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DIVISION OF DISTRIBUTIONS

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This paper deals with division in an associative commutative algebra containing the distributions in $\mathbf{R}^n$.

1. Introduction. In **[5]** and **[6]**, a family $(A_{p,\lambda} \mid p \in \bar{N}^n, \lambda \in \Lambda)$ of associative, commutative algebras with unit element were constructed, with the following main properties:

(1) $\mathscr{D}'(R^n) \subset A_{p,\lambda}$, $\forall p \in \bar{N}^n$, $\lambda \in \Lambda$,
(here, $N = \{0, 1, 2, \cdots\}$, $\bar{N} = N \cup \{\infty\}$ and $n \in N$, $n \geqq 1$);

(2) The multiplication in each of the algebras $A_{p,\lambda}$, $p \in \bar{N}^n$, $\lambda \in \Lambda$, induces on $\mathscr{C}^\infty(R^n)$ the usual multiplication of functions and the function $\psi \in \mathscr{C}^\infty(R^n)$, with $\psi(x) = 1$, $\forall x \in R^n$, is the unit element in the algebras;

(3) for each $\lambda \in \Lambda$, there exist linear mappings $D^p\colon A_{q+p,\lambda} \to A_{q,\lambda}$, with $p \in N^n$, $q \in \bar{N}^n$, such that

(3.1) D^p satisfies on $A_{q+p,\lambda}$ the Leibnitz rule of product derivative.

(3.2) D^p is the usual distribution derivative on $\mathscr{C}^\infty(R^n) \oplus \mathscr{D}'_\delta(R^n)$, where $\mathscr{D}'_\delta(R^n) = \{S \in \mathscr{D}'(R^n) \mid \operatorname{supp} S$ is finite$\}$;

(4) The following relations hold for the Dirac δ_{x_0} distribution, concentrated in $x_0 \in R^n$:

$$(x - x_0)^r \cdot D^q \delta_{x_0} = 0 \in A_{p,\lambda}, \quad \forall p \in N^n, \quad \lambda \in \Lambda,$$

if q, $r \in N^n$, $r \geqq p + e$, $r \geqq q + e$, where $e = (1, \cdots, 1) \in N^n$.

In the present paper, within the one dimensional case $n = 1$, necessary or sufficient conditions are given for $T \in A_{p,\lambda}$, in order to be a solution of one of the equations $x^m \cdot T = 0 \in A_{p,\lambda}$ and $x^m \cdot T = S \in A_{p,\lambda}$, with $m \in N$, $m \geqq 1$.

2. Notations. Several classes of sequences of complex valued smooth functions (see **[5]** and **[6]**) will be needed.

(1) $\mathscr{W} = N \to \mathscr{C}^\infty(R^1)$; if $s \in \mathscr{W}$, $\nu \in N$, $x \in R^1$, then $s(\nu) \in \mathscr{C}^\infty(R^1)$, $s(\nu)(x) \in C^1$; for $\psi \in \mathscr{C}^\infty(R^1)$ denote $u(\psi) \in \mathscr{W}$, where $u(\psi)(\nu) = \psi$, $\forall \nu \in N$; $\mathscr{W}$ is in a natural way an associative, commutative algebra (the vector spaces and algebras are considered over the field C^1 of

complex numbers), with the unit element $u(1)$ and zero element $u(0)$; thus, $\mathcal{O} = \{u(0)\}$ is the null space in $\mathcal{W}$;

(2) D: $\mathcal{W} \to \mathcal{W}$ is defined by $(Ds)(\nu)(x) = (Ds(\nu))(x)$, $\forall s \in \mathcal{W}$, $\nu \in N$, $x \in R^1$; for given $x_0 \in R^1$, define τ_{x_0}: $\mathcal{W} \to \mathcal{W}$ by $(\tau_{x_0}s)(\nu)(x) = s(\nu)(x - x_0)$, $\forall s \in \mathcal{W}$, $\nu \in N$, $x \in R^1$;

(3) $\mathcal{U} = \{u(\psi) \mid \psi \in \mathcal{C}^\infty(R^1)\}$;

(4) $\mathcal{S}_0$ is the set of $s \in \mathcal{W}$, weakly convergent in $\mathcal{D}'(R^1)$; $\mathcal{V}_0$ is the kernel of the linear surjection:

$$\mathcal{S}_0 \ni s \to \langle s, \cdot \rangle \in \mathcal{D}'(R^1),$$

where

$$\langle s, \psi \rangle = \lim_{\nu \to \infty} \int_{R^1} s(\nu)(x)\psi(x)dx, \qquad \forall \psi \in \mathcal{D}(R^1);$$

One of the basic ideas in the construction of the associative and commutative distribution multiplication in **[5]** and **[6]**, is the way the weakly convergent sequences of smooth functions representing the Dirac δ distribution are chosen:

(5) $\mathcal{Z}^0_\delta$ is the set of $s \in \mathcal{S}_0$, satisfying the conditions:

(5.1) $\langle s, \cdot \rangle = \delta$,

(5.2) $\forall \epsilon > 0$: $\exists \nu_\epsilon \in N$: $\forall \nu \in N$,
$\nu \geqq \nu_\epsilon, x \in R^1, |x| \geqq \epsilon$: $s(\nu)(x) = 0$

(5.3) $\forall p \in N$: $\exists \nu_p \in N$: $\forall \nu \in N$,
$\nu \geqq \nu_p$: $W(s(\nu), \cdots, s(\nu + p))(0) \neq 0$.

where $W(\psi_1, \cdots, \psi_m)(x)$, $x \in R^1$, denotes the Wronskian function of $\psi_1, \cdots, \psi_m \in \mathcal{C}^\infty(R^1)$.

The condition (5.3), called "*strong local presence of s in x* = 0" and replaced in **[6]** by a weaker form, plays a central role in the associative, commutative distribution multiplication presented in **[5]** and **[6]**.

(6) for $p \in \bar{N}$, denote by $\mathcal{V}^0_{\delta,p}$ the set of $v \in \mathcal{V}_0$, satisfying the above condition (5.2), as well as

$$\text{(6.1)} \quad \forall q \in N, q \leqq p: \exists \nu_q \in N: \forall \nu \in N: \nu \geqq \nu_q \Rightarrow D^q v(\nu)(0) = 0;$$

(7) $\mathcal{S}^0_\delta = \{s \in \mathcal{S}_0 \mid \text{supp}\langle s, \cdot \rangle \subset \{0\}\}$;

(8) $\mathcal{V}_{\delta,p}$, with $p \in \bar{N}$, and $\mathcal{S}_\delta$ are the vector subspaces generated in $\mathcal{W}$ by $\bigcup_{x \in R^1} \tau_x \mathcal{V}^0_{\delta,p}$, respectively $\bigcup_{x \in R^1} \tau_x \mathcal{S}^0_\delta$;

(9) $\mathcal{Z}_\delta = \mathsf{X}_{x \in R^1} \tau_x \mathcal{Z}^0_\delta$;

(10) for $\Sigma = (s_x \mid x \in R^1) \in \mathcal{Z}_\delta$, denote by $\mathcal{S}(\Sigma)$ the vector subspace generated in $\mathcal{S}_0$ by the sequences $D^p s_x$, with $x \in R^1$, $p \in N$.

And now, the definition of the associative, commutative algebras

$(A_{p,\lambda} \mid p \in \bar{N}, \lambda \in \Lambda)$, where Λ is the set of all $\lambda = (\Sigma, \mathscr{S}_1)$ with $\Sigma \in \mathscr{X}_\delta$ and $\mathscr{S}_1$ vector subspace in $\mathscr{S}_0$, such that $(\mathscr{U} + \mathscr{S}_\delta) \cap \mathscr{S}_1 = \mathscr{O}$ and $\mathscr{S}_0 = \mathscr{U} + \mathscr{S}_\delta + \mathscr{S}_1$.

Suppose $p \in \bar{N}$, $\lambda = (\Sigma, \mathscr{S}_1) \in \Lambda$ and denote

(11) $\mathscr{S}_{p,\lambda} = \mathscr{V}_{\delta,p} \oplus \mathscr{U} \oplus \mathscr{S}(\Sigma) \oplus \mathscr{S}_1$;

(12) $\mathscr{A}_{p,\lambda}$ the smallest subalgebra in $\mathscr{W}$, containing $\mathscr{S}_{p,\lambda}$ and invariant of the mapping $D\colon \mathscr{W} \to \mathscr{W}$;

(13) $\mathscr{I}_{p,\lambda}$ the vector subspace generated in $\mathscr{W}$ by $\mathscr{V}_{\delta,p} \cdot \mathscr{A}_{p,\lambda}$.

Then (see **[5]** and **[6]**)

(1) $A_{p,\lambda} = \mathscr{A}_{p,\lambda} / \mathscr{I}_{p,\lambda}$,

(2) $D\colon A_{p+1,\lambda} \to A_{p,\lambda}$ is given by

$$D(t + \mathscr{I}_{p+1,\lambda}) = Dt + \mathscr{I}_{p,\lambda}, \qquad \forall t \in \mathscr{A}_{p+1,\lambda}.$$

3. Multiplication by $1/x^m$, $m = 1, 2, \cdots$. It is shown (see Corollary 2) that in the algebras $A_{p,\lambda}$, the multiplication by $1/x^m$ does not represent the division by x^m.

THEOREM 1. *Suppose $T \in A_{p,\lambda}$, with given $p \in \bar{N}$, $\lambda \in \Lambda$. Suppose $\psi \in \mathscr{C}^\infty(R^1)$ such that for a certain $m \in \bar{N}$*

$$D^q\psi(0) = 0, \qquad \forall q \in N, \qquad q \leqq m.$$

If there exists $\chi \in \mathscr{C}^\infty(R^1)$ such that $\psi \cdot T = \chi$ in $A_{p,\lambda}$, then:

$$D^q\chi(0) = 0, \qquad \forall q \in N, \qquad q \leqq \min\{p, m\}.$$

Proof. Assume $T = t + \mathscr{I}_{p,\lambda}$, with $t \in \mathscr{A}_{p,\lambda}$. Then $\psi \cdot T = \chi$ in $A_{p,\lambda}$ implies $u(\chi) = u(\psi) \cdot t + w$, with $w \in \mathscr{I}_{p,\lambda}$. Therefore,

$$\forall q \in N,\ q \leqq p\colon \exists \nu_q \in N\colon \forall \nu \in N,\ \nu \geqq \nu_q\colon D^q w(\nu)(0) = 0.$$

Since $\chi = \psi \cdot t(\nu) + w(\nu)$, $\forall \nu \in N$, the proof is completed.

COROLLARY 1. *Suppose $T \in A_{p,\lambda}$, with given $p \in \bar{N}$, $\lambda \in \Lambda$.*

If $\psi \in \mathscr{C}^\infty(R^1)$ such that $\psi(0) \neq 0$, then, $x^m \cdot T \neq \psi$ in $A_{p,\lambda}$, $\forall m \in N$, $m \geqq 1$.

COROLLARY 2. *If $m \in N$, $m \geqq 1$, then, $x^m \cdot (1/x^m) \neq 1$, in each of the algebras $A_{p,\lambda}$, $p \in \bar{N}$, $\lambda \in \Lambda$.*

4. Division by x^m, $m = 1, 2, \cdots$. First, in Theorem 2, a sufficient condition is given for $T \in A_{p,\lambda}$, in order to be a solution of the equation $x^m \cdot T = 0 \in A_{p,\lambda}$, where $m \in N$, $m \geqq 1$.

For $p \in \bar{N}$ and $\lambda \in \Lambda$, denote by $B^0_{p,\lambda}$ all the elements $T \in A_{p,\lambda}$ of the form $T = t + \mathcal{I}_{p,\lambda}$, where $t \in \mathcal{A}_{p,\lambda} \cap \mathcal{V}_0$ and satisfies also (5.2) in §2.

PROPOSITION 1. *Suppose given $p \in \bar{N}$, $\lambda \in \Lambda$ and $\psi \in \mathcal{C}^\infty(R^1)$, such that, for a certain $q \in \bar{N}$, $q \geqq p$:*

$$D^r\psi(0) = 0, \qquad \forall r \in N, \qquad r \leqq q.$$

Then, $\psi \cdot B^0_{p,\lambda} = \{0\} \subset A_{p,\lambda}$.

Proof. Assume $T \in B^0_{p,\lambda}$ and $T = t + \mathcal{I}_{p,\lambda}$, with $t \in \mathcal{A}_{p,\lambda} \cap \mathcal{V}_0$ and satisfying (5.2) in §2. Then, $\psi \cdot T = u(\psi) \cdot t + \mathcal{I}_{p,\lambda}$. But, obviously, $u(\psi) \cdot t \in \mathcal{V}^0_{\delta,q} \subset \mathcal{V}^0_{\delta,p} \subset \mathcal{I}_{p,\lambda}$, hence, $T = 0 \in A_{p,\lambda}$.

THEOREM 2. *Suppose given $p \in N$, $\lambda \in \Lambda$ and $m \in N$, $m \geqq 1$. Then, any*

$$T_0 = \sum_{0 \leqq i \leqq k} x^{r_i} \cdot T_{1i} \cdot T_{2i} + \sum_{0 \leqq j \leqq h} x^{q_j} \cdot D^{p_j}\delta \cdot T_{3j},$$

with $k, h, r_i, q_j, p_j \in N$, $r_i > p - m$,
$q_j > \max\{p, p_j\} - m$,
and $T_{1i} \in B^0_{p,\lambda}$, $T_{2i}, T_{3j} \in A_{p,\lambda}$,
will be a solution in $A_{p,\lambda}$ of the equation $x^m \cdot T = 0$.

Proof. According to Proposition 1, $x^m \cdot x^{r_i} \cdot T_{1i} = x^{m+r_i} \cdot T_{1i} = 0 \in A_{p,\lambda}$, since $m + r_i > p$. According to (4) in §1 (see also 3) in Theorem 6, §8 **[5]**), $x^m \cdot x^{q_j} \cdot D^{p_j}\delta = x^{m+q_j} \cdot D^{p_j}\delta = 0 \in A_{p,\lambda}$, since $m + q_j > \max\{p, p_j\}$.

It results the following sufficient condition on $T \in A_{p,\lambda}$, solution of the equation $x^m \cdot T = S \in A_{p,\lambda}$.

COROLLARY 3. *Suppose $S \in A_{p,\lambda}$, with $p \in N$, $\lambda \in \Lambda$ given and $m \in N$, $m \geqq 1$.*

If T_1 is any solution in $A_{p,\lambda}$ of the equation $x^m \cdot T = S$ and T_0 is given as in Theorem 2, then $T = T_1 + T_0$ will be again a solution of that equation.

Before a necessary condition is given on $T \in A_{p,\lambda}$, solution of the equation $x^m \cdot T = 0 \in A_{p,\lambda}$, the notion of *support* of the elements in $A_{p,\lambda}$ will be defined.

Suppose $T \in A_{p,\lambda}$, with $p \in \bar{N}$, $\lambda \in \Lambda$ given and $E \subset R^1$. Then,

(1) *T vanishes on E*, only if $T = t + \mathcal{I}_{p,\lambda}$, with $t \in \mathcal{A}_{p,\lambda}$, such that $t(\nu)(x) = 0$, $\forall \nu \in N$, $\nu \geqq \nu_0$, $x \in E$.

(2) *T strictly vanishes on E*, only if T vanishes on a certain open set $G \subset R^1$, containing E.

(3) *T is supported by E*, only if for every open set $G \subset R^1$, containing E, one can write $T = t + \mathcal{I}_{p,\lambda}$, with $t \in \mathcal{A}_{p,\lambda}$, such that $\operatorname{supp} t(\nu) \subset G$, $\forall \nu \in N$, $\nu \geqq \nu_0$.

The *support of T* is defined as the closed set

$$\operatorname{supp} T = R^1 \backslash \{x \in R^1 \mid T \text{ strictly vanishes on } \{x\}\}.$$

Obviously, for the distributions in $\mathcal{C}^\infty(R^1) \oplus \mathcal{D}'_\delta(R^1)$, the above notion of support is identical with the usual one for distributions.

PROPOSITION 2. *Suppose $x_0 \in R^1$ and $q \in N$, then, $D^q\delta_{x_0} \in A_{p,\lambda}$, for $p \in \bar{N}$, $\lambda \in \Lambda$, and*

(1) *$D^q\delta_{x_0}$ is supported by $\{x_0\}$ and* $\operatorname{supp} D^q\delta_{x_0} = \{x_0\}$,

(2) *if $E \subset R^1$ and $x_0 \notin$ closure E, then $D^q\delta_{x_0}$ strictly vanishes on E,*

(3) *$D^q\delta_{x_0}$ does not vanish on $R^1\backslash\{x_0\}$,*

(4) *$D^q\delta_{x_0}$ does not vanish on $\{x_0\}$.*

Proof. (1), (2) and (3) follow easily.

(4) Assume $\lambda = (\Sigma, \mathcal{S}_1)$ and $\Sigma = (s_x \mid x \in R^1)$, then, $D^q\delta_{x_0} = D^q s_{x_0} + \mathcal{I}_{p,\lambda}$ and $s_{x_0} \in \tau_{x_0}\mathcal{Z}^0_\delta$. Suppose, $D^q\delta_{x_0}$ vanishes on $\{x_0\}$, then, there exists $t \in \mathcal{A}_{p,\lambda}$, such that $t - D^q s_{x_0} \in \mathcal{I}_{p,\lambda}$ and $t(\nu)(x_0) = 0$, $\forall \nu \in N$, $\nu \geqq \nu_0$. Denoting $v = t - D^q s_{x_0}$, the relation $v \in \mathcal{I}_{p,\lambda}$ implies $v(\nu)(x_0) = 0$, $\forall \nu \in N$, $\nu \geqq \nu_1$. Therefore, it results

$$D^q s_{x_0}(\nu)(x_0) = t(\nu)(x_0) - v(\nu)(x_0) = 0, \qquad \forall \nu \in N, \qquad \nu \geqq \nu_2.$$

But, that relation implies $W(s_{x_0}(\nu), \cdots, s_{x_0}(\nu + q))(x_0) = 0$, $\forall \nu \in N$, $\nu \geqq \nu_2$, which contradicts the assumption $s_{x_0} \in \tau_{x_0}\mathcal{Z}^0_\delta$.

REMARK. The property of the Dirac distributions that $D^q\delta_{x_0}$ *does not vanish on* $\{x_0\}$, $\forall x_0 \in R^1$, $q \in N$, is a direct consequence of the "*condition of strong local presence*" (see (5.3) in §2) and it is proper for the distribution multiplication presented in **[5]** and **[6]**. The "delta sequences" generally used (see **[2]**) do not necessarily prevent the vanishing of $D^q\delta_{x_0}$ on $\{x_0\}$.

THEOREM 3. *Suppose $T \in A_{p,\lambda}$ with $p \in \bar{N}$, $\lambda \in \Lambda$ given.*

If $x^m \cdot T = 0 \in A_{p,\lambda}$, for a certain $m \in N$, $m \geqq 1$, then T is supported by $\{0\}$, hence supp $T \subset \{0\}$.

Proof. Assume $T = t + \mathscr{I}_{p,\lambda}$, with $t \in \mathscr{A}_{p,\lambda}$. Then $x^m \cdot T = 0 \in A_{p,\lambda}$ implies $u(x^m) \cdot t \in \mathscr{I}_{p,\lambda}$, therefore, according to the definition of $\mathscr{I}_{p,\lambda}$ (see (13), §2), it results

$$u(x^m) \cdot t = \sum_{0 \leqq i \leqq k} v_i \cdot a_i$$

with $k \in N$, $v_i \in \mathscr{V}_{\delta,p}$, $a_i \in \mathscr{A}_{p,\lambda}$.

Now, due to the definition $\mathscr{V}_{\delta,p}$ (see (8) and (6), §2), it follows that: $\forall i \in \{0, \cdots, k\}$: $\exists X_i \subset R^1$, X_i finite: $v_i = \sum_{x \in X_i} v_{ix}$, where $v_{ix} \in \tau_x \mathscr{V}^0_{\delta,p}$.

Concluding, there exists $X \subset R^1$, X finite, such that

$$u(x^m) \cdot t = \sum_{x \in X} \sum_{0 \leqq j \leqq h} v_{xj} \cdot b_{xj} \quad \text{with} \quad h \in N, \quad v_{xj} \in \tau_x \mathscr{V}^0_{\delta,p}, \quad b_{xj} \in \mathscr{A}_{p,\lambda}.$$

It will be shown now, that in the above relation, one can consider $X = \{0\}$. Indeed, suppose $x_0 \in X \backslash \{0\}$, then $v_{x_0j} \in \tau_{x_0} \mathscr{V}^0_{\delta,p}$ with $0 \leqq j \leqq h$. The condition (5.2) in §2, results in the existence of $w_{x_0j} \in \mathscr{W}$, with $0 \leqq j \leqq h$, such that $v_{x_0j}(\nu)(x) = x^m \cdot w_{x_0j}(\nu)(x)$, $\forall 0 \leqq j \leqq h$, $x \in R^1$, $\nu \in N$, $\nu \geqq \nu_0$. Moreover, $w_{x_0j} \in \tau_{x_0} \mathscr{V}^0_{\delta,p}$, $\forall 0 \leqq j \leqq h$, since $v_{x_0j} \in \tau_{x_0} \mathscr{V}^0_{\delta,p}$, with $0 \leqq j \leqq h$, and $x_0 \neq 0$.

Denoting

$$v = \sum_{\substack{x_0 \in X \\ x_0 \neq 0}} \sum_{0 \leqq j \leqq h} w_{x_0j} \cdot b_{x_0j}$$

it results $v \in \mathscr{I}_{p,\lambda}$, hence, $T = t_1 + \mathscr{I}_{p,\lambda}$, where $t_1 = t - v \in \mathscr{A}_{p,\lambda}$. But $u(x^m) \cdot t_1 = u(x^m) \cdot t - u(x^m) \cdot v = \sum_{0 \leqq j \leqq h} v_{0j} \cdot b_{0j}$.

Since v_{0j}, with $0 \leqq j \leqq h$, satisfy (5.2) in §2, it follows that $u(x^m) \cdot t_1$ and, therefore t_1 satisfy the same condition. Thus, $T = t_1 + \mathscr{I}_{p,\lambda}$ is supported by $\{0\}$, which obviously results in supp $T \subset \{0\}$.

References

1. H. Kang, J. Richards, *A general definition of convolution for distributions*, (to appear).

2. J. Mikusinski, *On the square of the Dirac delta distribution*, Bull. Acad. Pol. Sci., **14**, 9, (1966), 511–513.

3. E. Rosinger, *Embedding the $\mathscr{D}'(R^n)$ distributions in pseudotopological algebras*, Stud. Cerc. Mat., **18**, 5, (1966), 687–729.

4. ———, *Pseudotopological spaces. Embedding the $\mathscr{D}'(R^n)$ distributions into algebras*, Stud. Cerc. Mat., **20**, 4, (1968), 553–582.

5. ———, *A distribution multiplication theory*, Haifa Technion's Preprint Series, AMT-31, October 1974 (to appear).
6. ———, *An associative, commutative distribution multiplication*, Technical Report, Haifa Technion, March 1976 (to appear).
7. L. Schwartz, *Sur l'impossibilité de la multiplication des distributions*, C. R. Acad. Sci. Paris, **239** (1954), 847–848.

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