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PSEUDO-VALUATION DOMAINS

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A domain R is called a pseudo-valuation domain if, whenever a prime ideal P contains the product xy of two elements of the quotient field of R then $x \in P$ or $y \in P$. It is shown that a pseudo-valuation domain which is not a valuation domain is a quasi-local domain (R, M) such that $V = M^{-1}$ is a valuation overring with maximal ideal M. The authors further show that the nonprincipal divisorial ideals of R coincide with the nonzero ideals of V. These ideas are then applied to the case of a Noetherian pseudo-valuation domain R. Such a domain R is shown to have all its nonzero ideals divisorial if and only if each ideal is two-generated. Examples include valuation rings, certain $D + M$ constructions, and certain rings of algebraic integers.

Introduction. The purpose of this paper is to study *pseudo-valuation domains*, a class of rings closely related to valuation rings. We define a *pseudo-valuation domain* to be a domain R in which every prime ideal P has the property that whenever a product of two elements of the quotient field of R lies in P then one of the given elements is in P. One shows easily that valuation rings are pseudo-valuation domains (Prop. 2.1). In the first section of the paper, several characterizations of pseudo-valuation domains are given. For example, a quasi-local domain (R, M) is a pseudo-valuation domain if and only if $x^{-1}M \subset M$ whenever x is an element of the quotient field of R, $x \notin R$ (Th. 1.4).

The name "pseudo-valuation domain" is justified in the second section, first by showing that these rings share many properties with valuation rings. More important is the characterization of a pseudo-valuation domain which is not a valuation domain as a quasi-local domain (R, M) with the property that $V = M^{-1}$ is a valuation overring with maximal ideal M. The second section is concluded with a study of the relationship between the ideals of R and the ideals of V; for example, the set of nonzero ideals of V and the set of nonprincipal, divisorial ideals of R are shown to be one and the same (Cor. 2.15).

In the final section, the authors study Noetherian pseudo-valuation domains. Such rings have Krull dimension $\leqq 1$. Also, a Noetherian pseudo-valuation domain has the 2-generator property if and only if every nonzero ideal is divisorial (Th. 3.5).

Besides valuation rings, two other classes of examples of pseudo-valuation domains are given. The first (Ex. 2.1) is obtained by taking a valuation ring of the form $V = K + M$, K a field, and taking R to be a subring of the form $F + M$, F a proper subfield of K. A second class of (Noetherian) pseudo-valuation domains is provided by localizing certain rings of algebraic integers (Ex. 3.6).

I. Definitions and properties.

DEFINITION. Let R be a domain with quotient field K. A prime ideal P of R is called strongly prime if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$.

DEFINITION. *A* domain R is called a pseudo-valuation domain if every prime ideal of R is strongly prime.

PROPOSITION 1.1. *Every valuation domain is a pseudo-valuation domain.*

Proof. Let V be a valuation domain, and let P be a prime ideal in V. Suppose $xy \in P$ where $x, y \in K$, the quotient field of V. If both x and y are in V, we are done. Suppose that $x \notin V$. Since V is a valuation domain, we have $x^{-1} \in V$. Hence $y = xy \cdot x^{-1} \in P$, as desired.

As we shall see in the next section, the converse of the above proposition is false. We turn now to some simple properties and characterizations of pseudo-valuation domains.

PROPOSITION 1.2. *Let P be a prime ideal of a domain R with quotient field K. Then P is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K - R$.*

Proof. Assume that P is strongly prime. If $x \in K - R$ and $p \in P$ then $p = px^{-1} \cdot x \in P$, whence $px^{-1} \in P$ or $x \in P$. Since $x \notin R$ we must have $px^{-1} \in P$. Thus $x^{-1}P \subset P$.

Conversely, assume $x^{-1}P \subset P$ whenever $x \in K - R$, and let $ab \in P$. If $a, b \in R$ there is nothing to prove. Hence we may assume $a \notin R$ so that $a^{-1}P \subset P$ and $b = a^{-1} \cdot ab \in P$. This completes the proof.

COROLLARY 1.3. *In a pseudo-valuation domain R, the prime ideals are linearly ordered. In particular R is quasi-local.*

Proof. Let P and Q be prime ideals, and suppose $a \in P -$

Q. Then for each $b \in Q$ we have $a/b \notin R$. Hence $(b/a)P \subset P$ by the proposition. Thus $b = b/a \cdot a \in P$ and we have $Q \subset P$.

THEOREM 1.4. *Let (R, M) be a quasi-local domain. The following statements are equivalent.*

(1) *R is a pseudo-valuation domain.*
(2) *For each pair I, J of ideals of R, either $I \subset J$ or $MJ \subset MI$.*
(3) *For each pair I, J of ideals of R, either $I \subset J$ or $MJ \subset I$.*
(4) *M is strongly prime.*

Proof. (1) $\Rightarrow$ (2). Assume $I \not\subset J$ and pick $a \in I - J$. For each $b \in J$ we have $a/b \notin R$, so that $(b/a)M \subset M$ and $Mb \subset Ma \subset MI$. It follows that $MJ \subset MI$.

(2) $\Rightarrow$ (3). This requires no comment.

(3) $\Rightarrow$ (4). Let $a, b \in R$ with $a/b \notin R$. We shall show that $(b/a)M \subset M$; by Proposition 1.2 this will suffice. Since $a/b \notin R$ we have $(a) \not\subset (b)$ whence $Mb \subset (a)$ and $Mb/a \subset R$. If $Mb/a = R$ then $M = Ra/b$ and $a/b \in R$, a contradiction. Hence $Mb/a \subset M$, as was to be shown.

(4) $\Rightarrow$ (1). Let x be an element of the quotient field of R, $x \notin R$, and let P be a prime ideal. Again, by Proposition 1.2, it is enough to show that $x^{-1}P \subset P$. Accordingly, let $p \in P$ and note that since $P \subset M$, we have $x^{-1}p \in M$. Hence $x^{-1}p \cdot x^{-1} \in M$, whence $(x^{-1}p)^2 = x^{-1}px^{-1} \cdot p \in P$. Since P is prime and $x^{-1}p \in R$, we therefore have $x^{-1}p \in P$.

In the following theorem we characterize pseudo-valuation domains without making the quasi-local assumption.

THEOREM 1.5. *Let R be a domain with quotient field K. The following statements are equivalent.*

(1) *R is a pseudo-valuation domain.*
(2) *For each $x \in K - R$ and for each nonunit a of R, we have $(x + a)R = xR$.*
(3) *For each $x \in K - R$ and for each nonunit a of R, we have $x^{-1}a \in R$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in K - R$ and let a be a nonunit of R. Then $a \in P$ for some prime ideal P, so that $x^{-1}a \in x^{-1}P \subset P \subset R$. Hence $(x + a)/x = 1 + a/x \in R$ and $(x + a)R \subset xR$. On the other hand, $x + a \notin R$ so that $(x + a)^{-1}P \subset P$ and $a/(x + a) \in R$. Since $x/(x + a) = 1 - a/(x + a)$, we have $x/(x + a) \in R$ and $xR \subset (x + a)R$.

(2) $\Rightarrow$ (3). By (2) $(x + a)/x = 1 + a/x \in R$, whence $x^{-1}a \in R$ also.

(3) $\Rightarrow$ (1). Let P be prime and take $ab \in P$ with $a, b \in K$. We

may assume $b \notin R$. By hypothesis since ab is a nonunit of R, $a = b^{-1} \cdot ab \in R$. We claim that a is a nonunit; otherwise $b = a^{-1} \cdot ab \in P$, a contradiction. We apply the hypothesis again to get $b^{-1}a \in R$. Thus $a^2 = b^{-1}a \cdot ab \in P$ and $a \in P$, as desired.

We close this section with a brief study of overrings of pseudo-valuation domains. (By an overring of a domain R, we mean a domain between R and its quotient field.)

LEMMA 1.6. *Let R be a pseudo-valuation domain and let T be an overring. If Q is prime in T, then every prime ideal of R contained in $Q \cap R$ is also a prime ideal of T.*

Proof. Let P be prime in R with $P \subset Q \cap R$. To show that P is an ideal of T, it suffices to show $tp \in P$ for all $t \in T$, $p \in P$. Now $p = tp \cdot t^{-1} \in P \Rightarrow tp \in P$ or $t^{-1} \in P$. However, if $t^{-1} \in P \subset Q \cap R$, we have that $t^{-1} \in Q$. This implies that t^{-1} is a nonunit of T, contradicting that $t \in T$. Thus $tp \in P$ and P is indeed an ideal of T. That P is a prime ideal of T follows easily from the fact that P is strongly prime in R.

THEOREM 1.7. *Let R be a pseudo-valuation domain with overring T. If the pair $R \subset T$ satisfies incomparability, then T is also a pseudo-valuation domain, and every prime ideal of T is a prime of R.*

Proof. Let Q be a prime ideal of T. We claim that Q is also prime in R. Clearly $Q \cap R$ is prime in R, whence $Q \cap R$ is prime in T by the lemma. Thus $Q \cap R \subset Q$ are primes of T lying over $Q \cap R$ in R. Since incomparability holds, we must have $Q = Q \cap R$, so that Q is a prime of R. Since R and T have the same quotient field and Q is strongly prime in R, it follows easily that Q is strongly prime in T. Thus T is a pseudo-valuation domain.

II. Valuation overrings. We begin this section with an example which anticipates most of the results in the section.

EXAMPLE 2.1. Let V be a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V. If F is a proper subfield of K, then $R = F + M$ is a pseudo-valuation domain which is not a valuation domain. To see this, note that by [**3**, Theorem A, p. 560] R and V have the same quotient field L and that M is the maximal ideal of R. Therefore, since valuation domains are pseudo-valuation domains, we see that M is strongly prime in V. It follows from the fact that R and V have the same quotient field that M is strongly prime in R. Thus by

Theorem 1.4 R is a pseudo-valuation domain. Note that R is not a valuation ring, again by [**3**, Theorem A, p. 560].

PROPOSITION 2.2. *If a GCD domain R is also a pseudo-valuation domain, then R is a valuation domain.*

Proof. By Theorem 1.3 the primes of R are linearly ordered. Thus the result follows from [**7**, Theorem 1].

REMARK 2.3. It is not enough in the above proposition to take R to be an integrally closed pseudo-valuation domain, for if in Example 2.1 we take F to be algebraically closed in K, then we have by [**3**, Theorem A, p. 560] that R is integrally closed.

As the following results show, pseudo-valuation domains enjoy many of the same properties that valuation domains do.

PROPOSITION 2.4. *If I is an ideal in a pseudo-valuation domain, then $P = \cap\{I^k : k = 1, 2, \cdots\}$ is a prime ideal.*

Proof. Let $xy \in P$ with $x \notin P$. Since $x \notin P$ we have that $x \notin I^n$ for some $n > 0$. Thus by Theorem 1.4 $I^{2n} \subset (x)$. Hence for each positive integer k, we have $(xy) \subset P \subset I^{2n+k} = I^{2n} \cdot I^k \subset xI^k$, whence $y \in I^k$. Therefore $y \in P$ and P is prime.

COROLLARY 2.5. *Let I, J be ideals in a pseudo-valuation domain R. If $I \subsetneqq \sqrt{J}$ then J contains some power of I.*

Proof. Suppose $I^k \not\subset J$ for all $k > 0$. Then by Theorem 1.4 we have $J^2 \subset I^k$ for all k so that $J^2 \subset \cap\{I^k : k = 1, 2, \cdots\} = P$, a prime ideal. Hence $J \subset P \subset I$ and $\sqrt{J} \subset P \subset I$, a contradiction.

PROPOSITION 2.6. *Let R be a pseudo-valuation domain with maximal ideal M. If P is a nonmaximal prime ideal of R, then R_P is a valuation domain.*

Proof. Let K denote the quotient field of R, and let $x \in K$. If $x \in R$ then $x \in R_P$. If $x \notin R$ then since R is a pseudo-valuation domain $x^{-1}M \subset M$. Choose $m \in M - P$. Then $x^{-1} = x^{-1}m/m \in R_P$.

We now characterize pseudo-valuation domains in terms of valuation overrings.

THEOREM 2.7. *The following statements are equivalent for a quasi-local domain (R, M).*

(1) *R is a pseudo-valuation domain.*

(2) *R has a (unique) valuation overring V with maximal ideal M.*

(3) *There exists a valuation overring V in which every prime ideal of R is also a prime ideal of V.*

Proof. (1) $\Rightarrow$ (2) By [**5**, Theorem 56] there is a valuation overring (W, N) with $N \cap R = M$. By Lemma 1.6 M is a prime ideal of W. Put $V = W_M$, then V is a valuation overring with maximal ideal M_M. Since M is strongly prime, it follows easily that $M = M_M$. The uniqueness of V follows from the fact that valuation overrings of R are determined by their maximal ideals [**3**, Theorem 14.6].

(2) $\Rightarrow$ (3). Let P be prime in R, $p \in P$, and $v \in V$. Then $p \in M$ so that $vp \in M$. Thus $v^2p \in M$, whence $(vp)^2 \in P$. Hence $vp \in P$ and P is an ideal of V. Now let $xy \in P$ with $x, y \in V$. If both x and y are in R then $x \in P$ or $y \in P$. Thus assume $x \notin R$ so that $x \notin M$ and $x^{-1} \in V$. Thus, since P is an ideal of V, we have $y = x^{-1} \cdot xy \in P$. Hence P is a prime ideal of V.

(3) $\Rightarrow$ (1). Let V be the given valuation overring. Then since every prime ideal P of R is also prime in V, and since V is a pseudo-valuation domain, P is strongly prime. Thus R is a pseudo-valuation domain.

In Theorem 2.10 we shall give more information about the valuation overring in the above theorem. We have need of the following:

PROPOSITION 2.8. *Let (R, M) be a pseudo-valuation ring which is not a valuation ring, and let (V, M) be the valuation overring (of Theorem 2.7). If I is a nonzero principal ideal of R, then I is not an ideal of V.*

Proof. Suppose $I = Ra$ is a nonzero ideal of V. Then $I = VI = VRa = Va$. Choose $v \in V - R$. Then $va \in I$ so that $va = ra$ with $r \in R$ and $v = r \in R$, a contradiction.

COROLLARY 2.9. *If a pseudo-valuation domain R has a nonzero principal prime ideal, then R is a valuation domain.*

Proof. Assume that R is not a valuation domain. Let $V \neq R$ be a valuation overring with the same maximal ideal. If P is a nonzero principal prime ideal of R then P is not an ideal of V by Proposition 2.8. This contradicts Lemma 1.6.

We now show that the valuation overring of Theorem 2.7 (2) is simply M^{-1}.

THEOREM 2.10. *Let (R, M) be a quasi-local domain which is not a* *valuation domain. Then R is a pseudo-valuation domain if and only if $V = M^{-1}$ is a valuation overring with maximal ideal M.*

Proof. Assume that R is a pseudo-valuation domain. Let $x \in V = M^{-1}$. We claim that $xM \subset M$. Otherwise $xM = R$, whence $M = Rx^{-1}$ is principal and R is a valuation domain by Corollary 2.9. Since R was assumed not valuation, our claim is verified. To show that V is an overring, it suffices to show that $xy \in V$ whenever $x, y \in V$. This follows from our claim since $x, y \in V$ implies $xyM \subset xM \subset M \subset R$ so that $xy \in M^{-1} = V$. To see that V is a valuation domain, let z be an element of the quotient field. If $z \in R$ then $z \in V$. Otherwise, $z^{-1}M \subset M$, whence $z^{-1} \in M^{-1} = V$. That M is an ideal of V also follows from $xM \subset M$ whenever $x \in V$. To see that M is the maximal ideal of V, let x be a nonunit of V. If $x \notin M$ then $x \notin R$, whence $x^{-1}M \subset M$ and $x^{-1} \in V$, a contradiction. Thus M is the maximal ideal of V.

Conversely, assume that $V = M^{-1}$ is a valuation ring with maximal ideal M. Then R is a pseudo-valuation domain by Theorem 2.7.

Throughout the rest of this section, (R, M) will denote a pseudo-valuation domain which is not a valuation ring, and $V = M^{-1}$ will denote the valuation overring with the same maximal ideal. As we have seen (Theorem 2.7), every prime ideal of R is also a prime ideal of V. Conversely, since every ideal of V is contained in M, it is clear that every ideal of V is an ideal of R. Thus R and V have the same set of prime ideals. As Proposition 2.8 shows, however, if A is a nonzero ideal of V then A is not a principal ideal of R; hence there are ideals of R which are not ideals of V. We shall now study further the relationship between ideals of R and ideals of V. This study is motivated by Bastida and Gilmer's investigation of divisorial ideals in rings of the form $D + M$ [**1**, §4]. In particular, compare [**1**, Theorem 4.1] with Lemma 2.12 and [**1**, Theorem 4.3 (1)] with Theorem 2.13.

PROPOSITION 2.11. *If A is an ideal of R, then either A is an ideal of V or AV is a principal ideal of V.*

Proof. Assume that A is not an ideal of V, and choose $x \in AV - A$. We shall show that $AV = xV$. Suppose, on the contrary, that $y \in AV - xV$. Then $y/x \notin V$, so that $x/y \in M$ and $x = x/y \cdot y \in M(AV) = MA \subset A$, a contradiction. Thus $AV = xV$ is a principal ideal of V.

To complete our discussion of ideals we have need of the v-operation, a discussion of which may be found in [**1**, p. 87]. To simplify our notation, we shall use "v" for the v-operation on R and "w" for the v-operation on V. Recall that an ideal A is called divisorial $\Leftrightarrow$ A is a

v-ideal $\Leftrightarrow A = A_v = (A^{-1})^{-1} =$ the intersection of principal fractional ideals containing A.

LEMMA 2.12. *M is divisorial.*

Proof. Otherwise $M^{-1} = R$, contradicting that M^{-1} is a valuation overring.

THEOREM 2.13. *If A is a nonzero ideal of V, then A is a divisorial ideal of R.*

Proof. We have already noted that A is an ideal of R. Assume that A is not divisorial in R, and pick $x \in A_v - A$. We assert that $Mx = MA$. Since $Rx \not\subset A$ we have $MA \subset Mx$ by Theorem 1.4. Furthermore, if $Mx \not\subset MA$ then $A \subset Rx$, also by Theorem 1.4. Hence if $a \in A$ then $a = rx$, whence $r \in M$ since $x \notin A$. Thus $a \in Mx$ and $A \subset Mx$. This implies that $Rx \subset A_v \subset (Mx)_v = M_v x = Mx$, the last equality following from the lemma. We have arrived at the absurdity that $Rx \subset Mx$;therefore, $Mx = MA$ as asserted.

Now in V either $M_w = V$ or M is principal [1, Lemma 4.2]. In either case M_w is principal. Thus $M_w x = (Mx)_w = (MA)_w = (M_w A_w)_w = M_w A_w$, the last equality following from the fact that M_w is principal. Again, since M_w is principal, we cancel M_w from the equation $M_w x = M_w A_w$, yielding $Vx = A_w$. If $A_w = A$ then $x \in A_w = A$, a contradiction. Thus A is not divisorial in V, whence by [1, Lemma 4.2], $A = bM$ for some $b \in K$, the quotient field of V. But then $A_v = (bM)_v = bM_v = bM = A$, and the theorem is established.

PROPOSITION 2.14. *If A is an ideal of R, then either A is principal in R or $A_v = AV$.*

Proof. Suppose A is not principal. Since AV is an ideal of V, AV is a divisorial ideal of R by the preceding theorem. Thus since $A \subset AV$ we have $A_v \subset (AV)_v = AV$. We must prove that $AV \subset A_v$; thus if $x \in A^{-1}$ we must show $AVx \subset R$. But $x \in A^{-1}$ implies that $xA \subset R$ whence $xA \subset M$ since A is not principal. Hence $VxA \subset VM = M \subset R$, as desired.

COROLLARY 2.15. *A is a divisorial ideal of R if and only if A is a nonzero principal ideal of R or A is a nonzero ideal of V.*

Proof. If A is a nonzero principal ideal of R, then A is clearly divisorial. If A is a nonzero ideal of V, then A is divisorial in R by Theorem 2.13.

Conversely, assume that A is a divisorial ideal of R. If A is not principal, then $A_v = AV$ by the preceding result. Hence $A = A_v = AV$ is an ideal of V.

REMARK. A summary of the results in 2.7–2.15 is in order. Let (R, M) be a pseudo-valuation domain which is not a valuation ring. Then $V(=M^{-1})$ is a valuation overring whose prime ideals coincide with those of R (Theorem 2.7 and 2.10). Recall that each nonzero ideal of $V(=M^{-1})$ is a nonprincipal ideal of R (Proposition 2.8). On the other hand, a nonprincipal ideal I of R is an ideal of $V \Leftrightarrow I$ is divisorial in R (Corollary 2.15). Thus the nonprincipal divisorial ideals of R coincide with the nonzero ideals of V.

III. Noetherian pseudo-valuation domains.

THEOREM 3.1. *Let R be a Noetherian domain with quotient field K and integral closure R'. Then R is a pseudo-valuation domain if and only if $x^{-1} \in R'$ whenever $x \in K - R$.*

Proof. Assume that R is a pseudo-valuation domain with maximal ideal M. If $x \in K - R$ then $x^{-1}M \subset M$. Since M is finitely generated, we have $x^{-1} \in R'$ by [**5**, Theorem 12].

Conversely, assume $x \in K - R$ and let P be prime in R. We must show $x^{-1}P \subset P$.

Let P' be a prime ideal of R' such that $P' \cap R = P$ [**5**, Theorem 44]. Since $x^{-1} \in R'$, $x^{-1}P \subset x^{-1}P' \subset P'$. We claim $x^{-1}P \subset R$, in which case $x^{-1}P \subset P' \cap R = P$, and we are done. To prove the claim, suppose there exists $p \in P$ with $x^{-1}p \notin R$. Then $xp^{-1} \in R'$ by hypothesis, whence $1 = xp^{-1} \cdot x^{-1}p \in P'$, a contradiction.

PROPOSITION 3.2. *If R is a Noetherian pseudo-valuation domain, then R has Krull dimension $\leqq 1$.*

Proof. This follows from [**5**, Theorem 144] and the fact that the primes of R are linearly ordered (Corollary 1.3).

COROLLARY 3.3. *If R is a Noetherian pseudo-valuation domain, then every overring of R is a pseudo-valuation domain.*

Proof. By the Krull-Akizuki Theorem [**5**, Theorem 93], every overring T has Krull dimension $\leqq 1$ (and is Noetherian). Hence the pair $R \subset T$ satisfies incomparability, and T is a pseudo-valuation domain by Theorem 1.7.

COROLLARY 3.4. *If R is a Noetherian pseudo-valuation domain, then the integral closure R' of R is a discrete rank one valuation ring.*

Proof. We noted in the proof of Corollary 3.3 that R' is a pseudo-valuation ring, hence R' is local of Krull dimension one and integrally closed. Thus R' is a discrete rank one valuation ring.

REMARK. A Noetherian pseudo-valuation domain which is a *GCD* domain is a discrete rank one valuation ring by Proposition 2.2.

In Theorem 3.5 we prove that each nonzero ideal of a Noetherian pseudo-valuation domain is divisorial if and only if every ideal of R requires at most two generators. The result is a consequence of Matlis [**6**, Theorems 40 and 57]. We include our direct proof due to the considerable simplification of the Matlis results in the case where R is a pseudo-valuation domain. It should be noted that the conditions on R in Theorem 3.5 do not imply that R is a pseudo-valuation domain, as one can show using the example in [**2**, Exercise 1, p. 81].

THEOREM 3.5. *Let (R, M) be a Noetherian pseudo-valuation domain with $V = M^{-1}(\neq R)$ its valuation overring. Then the following statements are equivalent.*

(1) *Each nonzero ideal of R is divisorial.*
(2) *Each ideal of R may be generated by two elements.*
(3) *M may be generated by two elements.*
(4) *V is a two-generated R-module.*
(5) *Each nonprincipal ideal of R is an ideal of V.*

Proof. (1) $\Leftrightarrow$ (5) This is a restatement of Corollary 2.15.

(1) $\Rightarrow$ (2) By [**4**, Lemma 2.2], $V = R + Rx$ with $x \in V - R$. Let I be a nonprincipal ideal of R. By (5) $I = IV = kV$ for some $k \in I$ since V is a discrete rank one valuation ring. Hence $I = kV = k(R + Rx) = Rk + Rkx$, and I is two-generated.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4). Let $M = (a, b)$. Then in V, M is generated by one of a and b, say $M = aV$. Then $V = 1/aM = 1/a\,(Ra + Rb) = R + Rb/a$, and V is two-generated.

(4) $\Rightarrow$ (5). Write $V = Rx + Ry$. We first reduce to the case $y = 1$. To this end pick $r, s \in R$ with $1 = rx + sy$. Then either r or s, say s, is a unit, and $y = s^{-1} - s^{-1}rx \in R + Rx$. Thus $V = R + Rx$. Now let I be a nonprincipal ideal of R. Then $IV = kV$ for some $k \in I$, and, since I is not principal in R, we may pick $i \in I - kR$. Now $i = kv = k(a + bx)$ for some $a, b \in R$, $v \in V$. If $b \in M$, then $bx \in M$ whence $a + bx \in R$ and $i \in kR$, a contradiction. Hence b is a unit of R, and we have

$kx = b^{-1}i - b^{-1}ka \in I$. Thus $IV = kV = kR + kxR \subset I$, proving (5).

We close this section with an example of a Noetherian pseudo-valuation domain which is not a valuation ring. The example given is easily seen to satisfy the equivalent conditions of Theorem 3.5.

EXAMPLE 3.6. Let m denote a square-free positive integer, $m \equiv 5 \pmod 8$. Let Z denote the ring of integers and set $D = Z[\sqrt{m}]$. Since $m \equiv 1 \pmod 4$, D does not contain the algebraic integers of the form $(a + b\sqrt{m})/2$, where a and b are odd integers. Thus, D is not integrally closed [8, Theorem 6.6]. It is routine to check that $(2, 1 + \sqrt{m}) = N$ is a maximal ideal of D. The desired example is $R = D_N$, which has $K = Q[\sqrt{m}]$ as its quotient field. R is not a valuation ring since neither $(1 + \sqrt{m})/2$ nor its inverse lies in R.

To show that R is a pseudo-valuation ring we apply Theorem 3.1 to the integral closure R' of R. Since $R' = (D_N)' = (D')_S$, where $S = D - N$ and $(')$ denotes integral closure, we must show $x \in K - R$ implies $1/x \in (D')_S$. Now $x = (a + b\sqrt{m})/c$ where $a, b, c \in Z$ and $gcd(a, b, c) = 1$. Since $x \notin R$, $c \in N \cap Z = 2Z$ so 2 divides c. But then a or b must be odd since $gcd(a, b, c) = 1$. Now $x^{-1} = c(a - b\sqrt{m}) \cdot (a^2 - b^2m)^{-1}$. If $a^2 - b^2m \notin S$ then $a^2 - b^2m \in N \cap Z = 2Z$, but $m \equiv 1 \pmod 4$; so a and b are both odd integers. It follows that $a^2 - b^2m \equiv 0 \pmod 4$, but $a^2 - b^2m \equiv 1 - m \equiv 4 \pmod 8$. Thus $a^2 - b^2m = 4t$ with t an odd integer, and so $x^{-1} = (c/2((a - b\sqrt{m})/2))/t \in D'_S = R$ because with a, b odd integers we have $(a - b\sqrt{m})/2$ an algebraic integer, hence an element of D'.

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