regular for $S(z)$ if $S(z)$ has analytic continuation over γ and for all $\lambda \in \gamma$, $S(\lambda)$ is unitary. Let $\sigma(T)$ and $\sigma(Z(A))$ denote the spectrum of T and $Z(A)$ respectively. Recall [8, Theorem VI, 4.1] that $\sigma(T) = \{|z| < 1 | S(z)$ is not boundedly invertible$\} \cup \{|\lambda| = 1 | \lambda$ lies on no regular arc of $S\}$.

THEOREM 3.2. *Under the assumptions of* 3.1, (i) $\sigma(Z(A)) = \{|\lambda| = 1 | \lambda$ lies on no regular arc of $S\} \cup \{|z| < 1 | (\Gamma(z) - A)$ is not boundedly invertible$\}$.

(ii) *$Z(A)^n$ and $Z(A)^{*n}$ both converge to zero in the strong operator topology if and only if $S(Z)$ is inner.*

REMARK 3.3. (a) If A is a strict contraction not satisfying (1), $\Theta_{Z(A)}(z)$ as defined above still has an interpretation as a characteristic function. (b) Note that $W \equiv Z(0)$ is the completely nonunitary partial isometry with initial space K_0 and final space k_0 and agreeing with T on K_0 With this choice of A, (1) is satisfied and Theorem 3.1 says $\Theta_{Z(0)}(z)D_y = d_{(I-S(0)S(0)^*)^{-1/2}\Gamma(z)(I-S(0)^*S(0))^{1/2}y}$. It is not difficult to see that $\Theta_{Z(0)}(z)$ coincides (see [8] page 192 for definition) with $\Gamma(z)$: $C \rightarrow C_*$. Hence the partial isometry W can be represented as the compressed right shift T' on the Sz.-Nagy-Foiaş space K' associated with $\Gamma(z)$ rather than with $S(z)$. (c) For A a contraction from C to C_*, let $Z'(A)$ be the associated perturbation of T' in K'. Since $\Gamma(0) = 0$, (1) is satisfied for any A. In particular, for $A = -S(0)$, Theorem 3.1 gives $\Theta_{Z'(-S(0))}(z)D'_y = d'_{S(z)y}$, and hence $\Theta_{Z'(-S(0))}(z)$ coincides with $S(z)$. Hence the operator T on K is unitarily equivalent to

$Z'(-S(0))$ on K'. It is then clear that the formula above for $\Theta_{Z(A)}$ (A not necessarily satisfying (1)), interpreted for D_y and d_z in K', gives the characteristic function for $Z'(A)$. In this sense, it is perhaps more natural to study perturbations of $Z'(-S(0))$ on K' rather than of T on K. It is now seen that (1) is the condition that $Z(A)$ and $Z'(A)$ be unitarily equivalent.

4. Unitary perturbations. Since the characteristic function of a unitary map is zero, the above method fails totally when $Z(A)$ is unitary. However, when A satisfies (1) we can still get spectral information about $Z(A)$ by adapting techniques of D. N. Clark [2] to a more general setting. We begin with two technical lemmas.

LEMMA 4.1. *If A is unitary and satisfies* (1), *then* $\alpha = \alpha_A = -(I + S(0)A^*)(S(0)^* + A^*)^{-1}$ *is unitary from C to C_*.*

Proof. α is a priori defined on some dense set $D_1 \subset C$ since $S(0)^*$ is a pure contraction and A^* is unitary. (1) implies that

$$(I + AS(0)^*)(I + S(0)A^*)(S(0)^* + A^*)^{-1} = (S(0) + A) .$$

Thus, for $x \in D_1$, $(\alpha x, \alpha x) = ((S(0)^* + A^*)^{-1}x, (S(0) + A)x) = (x, x)$, so α can be extended to an isometry on C. Similarly, α^* is an isometry on C_* so α is unitary.

Note that in fact, (1) is also necessary for α to be unitary, and α_A determines A by $A^* = -(\alpha + S(0))^{-1}(\alpha S(0)^* + I)$. Also, we have $(\alpha + S(0))^{-1} = -(S(0)^* + A^*)(I - S(0)S(0)^*)^{-1}$; our α corresponds to $-\alpha$ used in [2].

LEMMA 4.2. *For* $F = (f, g) \in K$,
(i) $(Z(A) - T)(F) = k_{0,x,0}$ *where* $x = -(\alpha^* + S(0)^*)^{-1}(\tau_1 F)(0)$
(ii) $(Z(A)^* - T)(F) = k_{0,0,y}$ *where* $y = -(\alpha + S(0))^{-1}f(0)$

Proof. Since $Z(A) = T(I - P_{K_0}) + Z(A)P_{K_0}$, we obtain

$$\begin{aligned}(Z(A) - T)(F) &= d_{(S(0)+A)(I-S(0)^*S(0))^{-1}(\tau_1 F)(0)} \\ &= d_x ,\end{aligned}$$

with x as in (i).

(ii) follows similarly.

For $|z| < 1$, define $\varphi(z): C_* \to C_*$ by $\varphi(z) = (I - S(z)\alpha^*)(I + S(z)\alpha^*)^{-1}$. Then straight-forward calculation gives

$$(3) \quad \varphi(\zeta) + \varphi(\eta)^* = 2(I + S(\zeta)\alpha^*)^{-1}(I - S(\zeta)S(\eta)^*)(I + \alpha S(\eta)^*)^{-1}$$

and hence (let $z = \zeta = \eta$) $\varphi(z)$ has nonnegative real part for $|z| < 1$. By the operator-valued version of the Herglotz theorem, there exists a non-negative operator-valued measure μ on $[0, 2\pi]$ such that $\varphi(z) = \int_0^{2\pi}(e^{i\theta}+z)(e^{i\theta}-z)^{-1}d\mu(\theta)$.

Thus

$$\varphi(\zeta)+\varphi(\eta)^* = 2\int(1-\zeta\bar{\eta})(1-e^{-i\theta}\zeta)^{-1}(1-e^{i\theta}\eta)^{-1}d\mu(\theta)\,. \tag{4}$$

Comparing (3) and (4) yields

$$\frac{I-S(\zeta)S(\eta)^*}{1-\zeta\bar{\eta}} = \int\frac{I+S(\zeta)\alpha^*}{1-e^{-i\theta}\zeta}d\mu(\theta)\frac{I+\alpha S(\eta)^*}{1-e^{i\theta}\bar{\eta}}\,. \tag{5}$$

Similar computations give

$$\begin{aligned}\frac{S(\zeta)-S(\bar{\eta})}{\zeta-\bar{\eta}} &= -\frac{1}{2}(I+S(\zeta)\alpha^*)\Big(\frac{\varphi(\zeta)-\varphi(\bar{\eta})}{\zeta-\bar{\eta}}\Big)(I+S(\bar{\eta})\alpha^*)\alpha\\ &= -\int\frac{I+S(\zeta)\alpha^*}{1-e^{-i\theta}\zeta}d\mu(\theta)\frac{I+S(\bar{\eta})\alpha^*}{e^{i\theta}-\bar{\eta}}\alpha\end{aligned} \tag{6}$$

and

$$\frac{I-\tilde{S}(\zeta)\tilde{S}(\eta)^*}{1-\zeta\bar{\eta}} = \int_0^{2\pi}\alpha^*\frac{I+\alpha S(\bar{\zeta})^*}{e^{-i\theta}-\zeta}d\mu(\theta)\frac{I+S(\bar{\eta})\alpha^*}{e^{i\theta}-\bar{\eta}}\alpha\,. \tag{7}$$

We define Hilbert space $L^2(\mu)$ as in Shulman [7]. For $f = x_1\chi_{E_1}+\cdots+x_n\chi_{E_n}$ a simple C_*-valued function, where $\chi_{E_1},\cdots,\chi_{E_n}$ are characteristic functions of disjoint Borel sets and $x_1,\cdots,x_n$ are corresponding elements of C_* define

$$\|f\|_\mu^2 = \int(d\mu(t)f(t), f(t)) = (\mu(E_1)x_1, x_1)+\cdots+(\mu(E_n)x_n, x_n)\,.$$

This does not depend on the representation of $f(t)$ in terms of characteristic functions. Let $\mathscr{A} = \Big\{f(t)\colon[0, 2\pi]\to C_* \mid f$ is Borel measurable, $\int\|f(t)\|^2 d(u(t)x, x) < \infty$ for all $x\in C_*$, the range of $f(t)$ is contained in a finite dimensional subspace of $C_*\Big\}$. For $f\in\mathscr{A}$ let $e_1, e_2, \cdots, e_k$ be a basis for the smallest subspace which contains the range of $f(t)$, and define

$$\alpha(f, t) = (\mu(t)e_1, e_1)+\cdots+(\mu(t)e_k, e_k)\,.$$

The definition is independent of the choice of basis for this subspace, and $\|f\|_\mu^2 \leqq \int\|f(t)\|^2 d\alpha(f, t)$ whenever f is a simple function. For $f\in\mathscr{A}$, there is a sequence of simple functions $\{f_n(t)\}$ such that the range of $f_n(t)$ is contained in the range of $f(t)$ for $n = 1, 2, \cdots$, and such that $\int\|f_n(t)-f(t)\|^2 d\alpha(f, t)\to 0$ as $n\to\infty$. We can define $\|f(t)\|_\mu^2$ unambiguously as

$$\|f\|_\mu^2 = \lim_{n\to\infty} \|f_n\|_\mu^2 .$$

By $L^2(\mu)$ is meant the Hilbert space completion of the inner product space of equivalence classes of functions with finite-dimensional range in μ-norm. The definition of $L^2(\mu)$ is such that explicit formulas can be written only for an element associated with the equivalence class of an element of $\mathscr{A}$. This, however, causes no difficulties for our purposes. It is clear, for example, that the transformation $h(t) \to e^{it}h(t)$ is unitary in $L^2(\mu)$, with spectrum equal to supp (μ) (the complement of the largest open set on which μ is zero).

We are now in position to define a unitary transformation of K onto $L^2(\mu)$ which transforms the operator $Z(A)$ on K to the operator of multiplication on e^{it} on $L^2(\mu)$.

THEOREM 4.3. *Define V on elements in K of the form $k_{\zeta,x,y}$ by*

$$V(k_{\zeta,x,y}) = \frac{I + \alpha S(\zeta)^*}{1 - e^{it}\bar{\zeta}} x - \frac{I + S(\bar{\zeta})\alpha^*}{e^{it} - \bar{\zeta}} \alpha y .$$

Then V is well-defined and extends uniquely to a unitary transformation (also V) *of K onto $L^2(\mu)$ such that $VZ(A) = e^{it}V$.*

Proof. We first check that V is an isometry on those vectors where it is defined. Note, for $x, y \in C_*$,

$$\begin{aligned}
(k_{\eta,y,0}, k_{\zeta,x,0})_K &= \left(\frac{I - S(\zeta)S(\eta)^*}{1 - \bar{\eta}\zeta} y, x\right)_{C_*} \\
&= \left(\int \frac{I + S(\zeta)\alpha^*}{1 - e^{-it}\zeta} d\mu(t) \frac{I + \alpha S(\eta)^*}{1 - e^{it}\eta} y, x\right)_{C_*} \text{ by (5)} \\
&= \left(\frac{I + \alpha S(\eta)^*}{1 - e^{it}\bar{\eta}} y, \frac{I + \alpha S(\zeta)^*}{1 - e^{it}\bar{\zeta}} x\right)_{L^2(\mu)} \\
&=: (Vk_{\eta,y,0}, Vk_{\zeta,x,0})_{L^2(\mu)} .
\end{aligned}$$

Also, for $x, y \in C$,

$$\begin{aligned}
(k_{\eta,0,y}, k_{\zeta,0,x})_K &= \left(\frac{I - S(\bar{\zeta})^* S(\bar{\eta})}{1 - \bar{\eta}\zeta} y, x\right)_C \\
&= \left(\int \alpha^* \frac{I + \alpha S(\bar{\zeta})^*}{e^{-it} - \zeta} d\mu(t) \frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}} \alpha y, x\right)_C \\
&= \left(\frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}} \alpha y, \frac{I + S(\bar{\zeta})\alpha^*}{e^{it} - \bar{\zeta}} \alpha x\right)_{L^2(t)} \\
&= (Vk_{\eta,0,y}, Vk_{\zeta,0,x})_{L^2(\mu)}
\end{aligned}$$

and finally, for $x \in C_*$ and $y \in C$,

$$
\begin{aligned}
(k_{\eta,0,y},\ k_{\zeta,x,0})_K &= \left(\frac{S(\zeta) - S(\bar{\eta})}{\zeta - \bar{\eta}} y,\ x\right)_{C_*} \\
&= \left(-\int \frac{I + S(\zeta)\alpha^*}{1 - e^{-it}\zeta} d\mu(t) \frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}} \alpha y,\ x\right)_{C_*} \text{ by (6)} \\
&= (Vk_{\eta,0,y},\ Vk_{\zeta,x,0})_{L^2(\mu)} .
\end{aligned}
$$

Hence V is isometric (and hence also well-defined) on its domain. Since elements of the form $k_{\eta,x,y}$ span a dense set in K, V extends by linearity and continuity to be an isometry of K into $L^2(\mu)$. Since the range of V contains all elements of the form $x/(1 - e^{it}\bar{w})$ and $x/(e^{it} - \bar{w})$ for $x \in C_*$ and $|w| < 1$, it follows that V is onto $L^2(\mu)$.

It remains to show $VZ(A) = e^{it}V$. By Lemmas 1.4 and 4.2,

$$
\begin{aligned}
Z(A)(k_{w,x,0}) &= \bar{w}^{-1}k_{w,x,0} - \bar{w}^{-1}k_{0,x,0} \\
&\quad + \bar{w}^{-1}k_{0,(\alpha^*+S(0)^*)^{-1}(S(0)^*-S(w)^*)x,0} \\
&= \bar{w}^{-1}(k_{w,x,0} - k_{0,(\alpha^*+S(0)^*)^{-1}(\alpha^*+S(w))x,0})
\end{aligned}
$$

and hence

$$
\begin{aligned}
VZ(A)k_{w,x,0} &= \bar{w}^{-1}(1 - e^{it}\bar{w})^{-1}(I + \alpha S(w)^*)x - \bar{w}^{-1}(I + \alpha S(w)^*)x \\
&= \bar{w}^{-1}[(1 - e^{it}\bar{w})^{-1} - 1](I + \alpha S(w)^*)x \\
&= e^{it}\frac{I + \alpha S(w)^*}{1 - e^{it}\bar{w}} x = e^{it}Vk_{w,x,0} .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
Z(A)k_{w,0,y} &= \bar{w}k_{w,0,y} - k_{0,S(\bar{w})y,0} \\
&\quad - k_{0,(\alpha^*+S(0)^*)^{-1}(I-S(0)^*S(\bar{w}))y,0} \\
&= \bar{w}k_{w,0,y} - k_{0,(\alpha^*+S(0)^*)^{-1}(I+\alpha^*S(\bar{w}))y,0} .
\end{aligned}
$$

So

$$
\begin{aligned}
VZ(A)k_{w,0,y} &= -\bar{w}(e^{it} - \bar{w})^{-1}(I + S(\bar{w})\alpha^*)\alpha y - (I + S(\bar{w})\alpha^*)\alpha y \\
&= -e^{it}\frac{I + S(\bar{w})\alpha^*}{e^{it} - \bar{w}}\alpha y \\
&= e^{it}Vk_{w,0,y} .
\end{aligned}
$$

The theorem follows.

We note the following inversion formula for V.

THEOREM 4.4. *Let* $V^*: L^2 \to K$ *be defined, for* F *in* $\mathscr{A}$, *by* $V^*F = (W_1F, W_2F)$ *where* $(W_1F)(z) = (I + S(z)\alpha^*)\int(1 - e^{-it}z)d\mu(t)F(t)$ *and* $(W_2F)(t) = \lim_{r\to 1}(I - S(re^{it})^*S(re^{it}))^{-1/2}$.

$$(\alpha^* + S(re^{it}))^* \int (e^{i(t-\theta)} - r)^{-1} d\mu(\theta)F(\theta)$$
$$- \int (S(re^{it})^* - S(re^{it})^* S(re^{it})\alpha^*)(1 - re^{i(t-\theta)})^{-1} d\mu(\theta)F(\theta) .$$

Then V^* is the adjoint of V defined in Theorem 4.3.

Proof. To obtain W_1, rewrite equation (5) substituting z for ζ and noting that

$$Vk_{\eta,x,0} = \frac{I + \alpha S(\eta)^*}{1 - e^{it}\bar{\eta}} x \text{ to obtain}$$
$$\frac{I - S(z)S(\eta)^*}{1 - \zeta\bar{\eta}} x = \int \frac{I + S(z)\alpha^*}{1 - e^{-it}z} d\mu(t)(Vk_{\eta,x,0})(t) .$$

Similarly, using equation (6),

$$\frac{S(z) - S(\bar{\eta})}{z - \bar{\eta}} y = \int \frac{I + S(z)\alpha^*}{1 - e^{-it}z} d\mu(t)(Vk_{\eta,0,y})(t) .$$

This proves the correctness of the formula for W_1 for all F of the form $Vk_{\eta,x,y}$, and hence by approximation for all $F \in \mathscr{K}$. To obtain the formula for W_2, we first find a formula for $(\tau_1 V^* F)(z)$. By an argument dual to that above, we find

$$(\tau_1 V^* F)(z) = -\alpha^*(I + \alpha S(\bar{z})^*) \int (e^{-it} - z)^{-1} d\mu(t)F(t) .$$

The formula for W_2 is then obtained by using the explicit formulas for τ and τ^* in Theorem 1.1.

THEOREM 4.5. *Let A be unitary and satisfy* (1). *Then* $\sigma(Z(A)) = \{|\lambda| = 1 | \lambda$ lies on no regular arc of $S\} \cup \{|\lambda| = 1 | \lambda$ lies on a regular arc of S but $(I + S(\lambda)\alpha^*)$ is not boundedly invertible$\}$.

Proof. Since $Z(A)$ has a representation as multiplication by $e^{i\theta}$ on $L^2(\mu)$, we have $\sigma(Z(A)) = \operatorname{supp}(\mu)$, the complement of the largest open set on which μ is zero. By the integral representation of φ, we see that the complement of $\operatorname{supp}(\mu)$ is the set of λ at which $\varphi(z)$ has analytic continuation with $\operatorname{Re}\varphi(\lambda) = 0$. Since $\varphi(z) = (I - S(z)\alpha^*)(I + S(z)\alpha^*)^{-1}$, we have $(I + \varphi(z)) = 2(I + S(z)\alpha^*)^{-1}$ and $S(z) = (I - \varphi(z))(I + \varphi(z))^{-1}\alpha$.

Now, suppose $\varphi(z)$ has continuation at λ and $\operatorname{Re}\varphi(\lambda) = 0$. Then $(I + \varphi(\lambda))$ is boundedly invertible, and hence $(I + \varphi(z))^{-1}$ extends to an analytic function in a neighborhood of λ. Thus, $S(z)$ has analytic continuation at λ and $(I + S(\lambda)\alpha^*)$ is boundedly invertible; since $\operatorname{Re}\varphi(\lambda) = 0$, $S(\lambda)$ is unitary. Conversely, suppose $S(z)$ has analytic continuation at λ, $(I + S(\lambda)\alpha^*)$ is boundedly invertible, and $S(\lambda)$ is unitary. Then $(I + S(z)\alpha^*)^{-1}$ is analytic in some neighborhood of λ, so $\varphi(z)$ has analytic continuation at λ; since $S(\lambda)$ is unitary, $\operatorname{Re}\varphi(\lambda) = 0$. By taking complements, the theorem now follows.

Since $(I + S(\lambda)\alpha^*) = [(I + S(0)A^*) - S(\lambda)(S(0)^* + A^*)](I + S(0)A^*)^{-1}$, we see that $(I + S(\lambda)\alpha^*)$ is boundedly invertible if and only if $B(\lambda) = -[(I + S(0)A^*) - S(\lambda)(S(0)^* + A^*)]$ is boundedly invertible. With Γ as in Theorem 3.1, we have, since A satisfies (1), $(\Gamma(\lambda) - A) = (I - S(0)S(0)^*)^{1/2}(I - S(\lambda)S(0)^*)^{-1}B(\lambda)A(I - S(0)^*S(0))^{-1/2}$. Thus, $(\Gamma(\lambda) - A)$ is invertible, but not necessarily boundedly, if and only if $B(\lambda)$ is invertible. Since boundedness follows immediately in the finite-dimensional case, we have the following generalization of [5, Theorem 3.6] to the case of general analytic contractions $S(z)$.

COROLLARY 4.6. *If A is unitary on C, C finite-dimensional, and A satisfies* (1), *then* $\sigma(Z(A)) = \{|\lambda| = 1 \,|\, \lambda$ lies on no regular arc of $S\} \cup \{|\lambda| = 1 \,|\, \lambda$ lies on a regular arc for S but $(\Gamma(\lambda) - A)$ is not invertible$\}$.

In the finite-dimensional case, $Z(A)$ is a compact perturbation of T. Hence by the known spectral behavior of T and Weyl's theorem, $\{|\lambda| = 1 \,|\, \lambda$ lies on a regular arc for S but $\Gamma(\lambda) - A$ is not invertible$\}$ must be eigenvalues for $Z(A)$.

We can also adapt Fuhrmann's calculations [5, page 174] to determine eigenvalues in our more general setting.

THEOREM 4.7. *If A is unitary and satisfies* (1), *and λ lies on a regular arc for S, then λ is an eigenvalue for $Z(A)$ if and only if the range of $\Gamma(\lambda) - A$ is not dense in C_*.*

REMARK 4.8 If A does not satisfy (1), all of the above results apply to $Z'(A)$, as in Remark 3.3. Also, we still have from Theorem 2.3 that $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$ for some unitary V. This implies that $\tilde{\alpha}_A = \tilde{\alpha} = (A^* + S(0)^*)^{-1}(I + A^*S(0))$ is unitary. (Note that if A satisfies (1), then $\tilde{\alpha} = \alpha$ used above.) In this case, the results of §4 still hold with $\tilde{\alpha}$ in place of α.

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