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REPRESENTATIONS OF WITT GROUPS

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This paper gives a tensor product theorem for the coordinate rings of the finite-dimensional Witt groups. This theorem leads to a demonstration of the equivalence of the representation theory of the Witt groups with that of certain truncated polynomial rings.

Introduction. The Steinberg tensor product theorem [1, Ch. A, §7] for a simply connected, semisimple algebraic group G in characteristic p displays irreducible G-modules as tensor products of Frobenius powers of infinitesimally irreducible G-modules (modules which are irreducible for the kernel G^1 of the Frobenius morphism of G).

A goal of modular representation theory is the expression of the coordinate ring of G in terms of tensor products of Frobenius powers of G-modules which are suitably elementary for G^1. In this paper, we give a tensor product theorem for the finite-dimensional Witt groups. We produce a subcoalgebra C of the coordinate ring A of the m-dimensional Witt group W_m which is isomorphic to the coordinate ring of the kernel W_m^1 of the Frobenius morphism. A is the inductive limit of tensor products of Frobenius powers of C [§3, Theorem].

One can see some things about the representations of W_m. First, every finite-dimensional representation of W_m^1 extends to a representation of W_m on the same representation space [§5]. Second, a representation of W_m on a finite-dimensional vector space V is determined by a family $\{f_1, \cdots, f_n\}$ of commuting endomorphisms of V such that $f_i^{p^m} = 0$. In other words, the representations of W_m on V may be studied via the representations of the algebras $\{k[x_1, \cdots, x_n]/(x_1^{p^m}, \cdots, x_n^{p^m})\}_n$ on V [Theorem, §4]. In particular, the representations of W_m which correspond to the representations of $k[x_1]/(x_1^{p^m})$ give canonical extensions for the representations of W_m^1.

This linear formulation of the representation theory of W_m leaves one with the apparently difficult problem of determining the representation theory of $k[x_1, \cdots, x_n]/(x_1^{p^m}, \cdots, x_n^{p^m})$.

For the definition of the Witt groups, see [2, Ch. 5, §1].

NOTATION. Let A denote the coordinate ring of the m-dimensional Witt group W_m, as a reduced, connected group scheme over the prime field $k = F_p$. For any subcoalgebra C of A which contains k, let $C^{(p^i)}$ be the image of C under the ith-power of the Frobenius morphism of A. We may form the inductive family of coalgebras $\{C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)}\}_{n=0}^{\infty}$, where $C \otimes \cdots \otimes C^{(p^n)} \hookrightarrow C \otimes \cdots \otimes C^{(p^n)} \otimes C^{(p^{n+1})}$ is the canonical morphism onto $C \otimes \cdots \otimes C^{(p^n)} \otimes k$. Let $\varinjlim_n C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)}$ be the coalgebra inductive limit of the family.

Let $\Pi: A \to A/M^{(p)}A$ be the quotient morphism, where $M^{(p)}$ is the image of the augmentation ideal M under the Frobenius morphism. We show in §3 that there is a coalgebra splitting $s: A/M^{(p)}A \to A$ of Π such that A, as a coalgebra, is isomorphic to $\varinjlim_n C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)}$ where $C =$ image s.

0. We require some facts from [3, Def. 6] of K. Newman. Let W_{m+1} be the $(m+1)$-dimensional Witt group over $k = F_p$, with coordinate ring A_{m+1}. As an algebra, A_{m+1} is the polynomial ring $k[X_1, X_p, X_{p^2}, \cdots, X_{p^m}]$ on $(m+1)$-variables. Grade A_{m+1} by letting X_{p^i} have degree p^i. The coproduct Δ of A_{m+1} is the following: $\Delta X_{p^i} = \sum_{j=0}^{p^i} Q_j \otimes Q_{p^i-j}$, where Q_j is a homogeneous (relative to the grading) polynomial of degree j. In particular, $Q_0 = 1$, $Q_{p^i} = X_{p^i}$ and $\{Q_j\}_{j=0}^{p^m}$ is a sequence of divided powers.

Since degree $Q_j = j$, Q_j lies in $k[X_1, X_p, \cdots, X_{p^{m-1}}]$ for $j < p^m$. The coordinate ring A of W_m may be identified with the sub-Hopf algebra $k[X_1, X_p, \cdots, X_{p^{m-1}}]$ of A_{m+1}.

1. The coalgebra splitting of Π. $M = (X_1, X_p, \cdots, X_{p^{m-1}})$ is the augmentation ideal of A. Let C be the k-span of $\{Q_j\}_{j=0}^{p^m-1}$. C is an irreducible coalgebra of dimension p^m, with $k \cdot X_1$ as its space of primitive elements. Since the coalgebra map $f: C \hookrightarrow A \xrightarrow{\Pi} A/M^{(p)}A$ has an injective restriction to $k \cdot X_1$, f is injective [5, Lemma 11.0.1]. Since $(A/M^{(p)}A)^*$ is the restricted universal enveloping algebra of $(M/M^2)^*$ [3, 13.2.3], $\dim_k (A/M^{(p)}A)^* = p^{\dim_k (M/M^2)^*} = p^m$. Therefore, $\dim_k (A/M^{(p)}A) = p^m$ and f is an isomorphism. $s = f^{-1}$ is the coalgebra splitting of Π that we use.

2. The value of Π at Q_j. Let $0 \leqq j < p^m$. Write $j = \sum_{i=0}^{m-1} a_i p^i$ where $0 \leqq a_i < p$.

LEMMA. *$\Pi(Q_j)$ is a nonzero scalar multiple of $\Pi(X_1^{a_0} X_p^{a_1} \cdots X_{p^{m-1}}^{a_{m-1}})$.*

Proof. Q_j is a linear combination of elements $X_1^{b_0} X_p^{b_1} \cdots X_{p^{m-1}}^{b_{m-1}}$ where $\sum b_i p^i = j$ by §0. If $\{b_i\}_i \neq \{a_i\}_i$, then $b_i \geqq p$ for some i, and $\Pi(X_1^{b_0} X_p^{b_1} \cdots X_{p^{m-1}}^{b_{m-1}}) = 0$. Therefore, $\Pi(Q_j) \in k \cdot \Pi(X_1^{a_0} X_p^{a_1} \cdots X_{p^{m-1}}^{a_{m-1}})$, where the coefficient of $\Pi(X_1^{a_0} X_p^{a_1} \cdots X_{p^{m-1}}^{a_{m-1}})$ is nonzero since the map

f of §1 is injective.

3. The coalgebra structure of the coordinate ring. Give the set of monomials in A the reverse lexicographic total order: $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}} > X_1^{b_0}X_p^{b_1}\cdots X_{p^{m-1}}^{b_{m-1}}$ if there is an index k such that $a_k > b_k$ and $a_i = b_i$ for $i > k$.

Let $\{a_i\}_0^{m-1}$ be a sequence where $0 \leqq a_i < p$, and let $\{b_i\}_0^{m-1}$ be a different sequence, where $0 \leqq b_i$.

Lemma. *If* $\sum_{i=0}^{m-1} a_i p^i = \sum_{i=0}^{m-1} b_i p^i$, *then* $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}} > X_1^{b_0}X_p^{b_1}\cdots X_{p^{m-1}}^{b_{m-1}}$.

Proof. Let k be the maximal index such that $a_k \neq b_k$. If $b_k > a_k$, then $\sum_{i=0}^{m-1} b_i p^i > \sum_{i=0}^{m-1} a_i p^i$ since $a_i < p$. Therefore, we must have $a_k > b_k$ and $X_1^{a_0}\cdots X_{p^{m-1}}^{a_{m-1}} > X_1^{b_0}\cdots X_{p^{m-1}}^{b_{m-1}}$.

Let C be the coalgebra formed in §1.

Theorem. *The map* $\varinjlim_n C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)} \to A$, *induced by multiplication;* $C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)}\to A$, *is an isomorphism of coalgebras.*

Proof. Denote the map by g.

Surjectivity of g. Suppose that monomials $X_1^{b_0}X_p^{b_1}\cdots X_{p^{m-1}}^{b_{m-1}}$ less than $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}}$ in the ordering lie in the image of g. We show that $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}}$ also lies there.

Write $a_i = \sum_j a_{ij}p^j$, where $0 \leqq a_{ij} < p$. Let $t_k = \sum_{i=0}^{m-1} a_{ik}p^i$. By the lemmas of §2 and §3,

$$Q_{t_k} = U_k \cdot X_1^{a_{0k}}X_p^{a_{1k}}\cdots X_{p^{m-1}}^{a_{m-1,k}} + Y_k\ ,$$

where Y_k is a linear combination of monomials of degree t_k and less than $X_1^{a_{0k}}\cdots X_{p^{m-1}}^{a_{m-1,k}}$ in the ordering, and where U_k is a nonzero scalar. Therefore,

$$\prod_{k=0}^{m=1} Q_{t_k}^{p^k} = \prod_{k=0}^{m-1} U_k^{p^k}\cdot X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}} + Y\ ,$$

where Y is a linear combination of monomials which are less than $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}}$. Since $\prod_{k=0}^{m-1} Q_{t_k}^{p^k}$ and Y lie in the image of g, so does $X_1^{a_0}X_p^{a_1}\cdots X_{p^{m-1}}^{a_{m-1}}$.

Injectivity of g. Since g is surjective, so is $\Pi\circ g\colon \varinjlim_n C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)} \xrightarrow{g} A \xrightarrow{\Pi} A/M^{(p^t)}A$ for any t; at the same time, $C^{(p^j)} \hookrightarrow A \xrightarrow{\Pi} A/M^{(p^t)}A$ has image $= k$ if $j \geqq t$. Therefore, $C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^{t-1})} \xrightarrow{\text{mult.}} A \xrightarrow{\Pi} A/M^{(p^t)}A$ is surjective. Since $\dim_k(A/M^{(p^t)}A) = p^{mt}$

by [4] or by inspection, and $\dim_k(C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^{t-1})}) = p^{mt}$, $\Pi \circ \text{mult.}$ is an isomorphism of coalgebras. In particular, $C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^{t-1})} \xrightarrow{\text{mult.}} A$ is injective. Hence, g is injective.

4. Representation theory of W_m.

The dual algebra $U = (A/M^{(p)}A)^*$ is the restricted universal enveloping algebra of the abelian p-Lie algebra $L = (M/M^2)^*$ [5, 13.2.3].

LEMMA. *There is a k-basis $f_0, \cdots, f_{m-1}$ for L, where $f_i^p = f_{i+1}$ for $i < m-1$ and $f_{m-1}^p = 0$.*

Proof. Define f_j on the k-basis $X_1, X_p, \cdots, X_{p^{m-1}}$ for M/M^2 by $f_j(X_{p^i}) = \delta_{ij}$. We have the following to complete the proof.

(1) If $i \neq j+1$, then $f_j^p(X_{p^i}) = (\bigotimes^p f_j)(\Delta^{p-1}X_{p^i})$ is 0, since $\Delta^{p-1}X_{p^i}$ is homogeneous of degree p^i under the grading of $\bigotimes^p A$ induced from the grading of A, while $\bigotimes^p f_j$ can be nonzero only at monomials in $\bigotimes^p A$ of degree p^{j+1}.

(2) One may check that $f_j^p(X_{p^{j+1}}) = 1$.

To proof is complete.

By this lemma, the algebra map from the polynomial ring $k[f]$ to U mapping f to f_1 induces an isomorphism of k-algebras $k[f]/(f^{p^m}) \cong U$.

Denote by R_n the set of isomorphism classes of finite-dimensional representations of W_m whose coefficients lie in $C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \hookrightarrow A$. The canonical map $C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \hookrightarrow C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \otimes C^{(p^{n+1})}$ induces $R_n \hookrightarrow R_{n+1}$. Then $R = \bigcup_n R_n$ is the set of isomorphism classes of finite-dimensional representations of W_m.

Let B denote the quotient of the polynomial ring $F_p[X_0, \cdots, X_n, \cdots]$ on generators $\{X_i\}_{i=0}^{\infty}$ by the ideal $(X_0^{p^m}, \cdots, X_n^{p^m}, \cdots)$. Denote by $\hat{B}$ the set of isomorphism classes among those finite-dimensional representations of B in which all but a finite number of the X_i act as the zero endomorphism. Denote by $\hat{B}_n$ the set of isomorphism classes of finite-dimensional representations of $k[X_0, \cdots, X_n]/(X_0^{p^m}, \cdots, X_n^{p^m})$. The map $k[X_0, \cdots, X_n, \cdots]/(X_0^{p^m}, \cdots, X_n^{p^m}, \cdots) \to k[X_0, \cdots, X_n]/(X_0^{p^m}, \cdots, X_n^{p^m})$, $X_i \mapsto X_i$ for $i \leqq n$ and $X_i \mapsto 0$ for $i > n$, induces $\hat{B}_n \hookrightarrow \hat{B}$, and $\hat{B} = \bigcup_n \hat{B}_n$.

THEOREM. *There is a canonical bijection $R \to \hat{B}$, under which R_n and $\hat{B}_n$ correspond.*

Proof. Since $C \cong A/M^{(p)}A$ as coalgebras, $C^* \cong U$ as algebras. Since A is reduced, the Frobenius morphism on A is injective, and $C \cong C^{(p^i)}$. Therefore,

(1) $(C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)})^*\cong\bigotimes^{n+1} U\cong k[X_0,\cdots,X_n]/(X_0^{p^m},\cdots,X_n^{p^m})$. The first isomorphism is induced by the maps $U\to(C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)})^*$ which are dual to the maps $C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)}\xrightarrow{\varepsilon_0\otimes\cdots\otimes\varepsilon_{i-1}\otimes I\otimes\varepsilon_{i+1}\otimes\cdots\otimes\varepsilon_n} C^{(p^i)}$, where ε_j is the counit of $C^{(p^j)}$; the second isomorphism is induced by $X_i\mapsto 1_0\otimes\cdots\otimes 1_{i-1}\otimes f_1\otimes 1_{i+1}\otimes\cdots\otimes 1_n$, where 1_j is the identity of U_j. Here u_j is the jth copy of u in $\otimes^{n+1}u$. Moreover,

(2) under dualization, the canonical map $C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)}\hookrightarrow C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^{n+1})}$ yields the map $k[f_0,\cdots,f_{n+1}]/(f_0^{p^m},\cdots,f_{n+1}^{p^m})\to k[X_0,\cdots,X_n]/(X_0^{p^m},\cdots,X_n^{p^m})$ where $X_i\mapsto X_i$ for $i\leqq n$ and $X_{n+1}\mapsto 0$.

The isomorphism $(C\otimes C^{(p)}\otimes\cdots\otimes C^{(p^n)})^*\cong k[X_0,\cdots,X_n]/(X_0^{p^m},\cdots,X_n^{p^m})$ of (1) induces a bijection $R_n\to\hat{B}_n$ such that $\begin{array}{ccc} R_n & \to & B_n \\ \downarrow & & \downarrow \\ R_{n+1} & \to & B_{n+1}\end{array}$ commutes by (2). Therefore, $R\xrightarrow{\sim}\hat{B}$.

5. **Representations of W_m^1.** The coalgebra C constructed in §1 is isomorphic to the coordinate ring $A/M^{(p)}A$ of W_m^1 under the mapping $\pi\colon A\to A/M^{(p)}A$ restricted to C. Therefore, the representations of W_m with coefficients in C correspond to the representations of W_m^1 via the isomorphism between the coefficient coalgebras C and $A/M^{(p)}A$, and very finite-dimensional representation of W_m^1 extends to a representation of W_m on the same representation space.

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