

OBSTRUCTIONS TO PSEUDOISOTOPY IMPLYING ISOTOPY FOR EMBEDDINGS

KENNETH C. MILLETT

A key theorem in efforts to determine the homotopy groups of spaces of automorphisms and embeddings has been Morlet's "disjonction theorem" or, equivalently, the author's "concordance implies isotopy theorem." Although it was known that the conclusions of these theorems could not be true without some restrictive hypotheses the only explicit nonzero obstructions occurred in the, possibly anomalous, cases of codimensions 0, 1, 2 and n . In this paper nonzero obstructions are determined for each of the remaining cases showing that the theorems are best possible.

Let $I = [-1, 1]$, $D^p = I^{\overline{p}}$, $S^p = \partial D^{p+1}$, and, for any compact piecewise linear manifold M , let the symbol \sqcup denote the subset $M \times \{-1\} \cup \partial M \times I$ of $M \times I$. Given piecewise linear manifolds $(M, \partial M) \subset (N, \partial N)$ and a subset A of M , let $E(M, N; \text{rel } A)$ denote the Δ -set of locally flat piecewise linear embeddings of $(M, \partial M, \text{int } M)$ in $(N, \partial N, \text{int } N)$ which are fixed on A .

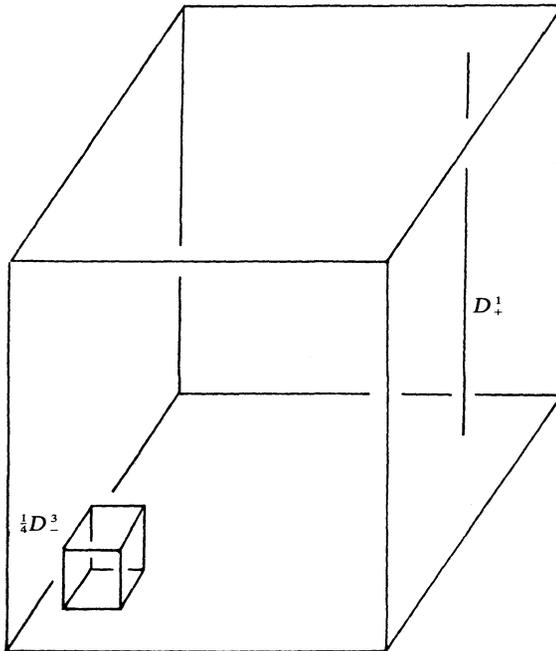


FIGURE 1. $D^1 \subset C D^3 \setminus \frac{1}{4} D^3$

In order to describe the obstructions we require an explicit description of the manifolds in which they arise. Thus let $D_+^p = (\frac{1}{2}, \dots, \frac{1}{2}) \times D^p \subset D^n$, $\frac{1}{4}D^n = \{(x_1, \dots, x_n) \in D^n \mid \max_i |x_i + \frac{1}{2}| \leq \frac{1}{4}\}$, and choose a piecewise linear homeomorphism $h: D^n \setminus \text{int } \frac{1}{4}D^n \rightarrow S^{n-1} \times I$. We shall consider the embedding of D^p as a submanifold of $S^{n-1} \times I$ given by

$$D^p \rightarrow D_+^p \hookrightarrow (D^n \setminus \text{int } \frac{1}{4}D^n) \xrightarrow{h} S^{n-1} \times I.$$

Finally, let $\Sigma_{i,n}^k: \Pi_i(S^n) \rightarrow \Pi_{i+k}(S^{n+k})$ denote the k -fold suspension homomorphism. The precise statement of the result is:

THEOREM 1.1. *If $n \geq 5$ and $n - p \geq 3$, there is an epimorphism of*

$$\prod_{2n-p-4} (E(D^p \times I, (S^{n-1} \times I) \times I; \text{rel } \sqcup))$$

onto the following nonzero groups; if $n \neq 8$, $\ker \Sigma_{2n-3, n-1}^{n-p-2}$, if $n = 8$ and $p = 5$, \mathbf{Z} , and if $n = 8$ and $p < 5$, \mathbf{Z}_2 .

Because the above group is never zero, the theorem implies that the following theorems are best possible.

THEOREM 1.2. (Morlet's "Lemme de Disjonction", [1, 7]). *Let D^p and D^q be disjoint submanifolds of N with $n - p \geq 3$, then*

$$\prod_i (E(D^p \times I, N \times I; \text{rel } \sqcup), E(D^p \times I, (N \setminus D^q) \times I; \text{rel } \sqcup)) = 0$$

for $i \leq 2n - p - q - 4$.

THEOREM 1.3 ("Concordance implies Isotopy", [5]). *Let D^p be a submanifold of N , $n - p \geq 3$, and $\Pi_j(N) = 0$ for $j \leq k$, then*

$$\prod_i (E(D^p \times I, N \times I; \text{rel } \sqcup)) = 0$$

for $i \leq n + k - p - 3$.

Letting $N = S^{n-1} \times I$, so that $k = n - 2$, Theorem 1.1 shows immediately that Theorem 1.3 is best possible. Letting $D^1 = * \times I \subset S^{n-1} \times I = N$, the fact (proved via an Alexander isotopy argument, cf. [1, 5]) that

$$\prod_i (E(D^p \times I, (N \setminus D^1) \times I; \text{rel } \sqcup)) = 0$$

implies, via the exact sequence of the pair, that Theorem 1.2 is best possible.

The proof of Theorem 1.1 consists of a reduction to a homotopy theoretic problem using a result of [6]. This homotopy problem is then analyzed via classical techniques involving the homotopy groups of wedges of spheres, Whitehead products, and the suspension homomorphism.

I wish to thank L. Larmore for several helpful conversations.

The Calculation.

Let $\text{Aut}(M; \text{rel } A)$ denote the Δ -set of piecewise linear automorphisms of a manifold M which are fixed on the subset A . Consider the fibrations

$$\begin{array}{ccc} \text{Aut}(D^n \times I; \text{rel } \sqcup \cup (* \cup D_+^p) \times I) & \text{Aut}(D^n \times I; \text{rel } \sqcup \cup (* \cup D_+^p) \times I) \\ \downarrow & \downarrow \\ \text{Aut}(D^n \times I; \text{rel } \sqcup \cup * \times I) & \text{Aut}(D^n \times I; \text{rel } \sqcup \cup D_+^p \times I) \\ \downarrow & \downarrow \\ E(D^p \times I, (D^n \setminus *) \times I; \text{rel } \sqcup) & E(I, (D_+^n \setminus D^p) \times I; \text{rel } \{-1\}) \end{array}$$

where $* = (-\frac{1}{2}, \dots, -\frac{1}{2})$ and $D_+^p = (\frac{1}{2}, \dots, \frac{1}{2}) \times D^p \subset D^n$. The base spaces

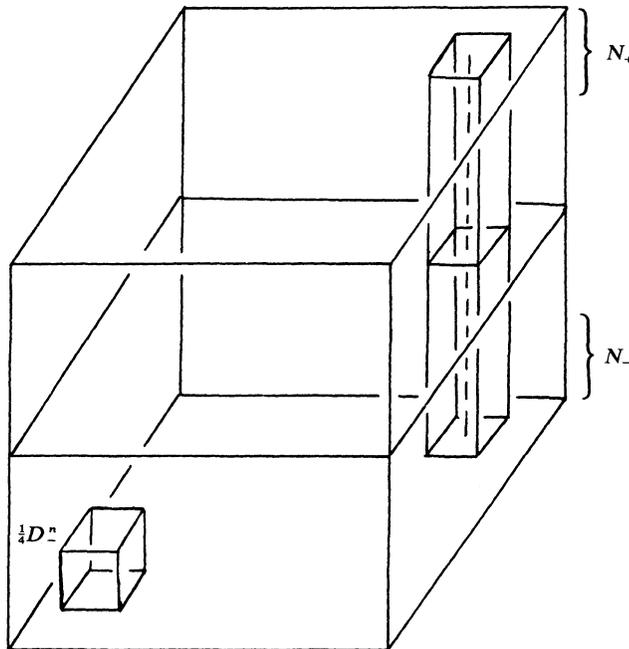


FIGURE 2. $* \in \text{Model for } (S^{n-p-1} \times D^{p+1}) \times I = N$.

are homotopy equivalent to $E(D^p \times I, (S^{n-1} \times I) \times I; \text{rel } \sqcup)$ and $E(I, (S^{n-p-1} \times D^{p+1}) \times I; \text{rel } \{-1\})$, respectively, where $D^p \subset S^{n-1} \times IC(D^n \setminus *,$ as described in the introduction, and $I = * \times IC(S^{n-p-1} \times D^{p+1}) \times IC(D^n \setminus D^p) \times I$. Since both total spaces are contractible and the fibers are equal, the homotopy groups of these two complexes are isomorphic. This it suffices to show that, for $n \geq 5$,

$$\prod_{2n-p-4} (E(I, (S^{n-p-1} \times D^{p+1}) \times I; \text{rel } \{-1\})) \neq 0.$$

Let $N = S^{n-p-1} \times D^{p+1}$, $N_+ = S^{n-p-1} \times D^p \times [0, 1]$, $N_- = S^{n-p-1} \times D^p \times [-1, 0]$, $* \in \text{interior } \frac{1}{4}D^n \subset \text{interior } N_-$, and $\hat{N} = N \setminus (\text{interior } \frac{1}{4}D^n)$.

It follows from the splitting of the exact homotopy sequence of the fibration,

$$\begin{array}{c} E(I, N \times I; \text{rel } \partial) \\ \downarrow \\ E(I, N \times I; \text{rel } \{-1\}) \\ \downarrow \\ E(\{+1\}, N) \end{array}$$

and the fact that $E(\{+1\}, N)$ has the homotopy type of N , that $\prod_i (E(I, N \times I; \text{rel } \partial))$ is isomorphic to $\prod_{i+1} (N) \oplus \prod_i (E(I, N \times I; \text{rel } \{-1\}))$.

By Theorem 3.18 [6] there is a homomorphism

$$\prod_i (E(I, N \times I; \text{rel } \partial)) \rightarrow \prod_{i+1} (h_\partial(N), h_\partial(N, *))$$

which is an isomorphism for $i \leq 2n - p - 5$ and an epimorphism for $i = 2n - p - 4$. Here $h_\partial(N)$ denotes the Δ -set of homotopy equivalences of N which are fixed on ∂N and $h_\partial(N, *)$ denotes the sub Δ -set of those homotopy equivalences which preserve the basepoint $*$ and its complement in N .

The homomorphism can be defined as follows: Let $f: D^i \times I \rightarrow D^i \times (N \times I)$ represent an element of $\prod_i (E(I, N \times I; \text{rel } \partial))$. By the isotopy extension theorem [4], there is an extension of f , $F_0: D^i \times (N \times I) \rightarrow D^i \times (N \times I)$ such that $F_0|_{D^i \times (N \times I)} = 1$ and $F_0|_{D^i \times \partial(N \times I)} = 1$. A homotopy, F_t , is then constructed, respecting the D^i -coordinates, such that $F_t|_{D^i \times (N \times I)} = 1$, $F_t|_{D^i \times \partial(N \times I)} = 1$, $F_t|_{D^i \times ((N \setminus *, *) \times I)} \rightarrow D^i \times ((N \setminus *, *) \times I)$, and such that F_1 respects the I -coordinate. F_1 then represents an element of

$\Pi_i(\Omega h_\partial(N), \Omega h_\partial(N, *))$ whose image via the isomorphism to $\Pi_{i+1}(h_\partial(N), h_\partial(N, *))$ is defined to be the image of $[f]$.

The remainder of the paper is devoted to showing that $\Pi_{i+1}(h_\partial(N), h_\partial(N, *))$ is isomorphic to $\Pi_{i+1}(N)$ for $i \leq 2n - p - 5$, verifying Theorems 1.2 and 1.3 in this special case, and showing that if $n \geq 5$, $\Pi_{2n-p-3}(h_\partial(N), h_\partial(N, *))$ is isomorphic to $\Pi_{2n-p-3}(N)$ plus a nontrivial group, proving Theorem 1.1.

The determination of $\Pi_i(h_\partial(N), h_\partial(N, *))$ is rather indirect and depends critically upon the specific N under consideration. The basic idea is to analyze the exact sequence,

$$\rightarrow \prod_{i+1} (h_\partial(N, *), h_\partial(\hat{N})) \rightarrow \prod_{i+1} (h_\partial(N), h_\partial(\hat{N})) \rightarrow \prod_{i+1} (h_\partial(N), h_\partial(N, *)) \rightarrow .$$

What we actually do is describe an exact sequence for $\Pi_{i+1}(h_\partial(N), h_\partial(\hat{N}))$ and an analogous sequence for $\Pi_{i+1}(h_\partial(N, *), h_\partial(\hat{N}))$ which includes as direct summand. The cokernel of this inclusion of sequences gives an exact sequence for $\Pi_{i+1}(h_\partial(N), h_\partial(N, *))$ and reduces the computation to classical homotopy theory.

The first step is to consider the fibration (onto the arc component of the identity)

$$h_{\partial \cup \text{nb}d(*)}(N, *) \rightarrow h_\partial(N, *) \xrightarrow{g} gh(D^n, *)$$

where g is “taking the germ at $*$ ”, and $gh(D^n, *)$ denotes the germs of homotopy equivalence of $(D^n, *)$ at $*$. $h_{\partial \cup \text{nb}d(*)}(N, *)$ is homotopy equivalent to $h_\partial(\hat{N})$ and $gh(D^n, *)$ is homotopy equivalent to $h(S^{n-1})$. Thus $\Pi_{i+1}(h_\partial(N, *), h_\partial(\hat{N}))$ is isomorphic to $\Pi_{i+1}(h(S^{n-1}))$ which occurs in the following exact sequence

$$\rightarrow \prod_{i+1} (h(S^{n-1})) \xrightarrow{ev} \prod_{i+1} (S^{n-1}) \xrightarrow{\partial} \prod_{i+n-1} (S^{n-1}) \rightarrow .$$

Here ev denotes the evaluation at a basepoint and the ∂ homomorphism is $\partial(\alpha) = [\alpha, \iota_{n-1}]$, the Whitehead product with the generator ι_{n-1} of $\Pi_{n-1}(S^{n-1})$.

To describe the sequence for $\Pi_{i+1}(h_\partial(N), h_\partial(\hat{N}))$ we consider the exact sequence

$$\rightarrow \prod_{i+1} (h_\partial(N), h_\partial(\hat{N})) \rightarrow \prod_i (h_\partial(\hat{N})) \xrightarrow{\ell} \prod_i (h_\partial(N)) \rightarrow$$

which splits via the homomorphism j induced by taking N to N_+ and extending homotopy equivalences of N_+ to \hat{N} by the identity on the complement. We shall show that there are exact sequences for $h_\partial(\hat{N})$ and $h_\partial(N)$ which respect this splitting and therefore determine a sequence for $\Pi_{i+1}(h_\partial(N), h_\partial(\hat{N}))$. Recall that $N = S^{n-p-1} \times D^{p+1}$ so that we have an exact sequence similar to the previous one,

$$\rightarrow \prod_i (h(N)) \xrightarrow{ev} \prod_{i+p+1} (S^{n-p-1}) \xrightarrow{\partial} \prod_{i+n-1} (S^{n-p-1}) \rightarrow$$

where ev is evaluation on $* \times D^{p+1}$ for a base point $*$ in S^{n-p-1} and ∂ is the Whitehead product with ι_{n-p-1} .

To develop a similar sequence for $\Pi_i(h_\partial(\hat{N}))$ we consider the two stage cofibration decomposition

$$\begin{array}{ccc} S^{i+p} \vee S^i & \xrightarrow{a_2^i} & X_2^i = S^i \times (S^{n-p-1} \times S^p \cup S^{n-1}) \cup * \times \hat{N} \\ & & \downarrow i_2 \\ S^{i+n-1} & \xrightarrow{a_1^i} & X_1^i = X_2^i \cup S^i \times (* \times D^{p+1} \cup \tilde{D}^1) \xrightarrow{p_2} S^{i+p+1} \vee S^{i+1} \\ & & \downarrow i_1 \\ & & X_0^i = S^i \times \hat{N} \xrightarrow{p_1} S^{i+n} \end{array}$$

illustrated in Figure 3, where i_j is the inclusion, p_j is the quotient map, and a_j^i is the attaching map of the cell(s). Also, given a map $f: A \rightarrow Y$ defined on a set $A \subset X$, let $[X, Y]_f$ denote the set of homotopy classes of extensions of f to X .

One then has an exact sequence,

$$\begin{array}{ccc} & \downarrow & \downarrow \\ [X_0^i, \hat{N}]_{p_{i1}} & \cong & \prod_{i+n} (\hat{N}) \\ & \downarrow & \downarrow \\ [X_0^i, \hat{N}]_{p_{i1}i_2} & \cong & \prod_i (h_\partial(\hat{N})) \\ & \downarrow & \downarrow \\ [X_1^i, \hat{N}]_{p_{i1}i_2} & \cong & \prod_{i+p+1} (\hat{N}) \oplus \prod_{i+1} (\hat{N}) \\ & \downarrow \partial & \downarrow \\ [X_0^{i-1}, \hat{N}]_{p_{i1}} & \cong & \prod_{i+n-1} (\hat{N}) \\ & \downarrow & \downarrow \end{array}$$

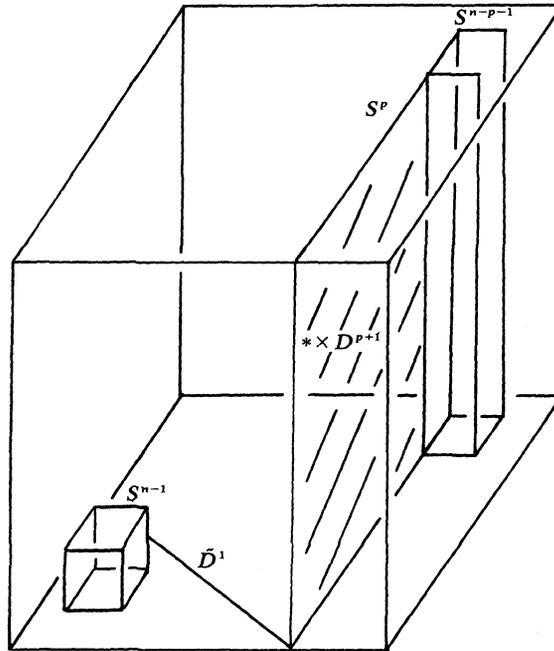


FIGURE 3. Model for $\hat{D}^1, * \times D^{p+1} \subset \hat{N}$.

where the two unlabeled functions are induced by inclusions and p is the projection to \hat{N} . From the first cofiber sequence we observe that $[X_0^{i-1}, \hat{N}]_{pi_1}$ is isomorphic to $\Pi_{i+n-1}(\hat{N})$, from the second we observe that $[X_1^i, \hat{N}]_{pi_2}$ is isomorphic to $\Pi_{i+p+1}(\hat{N}) \oplus \Pi_{i+1}(\hat{N})$, while $[X_0^i, \hat{N}]_{pi_2}$ is, by construction, $\Pi_i(h_\partial(\hat{N}))$. The “boundary” homomorphism takes $[f] \in [X_1^i, \hat{N}]_{pi_2}$ to $[f \circ \alpha_1^i] \in \Pi_{i+n-1}(\hat{N}) = [X_0^{i-1}, \hat{N}]_{pi_1}$.

To describe the associated sequence for $\Pi_i(h_\partial(N), h_\partial(\hat{N}))$ we require precise information relating the sequences and the splitting. First, \hat{N} , has the homotopy type of $S^{n-p-1} \vee S^{n-1}$ so that we must consider the sequence

$$\begin{array}{c}
 \downarrow \\
 \prod_{i+n} (S^{n-p-1} \vee S^{n-1}) \\
 \downarrow \\
 \prod_i (h_\partial(\hat{N})) \\
 \downarrow \\
 \prod_{i+1} (S^{n-p-1} \vee S^{n-1}) \oplus \prod_{i+p+1} (S^{n-p-1} \vee S^{n-1}) \\
 \downarrow \partial \\
 \prod_{i+n-1} (S^{n-p-1} \vee S^{n-1})
 \end{array}$$

and provide an explicit description of ∂ . To achieve this we employ the theorem of Hilton [3] describing the homotopy groups of wedges of spheres. Let ι_{n-p-1} and ι_{n-1} denote the homotopy classes of the inclusions of S^{n-p-1} and S^{n-1} into $S^{n-p-1} \vee S^{n-1}$, respectively. Following Hilton one defines basic Whitehead products of weight 1 as ι_{n-p-1} and ι_{n-1} with the ordering $\iota_{n-p-1} < \iota_{n-1}$. Given basic products of weight less than w , a basic product of weight w , greater than 1, is a product $[a, b]$ where a and b are basic products whose weights add to w , $a < b$, and if b is a Whitehead product $[c, d]$, then $c \leq a$. Basic products are ordered by weight and, except for $w = 1$, arbitrarily within a given weight. A Whitehead product ρ involving k ι_{n-p-1} 's and l ι_{n-1} 's is a specific map from $S^{h(\rho)+1}$ into $S^{n-p-1} \vee S^{n-1}$ where $h(\rho) = k(n-p-2) + l(n-2)$. Hilton's theorem states that, if $n-1$ and $n-p-1$ are at least 2,

$$\prod_i (S^{n-p-1} \vee S^{n-1}) \cong \bigoplus_{\text{basic products } \rho} \prod_i (S^{h(\rho)+1})$$

where the isomorphism is given by taking $\alpha \in \Pi_i(S^{h(\rho)+1})$ to $\rho \circ \alpha$.

The boundary homomorphism takes an element $\beta \in \Pi_{i+1}(S^{n-p-1} \vee S^{n-1})$ to $[\iota_{n-1}, \beta] \in \Pi_{i+n-1}(S^{n-p-1} \vee S^{n-1})$ and an element $\gamma \in \Pi_{i+p+1}(S^{n-p-1} \vee S^{n-1})$ to $[\iota_{n-p-1}, \gamma] \in \Pi_{i+n-1}(S^{n-p-1} \vee S^{n-1})$. We shall only describe the argument for β as the case for γ is quite similar. Given $\beta \in \Pi_{i+1}(S^{n-p-1} \vee S^{n-1})$, represented by a map $g: (D^i \times D^1, \partial(D^i \times D^1)) \rightarrow (S^{n-p-1} \vee S^{n-1}, *)$, one defines an associated map $\tilde{g}: (S^i \times D^1, S^i \times S^0) \rightarrow (S^{n-p-1} \vee S^{n-1}, *)$ since $S^{i-1} \times D^1$ goes to $*$. The element $[f] \in [X_i^i, \hat{N}]_{i \hat{k}}$ which corresponds to β is defined by taking \tilde{g} on $S^i \times D^1$, the constant map on $S^i \times D^{p+1}$, and projection to \hat{N} on the remainder of X_i^i . To show that the composition $[f \circ a_i^i]$ is $[\iota_{n-1}, \beta]$ we must use the geometry of X_i^i to provide a description of $[f \circ a_i^i]$ which makes this clear. The map a_i^i on $S^{i+n-1} = \partial(D^i \times D^n) = S^{i-1} \times D^n \cup D^i \times S^{n-1}$ is given by taking $S^{i-1} \times D^n$ to $* \times \hat{N}$ in X_i^i by the projection to the top cell of \hat{N} and by taking $D^i \times S^{n-1}$ into X_i^i by projecting D^i to S^i on the first factor and taking the attaching map of the top cell of \hat{N} on the second factor. When composed with f this gives projection onto S^{n-1} in $S^{n-p-1} \vee S^{n-1}$ on $S^{i-1} \times D^n = (S^{i-1} \times D^1) \times D^{n-1}$ where D^1 corresponds to the D^1 in the decomposition of \hat{N} . On $D^i \times S^{n-1} = (D^i \times S^0) \times D^{n-1} \cup D^{i+1} \times S^{n-2}$ the composition is given by projection to the S^{n-1} on the first part of the union, as above, and by β on the second part of the union. Noting that $(S^{i-1} \times D^1) \times D^{n-1} \cup (D^i \times S^0) \times D^{n-1} = S^i \times D^{n-1}$ we see that the composition is given by projection to the S^{n-1} on $S^i \times D^{n-1}$ and by β on $D^{i+1} \times S^{n-2}$, which is precisely the definition of $[\iota_{n-1}, \beta]$.

We are now able to describe $\Pi_{i+1}(h_\partial(N), h_\partial(\hat{N}))$ by considering the

commutative diagram of exact sequences

$$\begin{array}{ccc}
 & \downarrow & \downarrow \\
 & \prod_i (h_\partial(\hat{N})) \cong \prod_i (h_\partial(N)) & \\
 & \downarrow & \downarrow \\
 \bigoplus_\rho \prod_{i+1} (S^{h(\rho)+1}) \oplus \bigoplus_\rho \prod_{i+p+1} (S^{h(\rho)+1}) & \cong & \prod_{i+p+1} (S^{n-p-1}) \\
 & \downarrow & \downarrow \\
 \bigoplus_\rho \prod_{i+n-1} (S^{h(\rho)+1}) & \xrightarrow{\cong} & \prod_{i+n-1} (S^{n-p-1}) \\
 & \downarrow & \downarrow
 \end{array}$$

where the inclusion of the right hand sequence is given by the splitting homomorphism at the top and as the $\rho = \iota_{n-p-1}$ summand in the other cases. By the calculations above, this commutes with the boundary homomorphisms and is a splitting of the left sequence. As a consequence we have the following exact sequence

$$\begin{array}{c}
 \prod_{i+1} (h_\partial(N), h_\partial(\hat{N})) \\
 \downarrow \\
 \bigoplus_\rho \prod_{i+1} (S^{h(\rho)+1}) \oplus \bigoplus_{\rho \neq \iota_{n-p-1}} \prod_{i+p+1} (S^{h(\rho)+1}) \\
 \downarrow \partial \\
 \bigoplus_{\rho \neq \iota_{n-p-1}} \prod_{i+n-1} (S^{h(\rho)+1}) \\
 \downarrow
 \end{array}$$

with ∂ computed above. Finally, returning to the exact sequence of the triple $(h_\partial(N), h_\partial(N, *), h_\partial(\hat{N}))$ we have a commutative diagram of exact sequences,

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & \\
 \rightarrow \prod_{i+1} (h_\partial(N, *), h_\partial(\hat{N})) & \longrightarrow & \prod_{i+1} (h_\partial(N), h_\partial(\hat{N})) & \rightarrow & \\
 \downarrow & & \downarrow & & \\
 \prod_{i+1} (S^{n-1}) & \longrightarrow & \bigoplus_\rho \prod_{i+1} (S^{h(\rho)+1}) \oplus \bigoplus_{\rho \neq \iota_{n-p-1}} \prod_{i+p+1} (S^{h(\rho)+1}) & & \\
 \downarrow & & \downarrow & & \\
 \prod_{i+n-1} (S^{n-1}) & \longrightarrow & \bigoplus_{\rho \neq \iota_{n-p-1}} \prod_{i+n-1} (S^{h(\rho)+1}) & & \\
 \downarrow & & \downarrow & &
 \end{array}$$

where the lower two homomorphisms are the injection as the summand. Thus we have the following exact sequence for $\prod_{i+1}(h_\partial(N), h_\partial(N, *))$ which we wish to analyze when $i \leq 2n - p - 4$,

$$\begin{array}{c}
 \downarrow \\
 \prod_{i+1} (h_\partial(N), h_\partial(N, *)) \\
 \downarrow \\
 \bigoplus_{\rho \neq \iota_{n-1}} \prod_{i+1} (S^{h(\rho)+1}) \oplus \bigoplus_{\rho \neq \iota_{n-p-1}} \prod_{i+p+1} (S^{h(\rho)+1}) \\
 \downarrow \\
 \bigoplus_{\rho \neq \iota_{n-p-1}, \iota_{n-1}} \prod_{i+n-1} (S^{h(\rho)+1}) \\
 \downarrow
 \end{array}$$

The next lemma provides for a substantial simplification of the exact sequence in this range.

LEMMA. *If α or β are suspension elements $\Sigma\alpha'$ or $\Sigma\beta'$, respectively, then*

- (i) $[\iota_{n-1}, \rho \circ \alpha] = [\iota_{n-1}, \rho] \circ \Sigma^{n-1}\alpha'$, and
- (ii) $[\iota_{n-p-1}, \rho \circ \beta] = [\iota_{n-p-1}, \rho] \circ \Sigma^{n-p-2}\beta'$, respectively.

Proof. This follows immediately from the commutativity of the following diagram for α' ,

$$\begin{array}{ccc}
 S^{n+k-2} = D^{n-1} \times S^{k-1} \cup S^{n-2} \times D^k & \xrightarrow{\varphi} & S^{n-1} \vee S^k \\
 \downarrow 1 \times \alpha' \cup 1 \times c\alpha' = \Sigma^{n-1}\alpha' & & \downarrow \iota_{n-1} \vee \Sigma\alpha' \\
 S^{n+h(\rho)-1} = D^{n-1} \times S^{h(\rho)} \cup S^{n-2} \times D^{h(\rho)+1} & \xrightarrow{\psi} & S^{n-1} \vee S^{h(\rho)+1}
 \end{array}$$

where $c\alpha'$ denotes the cone on α' and the horizontal maps, φ and ψ , are those used to define the Whitehead products. The argument for β is just the same.

We shall also need the suspension theorem, cf. [2], which we paraphrase as follows:

Suspension Theorem. *The suspension homomorphism,*

$$\Sigma: \prod_i (S^k) \rightarrow \prod_{i+1} (S^{k+1})$$

is an isomorphism for $i < 2k - 1$ and an epimorphism for $i = 2k - 1$ with kernel generated by $[\iota_k, \iota_k]$. This kernel is infinite cyclic if k is even, is \mathbf{Z}_2 if k is odd unequal to 1, 3, or 7, and is zero if k is 1, 3, or 7.

Recall that $h(\rho) = k(n - p - 2) + l(n - 2)$ when ρ involves $k \iota_{n-p-1}$'s and $l \iota_{n-1}$'s. Thus an element of $\Pi_{j+1}(S^{h(\rho)+1})$ is a suspension if $j \leq 2h(\rho) - 1$. For $j = i$ or $i + p$ this can fail only for $\rho = \iota_{n-p-1}$ or ι_{n-1} . By the lemma the other elements are taken to the appropriate basic product summands by an iterated suspension which, by the suspension theorem, gives an isomorphism. Thus, for $i \leq 2n - p - 4$, we have the following exact sequence

$$\begin{array}{c} \downarrow \\ \prod_{i+1} (h_\partial(N), h_\partial(N, *)) \\ \downarrow \\ \prod_{i+1} (S^{n-p-1}) \oplus \prod_{i+p+1} (S^{n-1}) \\ \downarrow \\ \prod_{i+n-1} (S^{2n-p-3}). \\ \downarrow \end{array}$$

The isotopy extension theorem [4] for a point $* \subset N$ shows that $\Pi_{i+1}(h_\partial(N), h_\partial(N, *))$ is isomorphic to $\Pi_{i+1}(N) \oplus \Pi_{i+1}((h_\partial N)_*, h_\partial(N, *))$ where $(h_\partial N)_*$ is the fiber of the evaluation $h_\partial(N) \rightarrow N$. Moreover $\Pi_{i+1}(N)$ is isomorphic to $\Pi_{i+1}(S^{n-p-1})$ and goes isomorphically onto the summand in the sequence. The first case where the elements of $\Pi_{i+p+1}(S^{n-1})$ could fail to be suspensions occurs when $i + p + 1 = 2n - 3$, i.e., $i = 2n - p - 4$. By the suspension theorem ∂ takes $\Pi_{i+p+1}(S^{n-1})$ isomorphically onto $\Pi_{i+n-1}(S^{2n-p-3})$ for $i < 2n - p - 4$.

In the case $i = 2n - p - 4$ the suspension theorem implies that the suspension elements of $\Pi_{2n-3}(S^{n-1})$ go onto $\Pi_{3n-p-5}(S^{2n-p-3})$ so that $\Pi_{i+1}(h_\partial(N)_*, h_\partial(N, *)) = 0$ for $i < 2n - p - 4$. To complete the proof of the theorem we need only show that the boundary homomorphism from $\Pi_{2n-3}(S^{n-1})$ to $\Pi_{3n-p-5}(S^{2n-p-3})$ has a nontrivial kernel for $n \geq 5$, $n \neq 8$, and note that, when $n = 8$, the boundary homomorphism from $\Pi_{14}(S^7) \cong \mathbf{Z}_{120}$ to $\Pi_{20-p}(S^{13-p}) \cong \mathbf{Z}_{240}$, if $p > 5$, or $\mathbf{Z} + \mathbf{Z}_{120}$, if $p = 5$, is a monomorphism whose cokernel is \mathbf{Z}_2 and \mathbf{Z} , respectively, while the boundary homomorphism from $\Pi_{13}(S^7)$ to $\Pi_{19-p}(S^{13-p})$ is an isomorphism. If n is even all the elements of $\Pi_{2n-3}(S^{n-1})$ are suspensions and the suspension to $\Pi_{3n-p-5}(S^{2n-p-3})$ has a nontrivial kernel, $n \geq 5$, unless $n = 8$ when the kernel is trivial. If n is odd $\Pi_{2n-3}(S^{n-1})$ contains an element of infinite order while $\Pi_{3n-p-5}(S^{2n-p-3})$ is a finite group so that although we do not

know that the boundary homomorphism is a suspension as above we do know the kernel must be infinite. Thus the theorem is proved.

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UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106