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SOME NONOSCILLATION CRITERIA FOR HIGHER ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Sufficient conditions for an nth order nonlinear differential equation to be nonoscillatory are given. An essential part of the hypotheses is that a related linear equation be disconjugate.

The linear differential equation

$$x^{(n)} + p(t)x = 0, \tag{1}$$

where $p: [t_0, \infty) \to R$ is continuous, is said to be eventually disconjugate if there exists $T \geqq t_0$ such that no solution of (1) has more than $n-1$ zeros (counting multiplicities) on $[T, \infty)$. A solution $x(t)$ of (1) (or equation (2) below) will be called nonoscillatory if there exists $t_1 \geqq t_0$ such that $x(t) \neq 0$ for $t \geqq t_1$. Equation (1) (or (2)) will be called nonoscillatory if all its solutions are nonoscillatory. Clearly, disconjugacy implies nonoscillation. On the other hand, for $n = 2, 3$ or 4 and either $p(t) > 0$ or $p(t) < 0$, if equation (1) is nonoscillatory, then (1) is eventually disconjugate. Whether this is true for $n > 4$ remains an open question (see Nehari **[11]**).

In this paper we consider the nonlinear differential equation

$$x^{(n)} + q(t)f(t, x, x', \cdots, x^{(n-1)}) = 0 \tag{2}$$

where $q: [t_0, \infty) \to R$ and $f: [t_0, \infty) \times R^n \to R$ are continuous, and obtain some nonoscillation results by making assumptions on the disconjugacy of certain related linear equations. A discussion of disconjugacy criteria for linear differential equations can be found in Coppel **[2]**, Levin **[10]**, Nehari **[11]**, Trench **[12]**, or Willett **[13]**. For a discussion of nonoscillation criteria for second order nonlinear equations we refer the reader to the recent papers of Coffman and Wong **[1]**, Graef and Spikes **[3–5]**, Wong **[14]**, and the references contained therein. There appears to be no known sufficient conditions for nonoscillation of higher order nonlinear equations.

We will assume that there is a continuous function $W: [t_0, \infty) \times R^n \to R$ such that

$$|f(t, u_1, \cdots, u_n)| \leqq W(t, u_1, \cdots, u_n)|u_1| \tag{3}$$

for all $(t, u_1, \cdots, u_n) \in [t_0, \infty) \times R^n$, and

$$f(t, u_1, \cdots, u_n)/u_1 \to A \quad \text{as} \quad u_1 \to 0. \tag{4}$$

THEOREM 1. *Suppose that conditions* (3) *and* (4) *hold,* $W(t, u_1, \cdots, u_n) \leqq B$ *and* $M = \max\{|A|, B\}$. *If the equations*

$$x^{(n)} \pm M|q(t)|x = 0 \tag{5}$$

are eventually disconjugate, then equation (2) *is nonoscillatory.*

Proof. Suppose that equations (5) are disconjugate on $[T, \infty)$ where $T \geqq t_0$ and let $x(t)$ be a solution of (1). Define $Q: [T, \infty) \to R$ by

$$Q(t) = \begin{cases} q(t)f(t, x(t), \cdots, x^{(n-1)}(t))/x(t), & \text{if } x(t) \neq 0 \\ Aq(t), & \text{if } x(t) = 0. \end{cases}$$

It then follows that $Q(t)$ is continuous and $x(t)$ is a solution of

$$x^{(n)} + Q(t)x = 0. \tag{6}$$

Kondrat'ev **[9]** showed that if $p_1(t) \leqq p_2(t)$ and the equations

$$x^{(n)} + p_i(t)x = 0, \qquad i = 1, 2$$

are disconjugate on $[T, \infty)$, then for any $p(t)$ with $p_1(t) \leqq p(t) \leqq p_2(t)$ the equation

$$x^{(n)} + p(t)x = 0$$

is disconjugate on $[T, \infty)$. Here we have $|Q(t)| \leqq M|q(t)|$ so $-M|q(t)| \leqq Q(t) \leqq M|q(t)|$. Hence equation (6) is disconjugate and so $x(t)$ is nonoscillatory.

REMARK 1. If $q(t) \geqq 0$ and $u_1 f(t, u_1, \cdots, u_n) \geqq 0$, then $Q(t) \geqq 0$. Since the equation $x^{(n)} = 0$ is disconjugate on $[T, \infty)$ for any $T \geqq t_0$, we would only need to assume that equation (5) with "+" is eventually disconjugate. Note also that condition (4) is only needed to insure that Q is continuous.

REMARK 2. Equations (5) are eventually disconjugate if, for example,

$$\int_{t_0}^{\infty} t^{n-1}|q(t)|\,dt < \infty$$

(see Kiguradze **[8]**, Kondrat'ev **[9]**, or Willett **[13]**). In this regard we would then have a generalization of a result of Kartsatos [**7**; Theorem 2].

Willett [**13**; Theorem 1.4] has shown that if for each $i = 1, 2, \cdots, n$, $p_i : [t_0, \infty) \to R$ is continuous and

$$\int_{t_0}^{\infty} t^{i-1} |p_i(t)|\, dt < \infty, \tag{7}$$

then the equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0 \tag{8}$$

is eventually disconjugate. (Recently Gustafson **[6]** showed that even though nonoscillation implies disconjugacy for equation (8) with $n = 2$, this is not the case for $n > 2$.) Employing the method of proof used above we can obtain that all solutions of

$$x^{(n)} + p_1(t)f_1(x^{(n-1)}) + \cdots + p_n(t)f_n(x) = 0 \tag{9}$$

are nonoscillatory.

THEOREM 2. *Suppose that condition* (7) *holds and there are bounded continuous functions* W_i: $[t_0, \infty) \to R$, $i = 1, 2, \cdots, n$ *such that*

$$|f_i(u)| \leqq W_i(u)\,|u|$$

and

$$f_i(u)/u \to A_i \quad as \quad u \to 0.$$

Then all solutions of (9) *are nonoscillatory.*

Proof. If $x(t)$ is a solution of (9), then $x(t)$ is also a solution of

$$x^{(n)} + Q_1(t)x^{(n-1)} + \cdots + Q_n(t)x = 0 \tag{10}$$

where

$$Q_i(t) = \begin{cases} p_i(t)f_i(x^{(n-i)}(t))/x^{(n-i)}(t), & \text{if } x^{(n-i)}(t) \neq 0 \\ A_i p_i(t), & \text{if } x^{(n-i)}(t) = 0 \end{cases}$$

In addition, for each $i = 1, 2, \cdots, n$

$$\begin{aligned}\int_{t_0}^{\infty} t^{i-1}|Q_i(t)|\,dt &\leqq \int_{t_0}^{\infty} t^{i-1}[|p_i(t)|\,|f_i(x^{(n-i)}(t))|/|x^{(n-i)}(t)|]\,dt \\ &\leqq \int_{t_0}^{\infty} t^{i-1}|p_i(t)|\,W_i(x^{(n-i)}(t))dt \\ &\leqq K_i\int_{t_0}^{\infty} t^{i-1}|p_i(t)|\,dt \\ &< \infty\end{aligned}$$

where K_i is a constant. It follows from Willett's theorem that equation (10) is disconjugate and hence $x(t)$ is nonoscillatory.

Clearly various other forms of equation (9) can be handled in a similar fashion.

As an example of the above results, consider the equation

$$x^{(n)} + x^3(\sin t)/t^{n+1}(x^2+1) = 0. \tag{11}$$

The corresponding linear equation

$$x^{(n)} + x(\sin t)/t^{n+1} = 0$$

is disconjugate, so equation (11) is nonoscillatory.

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