PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No. 1, 1976

NORMALITY AND THE WEAK *cb* PROPERTY

K. HARDY AND I. JUHÁSZ

It is demonstrated that the Alexandroff duplicate of a Dowker space is again a Dowker space which is not weak *cb*, while the existence of weak *cb* Dowker spaces is made manifest.

A nonmetrizable, first countable compact space was created by Alexandroff in **[1]** and the construction has been subsequently generalized and employed (**[2]**, **[3]**, **[7]**, **[8]**). The present note concentrates on some properties of the Alexandroff duplicate $A(X)$ which, in particular, show that a (collectionwise) normal space need not have the weak *cb* property, thus resolving the open question in **[11**, p. 240**]**.

1. Preliminaries. No separation axioms are implicitly assumed for the topological space X. The Hewitt–Nachbin realcompactification of a Tychonoff space X is denoted by υX. We will write $A_n \searrow \emptyset$ to indicate that (A_n) is a decreasing sequence of subsets of X such that $\bigcap_n A_n = \emptyset$. **N** denotes the natural numbers. A set A is regular closed if $A = \mathrm{cl}_X \,\mathrm{int}_X A$, and ∂A denotes the boundary of A.

PROPOSITION 1.1. (**[10]**, **[11]**) *A space X is cb (weak cb) if and only if for each sequence $A_n \searrow \emptyset$ of closed (regular closed) subsets of X, there exists a sequence of zero sets (Z_n) with $A_n \subseteq Z_n$ and $\bigcap_n Z_n = \emptyset$.*

cb-spaces originated in **[5]** and were studied by Mack in **[10]**. Every normal, countably paracompact space is *cb* and every *cb*-space is countably paracompact. Weak *cb*-spaces were defined in **[11]**. They form a natural generalization of *cb*-spaces and include the Tychonoff pseudocompact spaces and all extremally disconnected spaces. Interest in weak *cb*-spaces is centered in the theorem (**[11]**) that for a Tychonoff space X, the Dedekind completion of $C(X)$ is isomorphic to $C(Y)$, for some space Y, if and only if υX is weak *cb*. It should be noted that if X is Tychonoff and weak *cb*, then any space T with $X \subseteq T \subseteq \upsilon X$ is weak *cb*. The converse fails in general (see **[4]**) but the following result is evident and will be needed below.

PROPOSITION 1.2. *Let X be Tychonoff and consider the statements*:

(a) *For any sequence $A_n \searrow \emptyset$ of regular closed sets in X we have $\bigcap_n \mathrm{cl}_{\upsilon X} A_n = \emptyset$.*

(b) *For any decreasing sequence* (A_n) *of regular closed sets in* X *we have* $\bigcap_n \mathrm{cl}_{\upsilon X} A_n = \mathrm{cl}_{\upsilon X} \bigcap_n A_n$.

(c) *If* υX *is weak cb, then any space* T, *with* $X \subseteq T \subseteq \upsilon X$ *is weak cb. Then* (a) *if and only if* (b); *and* (a) *or* (b) *implies* (c).

Proof. We merely recall that if X is dense in T and A is a regular closed subset of X then $\mathrm{cl}_T A = B$ is the unique regular closed subset of T with $A = B \cap X$.

According to a result of Ishikawa **[6]**, a space X is countably paracompact if and only if for each sequence $A_n \searrow \varnothing$ of closed subsets of X, there exists a sequence (G_n) of open sets such that $A_n \subseteq G_n$ and $\bigcap_n \mathrm{cl}_X G_n = \varnothing$. The following observation will be useful below and may have independent interest.

PROPOSITION 1.3. *The following statements are equivalent*:

(a) *X is countable paracompact.*

(b) *For each sequence* $F_n \searrow \varnothing$ *of closed nowhere dense subsets of* X, *there exists a sequence* (G_n) *of open sets such that* $F_n \subseteq G_n$ *and* $\bigcap_n \mathrm{cl}_X G_n = \varnothing$.

(c) *Each countable increasing cover* (**[10]**) *by dense open sets has a countable closed refinement whose interiors cover* X.

Proof. It is enough to show (b) implies (a). Let $A_n \searrow \varnothing$ be an arbitrary sequence of closed sets and define a sequence of open sets (G_n) with $A_n \subseteq G_n$ and $\bigcap_n \mathrm{cl}_X G_n = \varnothing$ in the following manner:

(i) If $\mathrm{int}_X A_m = \varnothing$ for some $m \geqq 1$, there exist open sets G_k with $A_k \subseteq G_k$, $k \geqq m$ and $\bigcap_k \mathrm{cl}_X G_k = \varnothing$; put $G_n = X$ for $1 \leqq n < m$.

Now assume that $\mathrm{int}_X A_n \neq \varnothing$ for all n.

(ii) If a subsequence (A_{n_k}) exists with $A_{n_{k+1}} \subseteq \mathrm{int}_X A_{n_k}$, let $G_{n_{k+1}} = \mathrm{int}_X A_{n_k}$ and $G_n = X$ otherwise.

(iii) If there exists $m \geqq 1$ such that $F_k = \partial A_k \cap \partial A_{k+1} \neq \varnothing$ for $k \geqq m$ then $F_k \searrow \varnothing$ is a sequence of closed nowhere dense sets and there exists a sequence of open sets (U_k) with $F_k \subseteq U_k$ and $\bigcap_k \mathrm{cl}_X U_k = \varnothing$. Define $G_{k+1} = \mathrm{int}_X A_k \cup U_k$ for $k \geqq m$ and $G_n = X$ for $1 \leqq n \leqq m$.

In order to exploit the use of nowhere dense closed subsets, we venture to make the following:

DEFINITION 1.4. *X is an nd-space if for each sequence* $F_n \searrow \varnothing$ *of closed nowhere dense sets, there exists a sequence of zero sets* (Z_n) *with* $F_n \subseteq Z_n$ *and* $\bigcap_n Z_n = \varnothing$.

Every *cb*-space is an *nd*-space. Since every zero set Z is a regular G_δ-set (a countable intersection of closed sets whose interiors contain Z), we may adapt the proof of Proposition 1.3 to conclude that every *nd*-space is countably paracompact. A space is *cb* if and only if it is both a weak *cb* and an *nd*-space. The example on p. 240 of **[11]** is countably paracompact but not an *nd*-space. It is conjectured that an *nd*-space need not be *cb*, although an example at the present time is not forthcoming.

2. Properties of $A(X)$. Recall the construction in **[7]**. Given an arbitrary topological space X, consider the set $A(X) = X \cup X'$, where X' is a disjoint copy of X. For any $x \in X$, let x' denote the corresponding point of X' and if $S \subseteq X$ define $S' = \{x' \mid x \in S\}$. A topology is introduced to $A(X)$ by defining a base $\{B(z) \mid z \in A(X)\}$ as follows:

$$B(x') = \{\{x'\}\} \quad \text{and} \quad B(x) = \{V \cup (V' \setminus \{x'\}) \mid V \in \mathcal{V}(x)\},$$

where $\mathcal{V}(x)$ is a neighbourhood base of x in X. The resulting space, also denoted by $A(X)$, generalizes the original construction in **[1]** and is called the Alexandroff duplicate of X. It is clear that X is a closed, C-embedded subspace of $A(X)$.

Many properties of X are shared with $A(X)$. It has been noticed that $A(X)$ is compact (**[2]**), α-compact (for any infinite cardinal α), realcompact and Tychonoff (**[7]**), if X has the corresponding property. We will now expand this list of properties.

Observe that a space is normal if and only if each pair of disjoint closed nowhere dense sets can be separated by disjoint open neighbourhoods.

PROPOSITION 2.1. *X is normal if and only if $A(X)$ is normal.*

Proof. Let A and B be disjoint closed nowhere dense subsets of $A(X)$. Then A and B are closed and disjoint in X and can be separated by disjoint open sets U and V in X. The sets $U \cup U'$ and $V \cup V'$ are open disjoint neighbourhoods of A and B in $A(X)$.

PROPOSITION 2.2. *X is countably paracompact if and only if $A(X)$ is countably paracompact.*

Proof. For the necessity, let $F_n \searrow \varnothing$ be a sequence of closed nowhere dense subsets of $A(X)$. Then $F_n \subseteq X$ and there exists a sequence (V_n) of open subsets of X with $F_n \subseteq U_n$ and $\bigcap_n \mathrm{cl}_X U_n = \varnothing$. Define $G_n = U_n \cup U_n'$ and note that $\mathrm{cl}_{A(X)} G_n = \mathrm{cl}_X U_n \cup U_n'$, so that $F_n \subseteq G_n$ and $\bigcap_n \mathrm{cl}_{A(X)} G_n = \varnothing$.

PROPOSITION 2.3. *If $A(X)$ is weak cb then both X and $A(X)$ are cb.*

Proof. To show that X is *cb*, take a sequence $A_n \searrow \emptyset$ of closed sets in X. Then $B_n = A_n \cup A'_n$ is regular closed in $A(X)$ and $B_n \searrow \emptyset$. There exist zero sets W_n in $A(X)$ with $B_n \subseteq W_n$ and $\bigcap_n W_n = \emptyset$. Then $Z_n = W_n \cap X$ is a zero set in X and $A_n \subseteq Z_n$ with $\bigcap_n Z_n = \emptyset$. If X is *cb* then both X and $A(X)$ are countably paracompact, hence $A(X)$ is *cb*.

One may show that $A(X)$ is countably compact if and only if X is. Furthermore, if X contains a C-embedded copy of $\mathbf{N}$, so does $A(X)$ so that $A(X)$ is pseudocompact implies that X is also. However, if X is pseudocompact (Tychonoff) but not countable compact then $A(X)$ is not weak *cb*, in particular, not pseudocompact.

3. Dowker spaces. A Dowker space is a normal Hausdorff space which is not countably paracompact. Such spaces exist within Zermelo–Fraenkel set theory; the axiom of choice implies the existence of a zero-dimensional P-space which is Dowker **[12]** and more recently the continuum hypothesis implies existence of a first countable, hereditarily separable Dowker space **[9]**.

The open question in [**11**, p. 240] may be phrased as follows: Must every Dowker space have the weak *cb* property? It follows from Propositions 2.1 and 2.2 that $A(X)$ is a Dowker space if and only if X is such. Since no Dowker space can be even an *nd*-space, 2.3 implies that for any Dowker space X, the space $A(X)$ answers the above question negatively. It may be of interest however that the (collectionwise normal) Dowker space of M. E. Rudin **[12]** is weak *cb*, as is now shown.

The reader is referred to **[12]** for details. With the same notation as in **[12]**, define

$$F = \{f\colon \mathbf{N} \to \omega_\omega \mid f(n) \leqq \omega_n \text{ for all } n \in \mathbf{N}\}.$$

$$X = \{f \in F \mid \omega_1 \leqq cf(f(n)) \leqq \omega_k \text{ for all } n \in \mathbf{N} \text{ and some } k \in \mathbf{N}\},$$

$$X' = \{f \in F \mid \omega_1 \leqq cf(f(n)) \text{ for all } n \in \mathbf{N}\}.$$

F carries a topology generated by the basic open-and-closed sets

$$(f, g] = \{h \in F \mid f(n) < h(n) \leqq g(n) \text{ for all } n \in \mathbf{N}\}.$$

Then $X \subseteq X' \subseteq F$ are subspaces and $\upsilon X = X'$ is paracompact, and hence a weak *cb*-space.

To show that X is weak *cb*, let $A_n \searrow \emptyset$ be a sequence of regular closed subsets of X and suppose $g \in \bigcap_n \mathrm{cl}_{\upsilon X} A_n$. We will define an increasing sequence $\{f_\alpha \in X \mid \alpha < \omega_1\}$ as follows:

(1) Choose any $f_0 \in \mathrm{int}_X A_1$ with $f_0 \leqq g$.

(2) Assume $f_\beta \in X$ is defined for all $\beta < \alpha$, and

(a) if $\alpha = \beta + 1$, let $i \in \mathbf{N}$ be the smallest integer with $f_\beta \notin \mathrm{int}_X A_i$ and choose $f_\alpha \in (\mathrm{int}_X A_i) \cap (f_\beta, g]$.

(b) if α is a limit ordinal, let $h_\alpha(n) = \sup\{f_\beta(n) \mid \beta < \alpha\}$ and choose $f_\alpha \in (\mathrm{int}_X A_1) \cap (h_\alpha, g]$.

Now define $f(n) = \sup\{f_\alpha(n) \mid \alpha < \omega_1\}$. Then $f \leqq g$ and $cf(f(n)) = \omega_1$ for all $n \in \mathbf{N}$ implies that $f \in X$. However, $f \in A_k$ for all $k \in \mathbf{N}$: let $h < f$ and for each $n \in \mathbf{N}$ there is $f_{\alpha_n} \in \{f_\alpha \mid \alpha < \omega_1\}$ with $h(n) < f_{\alpha_n}(n)$. Let $\beta = \sup\{\alpha_n \mid n \in \mathbf{N}\}$ and then $f_{\beta+k} \in (\mathrm{int}_X A_k) \cap (h, f]$, that is $f \in \mathrm{cl}_X\, \mathrm{int}_X A_k = A_k$.

We have a contradiction and so $A_n \searrow \varnothing$ implies $\bigcap_n \mathrm{cl}_{\upsilon X} A_n = \varnothing$. Finally, apply Proposition 1.2 to infer that X is weak *cb*.

4. Remarks. Since the Dowker space X in **[12]** is weak *cb*, it follows from **[4]** that $E(\upsilon X) = \upsilon E(X)$, where $E(X)$ denotes the absolute of X (see for example [**4**, p. 652]). Thus, $\upsilon E(X)$ is paracompact. However, it has been shown by E. K. van Douwen that $E(X)$ is not normal. It would seem natural therefore to pose the following questions. (1) Is there a normal space X with normal absolute $E(X)$; (2) Is there an extremally disconnected Dowker space; and ultimately (3) Is there a Dowker space X with Dowker absolute $E(X)$.

Added in proof. Regarding questions (2) and (3) above, M. Wage has proved that the set-theoretic hypothesis $\diamondsuit$ implies the existence of a separable extremally disconnected Dowker space.

References

1. P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam, **14** (1929), 1–96.

2. R. Engelking, *On the double circumference of Alexandroff*, Bull. Acad. Pol. Sci., **16** (1968), 629–634.

3. ———, *On functions defined on Cartesian products*, Fund. Math., **59** (1966), 221–231.

4. K. Hardy and R. G. Woods, *On c-realcompact spaces and locally bounded normal functions*, Pacific J. Math., **43** (1972), 647–656.

5. J. G. Horne, *Countable paracompactness and cb-spaces*, Notices Amer. Math. Soc., **6** (1959), 629–630.

6. F. Ishikawa, *On countably paracompact spaces*, Proc. Japan Acad., **31** (1955), 686–687.

7. I. Juhász, *On closed discrete subspaces of product spaces*, Bull. Acad. Pol. Sci., **17** (1969), 219–223.

8. I. Juhász and S. Mrówka, *E-compactness and the Alexandroff duplicate*, Indagationes Math., **73** (1970), 26–29.

9. I. Juhász, K. Kunen, and M. E. Rudin, *Two more hereditarily separable non-Lindelof spaces*, to appear.

10. J. Mack, *On a class of countably paracompact spaces*, Proc. Amer. Math. Soc., **16** (1965), 467–472.

11. J. Mack and D. Johnson, *The Dedekind completion of* $C(X)$, Pacific J. Math., **20** (1967), 231–243.

12. M. E. Rudin, *A normal space* X *for which* $X \times I$ *is not normal*, Fund. Math., **73** (1971), 179–186.

Received March 12, 1975. The first author gratefully acknowledges partial support from both the National Research Council of Canada and the Mathematisch Centrum, Amsterdam.

CARLETON UNIVERSITY, OTTAWA
MATHEMATISCH CENTRUM, AMSTERDAM
AND
MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCE, BUDAPEST