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MEASURES WITH CONTINUOUS IMAGE LAW

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Let M be a topological space, and X a metric space. Let $P(X)$ denote the collection of probability measures on X. Let $C(M, X)$ denote the set of continuous functions from M to X. Let $P(X)$ have the weak topology, and let $C(M, X)$ have the topology of uniform convergence. For a fixed measure $\mu \in P(C(M, X))$, and a member $t \in M$, define a measure $t\mu$ on X by

$$t\mu(A) = \mu\{f \in C(M, X): f(t) \in A\} .$$

In this paper, we consider the following problem: given a continuous function $T: M \to P(X)$, when is there a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t?

This problem has been introduced and studied by Blumenthal and Corson in [2] and [3].

The main results of this paper are as follows:

1. Let F be a closed subset of a totally disconnected compact metric space M, and let $i_F: F \to M$ be the natural inclusion. Let X be a complete separable metric space. Suppose that T is a continuous function from M to $P(X)$, and μ_F is a measure in $P(C(F, X))$ such that $T(t) = t\mu_F$ for all $t \in F$. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all $t \in M$ and $i_F\mu = \mu_F$. Consequently, the natural map $\phi: P(C(M, X)) \to C(M, P(X))$, defined as $\phi(\mu)(t) = t\mu(\mu \in P(C(M, X)), t \in M)$ is open, and there is a continuous function $\xi: C(M, P(X)) \to P(C(M, X))$ such that $\xi(T) \in \phi^{-1}(T)$.

2. Let M be a separable metric space, and let S^1 be the unit sphere in R^2. Let T be a continuous function from M into $P(S^1)$ such that $\operatorname{Supp} T(t)$ is a connected subarc of S^1 and of arc length $\leqq 2\pi - k$, for all t and some fixed $0 < k \leqq 2\pi$. Then there is a measure $\mu \in P(C(M, S^1))$ such that $T(t) = t\mu$ for all t.

3. Some generalizations of the results in [2] and [3] will be considered.

1. Introduction.

Let Y be a metric space. By a measure on Y, we mean a regular Borel measure on the class of all Borel subsets of Y. Let μ be a measure on Y. We define the support of μ to be the smallest nonempty closed subset $F \subseteq Y$ such that $\mu(U) > 0$ for every open subset U in Y such that $U \cap F \neq \varnothing$. We will denote this by Supp μ. For a fixed $y \in Y$, we denote by ε_y the unit point-mass measure putting mass 1 at the point y. Let $C(Y)$ denote the collection of all bounded continuous real-valued functions on Y. Let $P(Y)$ denote the collection of all measures on Y with total mass 1. Let $P(Y)$ have the weak topology as functionals on $C(Y)$. A measure in $P(Y)$ will be called a probability measure.

Let X be a metric space, and let $F \subseteqq P(X)$. We call F tight if for every $\varepsilon > 0$, there is a compact set $K \subseteqq X$ such that $\mu(K) \geqq 1 - \varepsilon$ for all $\mu \in F$. If $F = \{\mu\}$, we call μ a tight measure. Denote by $P_\tau(X)$ the collection of all tight measures in $P(X)$ with the relative topology. We call a metric space a Prohorov space if every compact subset of $P_\tau(X)$ is tight. Note that every complete metric space is Prohorov [1], and $P_\tau(X) = P(X)$ whenever X is complete separable metric.

Now consider a topological space M, and a metric space X. Let $C(M, X)$ denote the set of continuous functions from M into X. Let $C(M, X)$ have the topology of uniform convergence. Note that if M is not compact, the topology on $C(M, X)$ depends on the particular bounded metric used for X.

We will use the following notation throughout this paper. Let Y and X be metric and let π be a continuous function from Y into X. Then π induces a mapping, also denoted by π, from $P(Y)$ to $P(X)$, and defined by $\pi\mu(A) = \mu(\pi^{-1}(A))$. Also for $t \in M$, denote simply by t the mapping $f \rightarrow f(t)$ from $C(M, X)$ into X. Hence the mapping $\mu \rightarrow t\mu$ from $P(C(M, X))$ to $P(X)$ is defined and continuous by taking Y to be $C(M, X)$ and t to be π in the definition of $\mu \rightarrow \pi\mu$ above

Recall that, by a totally disconnected space, we mean a space which has a base consisting of sets which are both open and closed. By a Peano space, we mean a space which is a continuous image of the unit interval. For convenience, we will call a complete separable metric space a Polish space throughout this paper.

In this paper, we consider the following problem: given a continuous function $T: M \rightarrow P(X)$, when is there a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t? When this happens, we will say that μ represents T.

This problem was introduced by Blumenthal and Corson. They showed that such a measure exists in the following two cases:

(a) M is a compact totally disconnected space, and X is a complete metric space [2].

(b) M is a compact metric space, X is a Peano space, and Supp $T(t) = X$ for all t [3].

In §2, we let M be a totally disconnected compact metric space, the Cantor set for example, and let X be a Polish space. For a fixed $\mu \in P(C(M, X))$, one checks easily that $t \to t\mu$ is a continuous function from M into $P(X)$. Now define a mapping $\phi: P(C(M, X)) \to C(M, P(X))$ by

$$\phi(\mu)(t) = t\mu\,, \quad \mu \in P(C(M, X))\,, \quad t \in M\,.$$

Blumenthal and Corson (a) showed that ϕ maps $P(C(M, X))$ onto $C(M, P(X))$. We establish a certain extension theorem which has a consequence that the mapping ϕ is open, and that there is a continuous function $\xi: C(M, P(X)) \to P(C(M, X))$ such that $\xi(T) \in \phi^{-1}(T)$. (The openness of ϕ has been obtained independently by Eifler [7].)

In § 3, we establish a representation theorem when M is a separable metric space, and X is the unit sphere in R^2.

In § 4, we will see that the conditions on M or X in (a) and (b) can be relaxed and so more general results can be obtained.

2. Extension theorem. In this section, our main task is to give an affirmative answer to the following question:

Let $T \in C(M, P(X))$. Suppose that F is a closed subset of M, and that $\mu_F \in P(C(F, X))$ representing $T|_F$. Is there a measure $\mu \in P(C(M, X))$ representing T and $\mu\{f \in C(M, X): f|_F \in L\} = \mu_F(L)$ for all Borel sets $L \subseteq C(F, X)$?

Before giving the basic lemma, we review briefly some notations we need, and give one more piece of notation. Let M be a compact metric space, and let Y and X be Polish spaces. Let F be a closed subset of M. Let $i_F: F \to M$ be the natural inclusion map, and let $\pi: Y \to X$ be a continuous function. Then π induces a continuous mapping, also denoted by π, from $P(Y)$ to $P(X)$ and defined by $\pi\mu(E) = \mu(\pi^{-1}(E))$, and another continuous mapping, from $C(M, Y)$ to $C(M, X)$, by $\pi\phi(t) = \pi(\phi(t))$, $t \in M$, $\phi \in C(M, Y)$. We denote this mapping by π also. The mapping $i_F: F \to M$ induces a continuous mapping, also denote by i_F, from $C(M, Y) \to C(F, Y)$, by $i_F f(t) = f(i_F(t))$ for all $t \in F$, $f \in C(M, Y)$. If we regard $\pi: C(M, Y) \to C(M, X)$, $i_F: C(M, X) \to C(F, X)$, then $i_F\pi: C(M, Y) \to C(F, X)$. If we regard $i_F: C(M, Y) \to C(F, Y)$, $\pi: C(F, Y) \to C(F, X)$, then $\pi i_F: C(M, Y) \to C(F, X)$. It is easy to check that $i_F\pi = \pi i_F$.

LEMMA 2.1. *Let M be a totally disconnected compact metric space, let F be a closed subspace of M, and let $i_F: F \to M$ be the natural inclusion map. Let Y and X be countable discrete spaces and let π be a continuous mapping from Y onto X. Let T be a continuous function from M into $P(Y)$, and let $\nu \in P(C(M, X))$, $\mu_F \in P(C(F, Y))$ be such that*

(1) $t\mu_F = T(t)$ *for all* $t \in F$

(2) $t\nu = \pi T(t)$ *for all* $t \in M$

(3) $\pi\mu_F = i_F\nu$.

Then there is a measure $\mu \in P(C(M, Y))$ *such that*

(1) $t\mu = T(t)$ *for all* $t \in M$

(2) $\pi\mu = \nu$

(3) $i_F\mu = \mu_F$.

Proof. Let $A = \{\theta \mid \theta$ is a measure on $C(M, Y)$, $\pi\theta \leqq \nu$, $t\theta \leqq T(t)$, $i_F\theta \leqq \mu_F\}$. We will show that A contains a non-zero element. Indeed, $C(M, X)$ is a countable discrete space [6, p. 265] and so there is an element $f \in C(M, X)$ such that $\nu(\{f\}) = \varepsilon > 0$. Denote the restriction of f on F by f_F. Then $i_F\nu(\{f_F\}) \geqq \varepsilon$. Since $\pi\mu_F = i_F\nu$, there is a continuous function $g: F \rightarrow Y$ such that $\pi g = f_F$ and $\mu_F(\{g\}) = \delta > 0$. Suppose that g takes on values $x_1, \cdots, x_n$ on sets $M_1, \cdots, M_n$ respectively. Since $T(t) = t\mu_F$ for all $t \in F$, we have $T(t)(\{x_i\}) \geqq \delta$ for each $t \in M_i$. For each x_i, the function $t \rightarrow T(t)(\{x_i\})$ is continuous. Since M is totally disconnected and compact metric, we may pick pair-wise disjoint both open and closed subsets $U_1, \cdots, U_n$ of M such that $M_i \subseteqq U_i$, $1 \leqq i \leqq n$, $T(t)(\{x_i\}) \geqq \delta_0 = (1/2)\delta$, for each $t \in U_i$, and $f(U_i) = f(M_i) = \pi g(M_i) = \pi(x_i)$. Let $L = \bigcup_{i=1}^n U_i$, then L is a both open and closed subset of M. Define a continuous function h of L into Y by $h(t) = x_i$ if $t \in U_i$, $1 \leqq i \leqq n$. Now we consider the restriction of f on $L^c = M\backslash L$, denoted by f_{L^c}. We also denote the measure $i_{L^c}\nu$ by ν_{L^c}. Then $\pi T(t) = t\nu_{L^c}$ for all $t \in L^c$, and $\nu_{L^c}(f_{L^c}) \geqq \varepsilon$. By the lemma of [2], there is a continuous function $k: L^c \rightarrow Y$ such that $\pi k = f_{L^c}$ and a number $\delta' > 0$ such that if θ is a point-mass measure at k with mass δ', then $t\theta \leqq T(t)$ for each $t \in L^c$. Define $\hat{g}: M \rightarrow Y$ by $\hat{g}(t) = h(t)$ if $t \in L$ and $\hat{g}(t) = k(t)$ if $t \in L^c$. Then $\hat{g}$ is continuous, and $\pi\hat{g} = f$. Let $\delta_0' = \mathrm{Min}\,\{\delta_0, \delta', \varepsilon\}$, and let θ_0 be the measure putting mass δ_0' at the point $\hat{g} \in C(M, Y)$, then θ_0 is nonzero, and it is easy to check that $\theta_0 \in A$. Now return to the proof of the lemma. The set A is inductively ordered: indeed, if K is a totally ordered subset of A, we take $\mu_1 \leqq \mu_2 \leqq \cdots$ from K such that $\lim_{n\rightarrow\infty} \mu_n(C(M, Y)) = \mathrm{Sup}_{\mu \in K}\, \mu(C(M, Y))$, then $\alpha = \lim_n \mu_n$ is an element of A and $\alpha \geqq \rho$ for every $\rho \in K$. Let θ be a maximal element of A. If θ has mass 1, then $t\theta = T(t)$, $\pi\theta = \nu$ and $i_F\theta = \mu_F$. If θ has mass $\eta < 1$, then we may apply the first part of the proof to the mapping:

$$T'(t) = \frac{T(t) - t\theta}{1 - \eta} \in C(M, P(Y)) \quad \text{and}$$

measures:

$$\mu_F' = \frac{\mu_F - i_F\theta}{1 - \eta} \in P(C(F, Y))$$

$$\nu' = \frac{\nu - \pi\theta}{1 - \eta} \in P(C(M, X)) .$$

This will yield a strictly positive measure θ' with $t\theta' \leqq T'(t)$, $\pi\theta' \leqq \nu'$ and $i_F\theta' \leqq \mu'_F$, and then $\theta + (1 - \eta)\theta'$ will be an element of A strictly exceeding θ. This completes the proof.

LEMMA 2.2. *Let M be a totally disconnected compact metric space, and let F be a closed subset of M. Let $i_F: F \to M$ be the natural inclusion map. Let G be a totally disconnected complete separable metric space. Let $T \in C(M, P(G))$ and $\mu_F \in P(C(F, G))$ be such that $T(t) = t\mu_F$ for all $t \in F$. Then there is a measure $\mu \in P(C(M, G))$ such that $T(t) = t\mu$ for all $t \in M$ and $i_F\mu = \mu_F$.*

Proof. It is easy to check that G can be identified as a closed subspace of a countable product $X_{i=1}^{\infty} F_i$ of discrete spaces, with each F_i being countable, and F_1 consisting of a single point. For each $n > 0$, let $P_n: G \to F_1 \times \cdots \times F_n$ be defined by $P_n((g_1, \cdots, g_n, \cdots)) = (g_1, \cdots, g_n)$. Let $X_n = P_n(G)$. Then X_n is a subspace of $F_1 \times \cdots \times F_n$, and so is a countable discrete space. Let $\pi_n: X_n \to X_{n-1}$ be the mapping that sends $(g_1, \cdots, g_{n-1}, g_n)$ to $(g_1, \cdots, g_{n-1})$. Then $\pi_n P_n = P_{n-1}$. Let $T^{(n)}(t) = P_n T(t)$. Then $T^{(n)}$ is a continuous function from M to $P(X_n)$, and $\pi_n T^{(n)} = T^{(n-1)}$. Let $\mu_F^{(n)} = P_n \mu_F \in P(C(F, X_n))$, and let $T_F^{(n)}(t) = t\mu_F^{(n)}$. Then $\pi_n \mu_F^{(n)} = \mu_F^{(n-1)}$, and $T^{(n)}(t) = T_F^{(n)}(t)$ if $t \in F$. When $n = 1$, we have of course the trivial measure $\mu^{(1)}$ putting mass 1 on the one point of $C(M, X_1)$ so that (a) $t\mu^{(1)} = T^{(1)}(t)$ and (b) $i_F\mu^{(1)} = \mu_F^{(1)}$. Consequently, by repeatedly applying Lemma 2.1, we obtain a sequence of measures $\mu^{(n)} \in P(C(M, X_n))$ such that (a) $t\mu^{(n)} = T^{(n)}(t)$, (b) $i_F\mu^{(n)} = \mu_F^{(n)}$ and (c) $\pi_n\mu^{(n)} = \mu^{(n-1)}$ for all n. By Kolomogorov's consistency theorem [4, p. 120], there is a measure $\mu \in P(C(M, G))$ such that $P_n\mu = \mu^{(n)}$ for all n. Since for all n, $P_n t\mu = t\mu^{(n)} = P_n T(t)$, and $P_n i_F \mu = i_F \mu^{(n)} = P_n \mu_F$, so that $t\mu = T(t)$ for all $t \in M$ and $i_F\mu = \mu_F$. This completes the proof.

THEOREM 2.3. *Let M be a totally disconnected compact metric space and let G be a totally disconnected complete separable metric space. Suppose that $\{T_n\}$ is a sequence converging to T in $C(M, P(G))$ and that μ is a measure in $P(C(M, G))$ such that $t\mu = T(t)$ for all t. Then there is a sequence $\{\mu_n\}$ in $P(C(M, G))$ such that $\mu_n \to \mu$ and $t\mu_n = T_n(t)$ for all t. This is equivalent to saying that the natural map $\phi: P((M, G)) \to C(M, P(G))$ as described in the introduction is open.*

Proof. Let $D = \{T_n: n = 1, 2, \cdots\} \cup \{T\}$ with the subspace topology. Then the product space $D \times M$ is totally disconnected compact metric, and the function $\psi: D \times M \to P(G)$ defined as $\psi(S, t) = S(t)$ is continuous. Consider the closed subspace $F = \{T\} \times M$ of $D \times M$. The measure μ, regarded as a measure on $C(\{T\} \times M, G)$, satisfies $\psi(s) = s\mu$ for every $s \in F$. Therefore, by Lemma 2.2, there is $\tilde{\mu} \in P(C(D \times M, G))$ such that $s\tilde{\mu} = \psi(s)$ for all $s \in D \times M$, and $i_F\tilde{\mu} = \mu$. For each $S \in D$, define a measure $S\tilde{\mu} \in P(C(M, G))$ by

$$S\tilde{\mu}(K) = \tilde{\mu}\{f \in C(D \times M, G): f(S, \cdot) \in K\} .$$

Then if we set $\mu_n = T_n\tilde{\mu}$, we obtain $t\mu_n = T_n(t)$ for all $t \in M$, and $\mu_n \to \mu$ in $P(C(M, G))$. This completes the proof.

Recall that the natural map $\phi: P(C(M, G)) \to C(M, P(G))$ in the introduction is defined as $\phi(\mu)(t) = t\mu$, for $\mu \in P(C(M, G))$, $t \in M$. By Theorem 2.3, ϕ is an open mapping. Moreover, we have the following:

THEOREM 2.4. *There is a continuous function* $\xi: C(M, P(G)) \to P(C(M, G))$ *such that* $\xi(T) \in \phi^{-1}(T)$ *for all* T.

Proof. Let $\mathscr{M}(C(M, G))$ be the space of all finite signed measures on $C(M, G)$ with the weak topology as functionals on $C(C(M, G))$. Then, $P(C(M, G))$ is a closed subspace of $\mathscr{M}(C(M, G))$. It is clear that the closed convex hull of every compact subset of $P(C(M, G))$ is compact. Let $2^{P(C(M,G))}$ be the collection of all non-empty subsets of $P(C(M, G))$. Define a function $\tilde{\phi}: C(M, P(G)) \to 2^{P(C(M,G))}$ by

$$\tilde{\phi}(T) = \phi^{-1}(T) .$$

Since ϕ is an open mapping, it follows that $\tilde{\phi}$ is lower semi-continuous [8]. Therefore, there is a continuous function $\xi: C(M, P(G)) \to P(C(M, G))$ such that $\xi(T) \in \tilde{\phi}(T) = \phi^{-1}(T)$ [8]. This completes the proof.

In the following, we will establish two lemmas from which we will obtain the same results of Lemma 2.2, Theorem 2.3, and Theorem 2.4 in replacing the space G by an arbitrary Polish space X.

LEMMA 2.5. *Let X be a Polish space. Then there is a totally disconnected complete separable metric space G, a continuous function $\phi: G \to X$ and a continuous function $\tilde{\phi}: P(X) \to P(G)$ such that $\phi\tilde{\phi}(\mu) = \mu$ for all $\mu \in P(X)$. Moreover, $\tilde{\phi}$ is an affine map:*

$$\tilde{\phi}\Big(\frac{1}{2}\mu + \frac{1}{2}\nu\Big) = \frac{1}{2}\tilde{\phi}(\mu) + \frac{1}{2}\tilde{\phi}(\nu) .$$

Proof. Let F_n be a countable partition of unity of X, subordinated by a cover of diameter $\leqq 1/n$. Give F_n the discrete topology. Let

$$X_n = \{(g_1, \cdots, g_n) \mid g_i \in F_i,\ g_1 \cdots g_n \neq 0\}$$

and

$$G = \{(g_1, \cdots, g_n, \cdots) \mid (g_1, \cdots, g_n) \in X_n \quad \text{for all } n\} .$$

Give G the subspace topology of the product space $X_{i=1}^{\infty} F_i$. Then G is a totally disconnected, complete separable metric space. Define $\phi\colon G \to X$ by $\phi(g_1, \cdots, g_n, \cdots)$ to be the unique point $x \in \operatorname{Supp} g_n$ for all n. Then ϕ is continuous. Let $P_n\colon G \to X_n$ be the canonical projection. Define $\tilde{\phi}\colon P(X) \to P(G)$ by setting $\tilde{\phi}(\mu)$ to be the unique measure $\tilde{\mu} \in P(G)$ such that $P_n\tilde{\mu}\{(g_1, \cdots, g_n)\} = \int g_1 \cdots g_n d\mu$, for all $(g_1, \cdots, g_n)$ in X_n and for all n (via the Kolomogorov consistency theorem [4, p. 120]). It is clear that $\tilde{\phi}$ is continuous and $\phi\tilde{\phi}(\mu) = \mu$ for all $\mu \in P(X)$. It is also easy to check that

$$\tilde{\phi}\Big(\frac{1}{2}\mu + \frac{1}{2}\nu\Big) = \frac{1}{2}\tilde{\phi}(\mu) + \frac{1}{2}\tilde{\phi}(\nu) .$$

The proof is complete.

LEMMA 2.6. *Let X, G, ϕ, $\tilde{\phi}$ be as in Lemma 2.5. Let $T \in C(M, P(X))$, $\mu \in P(C(M, X))$ be such that $T(t) = t\mu$ for all t. Then there is a measure $\hat{\mu} \in P(C(M, G))$ such that $\tilde{\phi}T(t) = t\hat{\mu}$ for all t, and $\phi\hat{\mu} = \mu$.*

Proof. Since $C(M, X)$ is a polish space, there is a compact subset L of $C(M, X)$ such that $\mu(L) \geqq 1 - \varepsilon$. We may assume that $\operatorname{Supp} \mu$ is a compact subset L of $C(M, X)$. Each $f \in L$ is regarded as a continuous function from M to $P(X)$. Let $\tilde{f} = \tilde{\phi}f$, then $\tilde{f} \in C(M, P(G))$. Let $\xi\colon C(M, P(G)) \to P(C(M, G))$ be a continuous function as in Theorem 2.4. Denote $\xi(\tilde{f})$ by $\hat{\mu}_f$, then $t\hat{\mu}_f = \tilde{f}(t)$ for all t. Let $\mu_f = \phi\hat{\mu}_f$, then $t\mu_f = f(t)$ for all t. Therefore, $\mu_f = \varepsilon_f$, the unit point-mass measure putting mass 1 on the point $f \in C(M, X)$. Since L is compact and the function $f \to \hat{\mu}_f$ is continuous, the set $\{\hat{\mu}_f \mid f \in L\}$ is compact. Hence for $\varepsilon > 0$, there is a compact set $\tilde{L} \subseteqq C(M, G)$ such that $\hat{\mu}_f(\tilde{L}) > 1 - \varepsilon$ for all $f \in L$ [9, p. 47]. Now $\mu \in P(L)$, there is a sequence $\mu_n \to \mu$ in $P(L)$ with $\mu_n = \sum_{i=1}^{k} \lambda_i \varepsilon_{f_i}$, for some $\lambda_i > 0$, $\sum \lambda_i = 1$ and $f_i \in L$. Let $\hat{\mu}_n = \sum_{i=1}^{k} \lambda_i \hat{\mu}_{f_i}$, then

$\phi\hat{\mu}_n = \mu_n$ and

$$t\hat{\mu}_n = \sum_{i=1}^{k} \lambda_i \tilde{f}_i(t) = \sum_{i=1}^{k} \lambda_i \tilde{\phi}(f_i(t))$$
$$= \tilde{\phi}\left(\sum_{i=1}^{k} \lambda_i \varepsilon_{f_i(t)}\right) = \tilde{\phi}(T_n(t))$$

for all t, where $T_n(t) = t\mu_n$. Since $\mu_n \to \mu$, so that $T_n(t) \to T(t)$ in $P(X)$. Also, $\hat{\mu}_n(\tilde{L}) > 1 - \varepsilon$ for all n. Hence $\{\hat{\mu}_n: n = 1, 2, 3, \cdots\}$ is relative compact. Therefore some subsequence $\hat{\mu}_{n_k} \to \hat{\mu}$ in $P(C(M, G))$. Since $\tilde{\phi}(T_{n_k}(t)) = t\hat{\mu}_{n_k}$ and $\phi(\hat{\mu}_{n_k}) = \mu_{n_k}$, letting $k \to \infty$, we obtain $\tilde{\phi}(T(t)) = t\hat{\mu}$ for all t, and $\phi\hat{\mu} = \mu$. This completes the proof.

THEOREM 2.7 (Extension theorem). *Let M be a totally disconnected compact metric space, F be a closed subset of M, and $i_F: F \to M$ be the natural inclusion. Let X be a Polish space. Let T be a continuous function from M into $P(X)$, and μ_F be a measure in $P(C(F, X))$ such that $t\mu_F = T(t)$ for all $t \in F$. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all $t \in M$ and $i_F\mu = \mu_F$.*

Proof. By Lemmas 2.5 and 2.6, there is a totally disconnected complete separable metric space G and there are continuous functions $\phi: G \to X$, $\tilde{\phi}: P(X) \to P(G)$ and a measure $\hat{\mu}_F \in P(C(F, G))$ such that $\phi\hat{\mu}_F = \mu_F$ and $t\hat{\mu}_F = \tilde{\phi}(T(t))$ for all $t \in F$. By Lemma 2.2, there is a measure $\hat{\mu} \in P(C(M, G))$ such that $t\hat{\mu} = \tilde{\phi}(T(t))$ for all $t \in M$, and $i_F\hat{\mu} = \hat{\mu}_F$. Let $\mu = \phi\hat{\mu}$. Then μ is a measure in $P(C(M, X))$ such that $t\mu = \phi(\tilde{\phi}T(t)) = T(t)$ for all $t \in M$ and $i_F u = \mu_F$. This completes the proof.

Therefore, by following the same arguments as in Theorem 2.3 and Theorem 2.4, we obtain:

THEOREM 2.8. *Let M be a totally disconnected compact metric space, and let X be a Polish space. Then the natural map $\phi: P(C(M, X)) \to C(M, P(X))$, defined as $\phi(\mu)(t) = t\mu$, is open, and there is a continuous function $\xi: C(M, P(X)) \to P(C(M, X))$ such that $\xi(T) \in \phi^{-1}(T)$ for all T.*

As an application of Theorem 2.7, we obtain:

COROLLARY 2.9. *Let M and X be as in Theorem 2.8, and let $T: M \times M \to P(X \times X)$ be continuous. Let $\Delta_M = \{(t, t): t \in M\}$. For any subset $A \subseteq X$, let $\Delta_A = \{(a, a): a \in A\}$. Suppose μ_Δ is a measure in $P(C(M, X))$ such that $t\mu_\Delta(A) = T(t, t)(\Delta_A)$ for all $t \in M$. Regard μ_Δ*

as a measure on $C(\Delta_M, \Delta_X)$. *Then there is a measure* $\mu \in P(C(M \times M, X \times X))$ *representing* T *such that* $i_\Delta \mu = \mu_\Delta$, *where* $i_\Delta \colon \Delta_M \to M \times M$ *is the natural inclusion.*

We remark that the function ξ in Theorem 2.8 is a continuous selection function which selects for each T in $C(M, P(X))$ a measure $\mu_T(=\xi(T))$ in $P(C(M, X))$ representing T.

3. Representation when $X = S^1$

THEOREM 3.1. *Let* M *be a separable metric space, and let* S^1 *be the unit sphere in* R^2. *Let* $0 < k \leqq 2\pi$, *and let* $T\colon M \to P(S^1)$ *be a continuous function such that for each* $t \in M$, $\operatorname{Supp} T(t)$ *is connected and of arc length* $\leqq 2\pi - k$. *Then there is a measure* $\mu \in P(C(M, S^1))$ *such that* $T(t) = t\mu$ *for all* t.

Proof. Let $P_k(S^1) = \{\mu \in P(S^1)\colon \operatorname{Supp}(\mu)$ is connected and has arc length at most $2\pi - k\}$. Given $r < t$, set $(r, t) = \{e^{is}\colon r < s < t\}$ and set $[r, t] = \{e^{is}\colon r \leqq s \leqq t\}$. Following an idea in [3, Lemma 2.1], we construct a mapping $Z\colon P_k(S^1) \times (0, 1) \to S^1$ such that

$$(*) \qquad P\{\omega\colon Z(\mu, \omega) \in (a, b)\} = \mu(a, b)$$

for each $a < b$ where P is Lebesgue measure. Namely, given $\mu \in P_k(S^1)$, choose a_μ and b_μ such that $\operatorname{Supp}(\mu) = [a_\mu, b_\mu]$ and set $Z(\mu, \omega) = e^{is}$ where $s = \inf\{t \geqq a_\mu\colon \mu[a_\mu, t] \geqq \omega\}$. We see that $Z(\mu, \omega)$ is well-defined since $\operatorname{Supp}(\mu) \neq S^1$. Suppose $\mu_n \to \mu$ where $\mu_n \in P_k(S^1)$. Set $I_n = S^1 \backslash \operatorname{Supp}(\mu_n)$ and set $S^1 \backslash \operatorname{Supp}(\mu) = (a, b)$. Set $J = (a + k/3, b - k/3)$. By considering the geometry of S^1, we see that $J \subseteqq \operatorname{Supp}(\mu_n)$ for each n is impossible. Hence, $J \cap I_n \neq \varnothing$ for n large. One may check that if $\mu_n \to \mu$, that $Z(\mu_n, \omega) \to Z(\mu, \omega)$. One checks directly that $(*)$ holds.

Let Ω be the open interval $(0, 1)$. For each $\omega \in \Omega$, $t \in M$, define $X_t(\omega)$ by $X_t(\omega) = Z(T(t), \omega)$. Thus, for a fixed $\omega \in \Omega$, the function $t \to X_t(\omega)$ is a continuous function from M into S^1. Therefore, we may define a mapping $X\colon \Omega \to C(M, S^1)$ by $X(\omega)(t) = X_t(\omega)$. One checks easily that X is Borel measurable and $P\{\omega\colon X_t(\omega) \in (a, b)\} = T(t)(a, b)$ for any arc (a, b) in S^1. Define a measure $\mu \in P(C(M, S^1))$ by $\mu(A) = P(X^{-1}(A))$. Then $t\mu = T(t)$ for all t. This completes the proof.

EXAMPLE 3.2.

(a) This example shows that the requirement of an upper bound $2\pi - k < 2\pi$ for the lengths of all $\operatorname{Supp} T(t)$ cannot be entirely eliminated.

Let $M = D^1 = \{x \in R^2 \colon |x| \leqq 1\}$, $X = S^1$, and let T be a continuous function from M into $P(S^1)$ such that $T(t) = \varepsilon_t$, the unit point mass measure at the point t, for each $t \in S^1$. Then any measure representing T must attribute all its mass to those continuous functions from D^1 to S^1 which are the identity on S^1. Since there is no such function, no such T has a representing measure. An example of such a T in which each $T(t)$ is supported by a subarc of S^1 is:

$$T(re^{it}) = r^2\varepsilon_t + \left[(1 - r^2) - \left(\frac{2\pi - t}{2\pi}\right)(r - r^2)\right]\varepsilon_0$$
$$+ \left(\frac{2\pi - t}{2\pi}\right)(r - r^2)l_{[0,t]}\,, \quad 0 \leqq r \leqq 1\,, \quad 0 \leqq t \leqq 2\pi\,,$$

where $l_{[0,t]}$ is the probability measure uniformly distributed on the arc $[0, t] = \{e^{is} \colon 0 \leqq s \leqq t\}$.

Note that there is no bound on the arc lengths of the supports as required for Theorem 3.1.

(b) The following example shows that for a continuous function T from S^1 into $P(S^1)$, all supports of $T(t)$ connected is not enough for representation.

Define $T \colon S^1 \to P(S^1)$ as follows:
We regard S^1 as the quotient space by identifying 0 and 2π in the closed interval $[0, 2\pi]$.

Let

$$t_n = \left(\frac{2^n - 2}{2^n}\right)2\pi \quad n = 1, 2, 3, \cdots .$$

Define $T(t_n) = \varepsilon_{t_n}$, and

$$T\left(\frac{t_n + t_{n+1}}{2}\right) = \frac{1}{2^n}l_{(S^1\setminus J_n^\circ)} + \frac{\left(1 - \frac{1}{2^n}\right)}{2}T(t_n)$$
$$+ \frac{\left(1 - \frac{1}{2^n}\right)}{2}T(t_{n+1})\,,$$

where $J_n^\circ = \{e^{it} \colon t_n < t < t_{n+1}\}$ and $l_{S^1\setminus J_n^\circ}$ is the probability measure uniformly distributed on $S^1\setminus J_n^\circ$. Define T piece-wise linear on $[t_n, (t_n + t_{n+1})/2]$ and on $[(t_n + t_{n+1})/2, t_{n+1}]$. Define $T(2\pi) = T(0) = \varepsilon_0$. Then T is continuous and $\operatorname{Sup} T(t)$ is connected for each t. Let $A_n = \{f \in C(S^1, S^1) \mid f(t_n) = t_n,\ f(t_{n+1}) = t_{n+1},\ f(t_n, t_{n+1}) \cap (t_n, t_{n+1}) = \varnothing\}$. Note that if μ represents T, then $\mu(A_n) = 1$. Therefore, $\mu(\overline{\lim}_n A_n) = 1$. But $\overline{\lim}_n A_n = \varnothing$. Therefore, T cannot be represented by a measure.

4. Some generalizations. We will present here some results

for our representation theorem. As you will see, most of them generalize the results of (a) and (b) as described in the introduction. The proofs of these theorems are, by no means, difficult, and they can be found in [5].

THEOREM 4.1. *Let M be a compact extremally disconnected space, and let X be a Prohorov space. Let $T: M \to P_\tau(X)$ be a continuous function. Then there is a tight measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t.*

THEOREM 4.2. *Let M be a separable metric space, and let I be the unit interval. Let $T: M \to P(I)$ be continuous such that* $\operatorname{Supp} T(t)$ *is a connected subset of I for all t. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t.*

THEOREM 4.3. *Let M be a separable metric space, and let X be a Peano space. Let $T: M \to P(X)$ be continuous such that the set $M_1 = \{t \in M: \operatorname{Supp} T(t) \neq X\}$ is at most countable. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t.*

THEOREM 4.4. *Let M be a separable metric space, and let X be a metric space such that there is a sequence of open sets U_i with compact closures K_i satisfying*

(a) *$U_i \subseteqq K_i \subseteqq U_{i+1}$ for all i,*

(b) *K_i, with the relative topology, is a Peano space, and*

(c) *$X = \bigcup_{i=1}^{\infty} K_i$.*

(e.g., $X = R^n$, for any integer $n > 0$.)

Let T be a continuous function from M into $P(X)$ such that $\operatorname{Supp} T(t) = X$ for all t. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t.

THEOREM 4.5. *Let M be a countable metric space, and let X be a metric space. Let $T: M \to P_\tau(X)$ be continuous. Then there is a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t. As a consequence, for any countable closed subset F of $P_\tau(X)$, F is compact if and only if F is tight.*

One may wonder how far we can go in this representation theorem. Among the difficulties, one notes that the functions in $C(M, X)$ are not enough in general. However, the following question remains unknown, and is worthy to work with:

Let M be a totally disconnected compact metric space, and let X be a Prohorov space. Let T be a continuous function from M

into $P_\tau(X)$. Is there a measure $\mu \in P(C(M, X))$ such that $T(t) = t\mu$ for all t?

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