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REGULAR FPF RINGS

S. PAGE

It is shown that a von Neumann regular ring is FPF (i.e., very faithful finitely generated module is a generator) iff it is self-injective of bounded index.

1. **Introduction.** An associative ring R is called a left (F)PF ring if every (finitely generated) faithful module generates the category of left R-modules. Azumaya [1], Osofsky [7], and Utumi [9, 12] characterized the left PF rings as those rings for which any one of the following equivalent conditions holds:

(PF_1) R is left self-injective, semiperfect, and has essential left socle.

(PF_2) R is left self-injective with finitely generated essential left socle.

(PF_3) $R = \bigoplus \sum_{i=1}^{n} Re_i$, $e_i^2 = e_i$ and Re_i is injective with simple essential socle.

(PF_4) R is an injective cogenerator in R-mod.

(PF_5) R is left self-injective and every simple left R-module embeds in R.

C. Faith in [3, 4] has studied semiperfect left FPF rings. In this note we are concerned with von Neumann regular rings which are left FPF. As the conditions PF_1-PF_5 readily point out a von Neumann regular ring which is PF must be semi-simple artinian. In this note we show that if R is von Neumann regular, then R is FPF iff R is of bounded index and left self-injective. It follows that for regular rings left FPF implies right FPF also.

II. **Preliminaries.** In what follows R will denote an associative ring with unity and all modules will be unitary left R-modules unless otherwise noted.

A ring R is von Neumann regular if for every $a \in R$ there is an $x \in R$ such that $axa = a$. We will just say R is regular.

DEFINITION. For a set $S \subset M$, M an R-module, let ${}^{\perp}S = \{r \in R: rs = 0 \text{ for all } s \in S\}$. If M is a right R-module, define $S^{\perp} = \{r \in R: sr = 0 \text{ for all } s \in S\}$.

DEFINITION. Let M be an R-module. Let $Z(M)$ be the left singular submodule of M i.e., $Z(M)$ is the set of elements of M whose annihilators are essential left ideals of R. M is called non-singular if $Z(M) = 0$.

DEFINITION. A ring R is of bounded index if there exists an integer $N > 0$ such that if $x^n = 0$ then $x^N = 0$.

DEFINITION. Let M and N be R-modules. Let $N - \dim M = \sup\{n: \bigoplus \sum_{i=1}^{n} N_i \subset M,\ N_i \cong N,\ i = 1, \cdots, n\}$. Also, let $D(M) = \sup\{N - \dim M,\ N \in R\text{-mod}\}$.

The following result of Utumi [10] gives the connection between rings of bounded index and FPF rings. We include the proof for completeness.

THEOREM 1. *Let R be a ring with zero singular left ideal. Then R is of bounded index if $D(R) < \infty$ and in case R is regular $D(R)$ equals the smallest bound on the index of nilpotence.*

Proof. We can suppose R is regular for the maximal ring of quotients, $Q(R)$, is regular and R is an essential submodule of $Q(R)$. Suppose $x^n = 0$ but $x^{n-1} \neq 0$, for some $x \in R$. Let $K_1 = {}^{\perp}(x^{n-1})$ and consider $0 \to K_1 \to R \xrightarrow{x^{n-1}} Rx^{n-1} \to 0$. The sequence splits by regularity of R, so $R \supseteqq W_1 \cong Rx^{n-1}$ and $W_1 \cap K_1 = 0$. Let $K_2 = {}^{\perp}\{x^{n-2}\} \cap Rx$ and form $0 \to K_2 \to Rx \to Rx^{n-1} \to 0$ which also splits. Therefore there exists $W_2 \subseteqq Rx$ with $W_2 \cap K_2 = 0$ and $w_2 \cong Rx^{n-1}$ so that $w_2 \cong W_1$. Also since $K_1 \cap W_1 = 0$ and $Rx \subset K_1 W_2 \cap W_1 = 0$.

By $n - 1$ applications of the above technique we obtain $W_1 \cong W_2 \cong \cdots \cong W_{n-1}$ with $Rx^{n-1} \subseteqq K_i = {}^{\perp}\{x^{n-i}\} \cap Rx^i$, and $W_i \cap K_i = 0$. It follows that $D(R) \geqq n$ since $(\bigoplus \sum_{i=1}^{n-1} W_i) \bigoplus Rx^{n-1} \subset R$.

Next suppose $\{L_i\}_{i=1}^{n}$ is an independent set of left ideals in R with $L_i \cong L_j$ for all i and $j \leqq n$. Since R is regular we can assume the L_i are all idempotent generated, by $e_1, e_2, \cdots, e_n$, say, with $e_i e_j = 0$ for $i, j = 1, \cdots, n, i \neq j$. Let $\phi_{ij}: Re_i \cong Re_j$. Then ϕ_{ij} is right multiplication by $e_i r_{ij} e_j$ for some $r_{ij} \in R$. Let $x = \Sigma e_i r_{i,i+1} e_{i+1}$. Then $x^n = 0$ but $x^{n-1} \neq 0$.

COROLLARY 1.1. *If R is a domain which is not a left Ore domain, $Q(R)$ is of unbounded index, where $Q(R)$ is the maximal left quotient ring of R.*

Another fundamental result is the following of Bumby [2].

PROPOSITION 1.2. *Let M_1 and M_2 be injective modules with $0 \to M_1 \to M_2$ and $0 \to M_2 \to M_1$. Then $M_1 \cong M_2$.*

III. Regular FPF rings. We start with commutative rings, then using Morita equivalence build up to the more general case.

THEOREM 2. *The following are equivalent for a commutative regular ring R.*

(i) *R is self-injective.*

(ii) *R is FPF.*

(iii) *The trace of every finitely generated faithful module is finitely generated.*

Proof. If R is injective and M is a finitely generated faithful module, then R embeds in a finite direct sum of copies of M as a direct summand. This gives (i) $\Rightarrow$ (ii).

That (ii) implies (iii) is trivial.

Assume (iii) and let $q \in Q$, the injective hull of R. Form $Rq + R = M$. Now trace (M) is finitely generated since M is finitely generated and faithful. Since R is regular and trace (M) is finitely generated, we have that trace $(M) = Re$, $e^2 = e$. Let $i \in I = \{r \in R: rq \in R\}$, an essential ideal. Then multiplication by i defines a map of M into R and this map sends 1 into i so $I \subset \text{trace}\,(M)$. Now take $f: M \to R$. Let $f(q) = x_0$ and $f(1) = y_0$. Then for every $z \in I$ we have $f(zq) = zqy_0$ so $z(x_0 - qy_0) = 0$, hence $x_0 = qy_0$ and $y_0 \in I$. I is generated by idempotents so we can take $y_0 = y_0^2$ so that $x_0 = x_0 y_0$, that is, trace $(M) \subsetneqq I$ too. Since $I = Re$ and I is essential, $I = R$ and hence $q \in R$.

COROLLARY 2.1. *If R is a strongly regular ring (all idempotents are central) then R is FPF iff R is self-injective.*

Proof. If R is strongly regular left ideals are ideals and are generated by idempotents. Also if M is finitely generated by $x_1, \cdots, x_n$ say $M = \bigcap_{i=1}^{n} {}^{\llcorner}\{x_i\}$ for strongly regular rings. With these observations the previous proof goes through.

If D is a division ring and $R = \text{End}_D(\gamma)$ then R is FPF iff γ is finite dimensional over D, but R is always self-injective and regular. The significant observation is that if γ is infinite dimensional over D and $f \in R$ is a map with one dimensional range Rf is finitely generated and faithful but can not generate R because roughly R contains infinitely many copies of Rf i.e., $Rf - \dim R = \infty$.

We do have the following.

PROPOSITION 3. *Let R be a ring with $Z(R) = 0$. If R is left FPF then every left ideal is an essential submodule of a direct summand of R.*

Proof. Let L be any left ideal and B a left ideal maximal with respect to $L \cap B = 0$. Form $R/L \oplus R/B = M$. M is faithful and finitely generated so generates R. Now if $f: M \to R$, let $f((1+L, 0)) = x_0$ and $f((0, 1+B)) = y_0$. Then $x_0 \in L^{\perp}$ and $y_0 \in B^{\perp}$ so since M is faithful $L^{\perp} + B^{\perp} = R$. This gives ${}^{\perp}(L^{\perp} + B^{\perp}) = 0$ or ${}^{\perp}(L^{\perp}) \cap {}^{\perp}(B^{\perp}) = 0$. Since $L \subseteqq {}^{\perp}(L^{\perp})$ and $B \subseteqq {}^{\perp}(B^{\perp})$ the maximality of B gives $B = {}^{\perp}(B^{\perp})$. Also, if we now take $L_1 \supset L$ and maximal with respect to $L_1 \cap B = 0$, L_1 is an essential extension of L, and ${}^{\perp}(L_1^{\perp}) = L_1$ as we have just seen. Now we have $0 = (L_1 + B)^{\perp}$ since $L_1 + B$ is essential, hence $L_1^{\perp} \cap B^{\perp} = 0$. Also $L_1^{\perp} + B^{\perp} = R$ by the above which yields $L_1^{\perp} = eR$, $e^2 = e$ so that ${}^{\perp}(L_1^{\perp}) = R(1-e)$ a direct summand, as promised.

PROPOSITION 4. *If R is a regular ring which is left FPF, then R is left self-injective.*

Proof. If R is regular then certainly $Z(R)=0$ and by Proposition 3 each left ideal is essential in a direct summand of R. In regular rings it is trivial that a left ideal isomorphic to a direct summand is a direct summand. These two properties constitute the definition of left continuous and the last corollary of Utumi [11, Corollary 8.4] states that if R and any matrix ring over R are both continuous R is self-injecture. Since both FPF and regularity are easily checked to be Morita invariant properties, it follows that R is left self-injective.

REMARK. The integers are FPF but lack the second part of the definition of left continuous.

PROPOSITION 5. *Let $\{R_i\}_{i \in I}$ be a collection of rings. Let $R = \prod_{i \in I} R_i$ as rings. Then R is left FPF iff each R_i is left FPF and for each collection $\{M_i : M_i$ a finitely generated faithful R_i-module $i \in I\}$ such that πM_i is a finitely generated R-module, there exists an integer $N > 0$ such that R_i is a homomorphic image of a direct sum of N copies of M_i for each $i \in I$.*

Proof. Routine coordinate wise computation yields the proposition.

The previous proposition points out that if R is a product of matrix rings over division rings in order that R be left FPF the matrix rings had better not become to "large". It also suggests we look at the types given by Kaplansky and refined by Goodearl and Boyle [5].

DEFINITION. A regular left self-injective ring R is called type

I if for every direct summand L of R, $L \supseteqq L^1 \neq 0$, a left ideal, such that for any left ideals $A \neq 0$ and $B \neq 0$ contained in L^1, $\operatorname{Hom}(A, B) \neq 0$. If $L = L^1$ L is called abelian.

DEFINITION. A ring R is called Dedekind finite if $xy = 1$ iff $yx = 1$, otherwise we say R is Dedekind infinite.

DEFINITION. A regular left self-injective ring R is called type II if R contains an idempotent e such that Re is faithful, eRe is Dedekind finite but R contains no abelian left ideals.

DEFINITION. A regular left self-injective ring R is type III if $0 \neq e^2 = e$ then eRe is not Dedekind finite.

Type III rings are characterized by the fact that for any direct summand, L, then $L \cong L \oplus L$.

THEOREM [*Kaplansky* [6], *Goodearl, Boyle* [5, *Corollary* 7.7, *p.* 48]. *If R is a regular left self-injective ring, then $R = \prod_{i=1}^{5} R_i$, where R_1 is type I and Dedekind finite, R_2 is type I and Dedekind infinite, R_3 is type II and Dedekind finite, R_4 is type II and Dedekind infinite, and R_5 is type III.*

REMARK. All type III rings are Dedekind infinite. Also, we will adopt Kaplansky [6, p. 11] notation and say R is type I_f if R is type I and Dedekind finite, type I_∞ if type I and Dedekind infinite, type II_f if$\cdots$, type $II_\infty\cdots$.

PROPOSITION 6. *If R is regular and FPF then R is biregular.*

Proof. Let $x \in R$. We wish to show RxR is generated by a central idempotent. Let $H = {}^{\perp}(RxR)$. If $H = 0$, then Rx generates so $RxR = R$. If $H \neq 0$, then H is the left ideal maximal with respect to $H \cap RxR = 0$. It follows that H is a direct summand of R because R is self-injective. Now $H \oplus Rx$ is a finitely generated faithful module, hence a generator, so trace $(H \oplus Rx) = H \oplus RxR = R$.

PROPOSITION 7. *If R is regular left FPF, R is Dedekind finite.*

Proof. If not, then by [5, Prop. 7.4, p. 48] $R = R_1 \times R_2$ with $R_2 \neq 0$ and purely infinite, i.e., for every $0 \neq e$, a central idempotent in R_2, eR_2e is not Dedekind finite. So assume $R \neq 0$ and purely infinite.

By [5, Thm. 6.2, p. 41] there is in R a sequence of idempotents $e_1, e_2, \cdots$ such that for each i, $Re_i \cong R$, and $\sum_{i=1}^{\infty} Re_i$ is direct, essential and $R = E(\sum_{i=1}^{\infty} Re_i)$. Let $M = R/\sum_{i=1}^{\infty} Re_i$. We claim M is faithful. If not, there exist $x \in R$ such that $R \times R \subseteqq M$. By Proposition 6, $RxR = Re$ for some central idempotent e. Since $eM = 0$ it follows that $Re \subseteqq \sum_{i=1}^{\infty} Rx_i$. But then $Re \subseteqq \sum_{i=1}^{N} Rx_i$ for some N large enough. This implies $Re \cap Rx_j = 0$ for $j > N$, which implies $ex_j = 0$ $j > N$ since e is central. However, since $Rx_i \cong Rx_j$ for all i and j and e is central, then $ex_i = 0$ for all i, a contradiction.

Thus M is faithful. M is also singular, hence R is singular so must be zero.

COROLLARY 7.1. *If R is regular FPF type I, then R is of bounded index.*

Proof. By [5, p. 30] we see that if R is type I, R contains an idempotent such that eRe is strongly regular and Re is faithful. It follows that R is Morita equivalent to a strongly regular ring. Then using Tominaga [8, Lemma 1, p. 139] we see that R is of bounded index.

PROPOSITION 8. *Let R be a regular left FPF ring of type II_f. Then $R = \{0\}$.*

Proof. Let $0 \neq R$ be as above. We claim R can not be a simple ring. If R were a simple ring since it is type II it cannot be a semi-simple ring, hence must have an essential left E. But then R/E is faithful by the simplicity of R hence a generator of R. This says $Z(R) = R$, ridiculous. Since R is not simple there must exist an idempotent $e_1 \in R$ such that $0 \neq Re_1R \neq R$. Now let $H_1 = {}^{\perp}(Re_1R)$. If $H_1 = 0$ then Re_1 generates R which it does not, so $H_1 \neq 0$. H_1 is the left ideal maximal with respect to $H_1 \cap Re_1R = 0$, so H_1 is a summand by injectivity of R. It follows that $H_1 \oplus Re_1R = R$ as above. Now H_1 and Re_1R are type II_f left FPF rings so we can repeat the process to Re_1R to obtain an ideal $H_2 \subseteqq Re_1R$. Continuing in this way we obtain $H_1 \oplus H_2 \oplus \cdots \subseteqq R$ each H_i a nonzero two sided direct summand of R. Since each H_i is type II_f we can choose an idempotent $f_i \in H_i$ such that $H_i = \bigoplus \sum_{j=1}^{i} Rf_{ij}$, $Rf_i \cong Rf_{ij}$ for all $j \leqq i$. Next take $Rg = E(\bigoplus \Sigma H_i)$. Rg is a two sided ideal for the hull of any two sided ideal in a semiprime left self-injective ring is complemented by its left annihilator which is a two sided ideal. We can assume then that g is a central idempotent. Form $\prod_{i=1}^{\infty} Rf_i$ and let M be the cyclic submodule generated by $R((f_i)_{i\in I})$. Let $N = M \oplus R(1-g)$. Then $yN = 0$ iff $y(g-1) = 0$ and $yRf_i = 0$ for all i, so $yRf_iR = 0$ for all i. Then $y(\sum_{i=1}^{\infty} H_i) = 0$. But since $yg = y$ there exists an essential left ideal E such that $Ey \subseteq \sum_{i=1}^{\infty} H_i$ and $(Ey)^2 = 0$ implies $y = 0$ so N is faithful. Since R is left FPF, N generates R so $R((f_i)_{i\in I})$ must generate Rg. It follows that for a fixed $n > 0$ there are maps $\sum_{j=1}^{n} Rf_{i_j} \to H_i \to 0$ for every i. But if $i > n$ we see by Bumbys result $H_i \oplus Rf_i \cong H_i$ and R is not Dedekind finite.

Putting the above facts together gives:

THEOREM 9. *A regular ring is left FPF iff it is left self-injective of bounded index.*

COROLLARY 9.1. *A regular ring is left FPF iff it is Morita equivalent to a strongly regular left self-injective ring.*

COROLLARY 9.2. *A regular ring is left FPF iff it is right FPF.*

Proof. By Utumi [13, Thm. 1.4] a strongly regular ring is left self-injective iff it is right self-injective.

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