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COMMUTANTS AND THE OPERATOR EQUATION $AX=\lambda XA$

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Suppose A is a bounded operator on the Banach space $\mathscr{B}$ such that A or A^* is one-to-one. In this note, we point out a relation between the commutant of A, the commutants of its powers, and operators which intertwine A and λA, where λ is a root of unity. A consequence of this relation is that the commutants of A and A^n are different if and only if there is an operator Y, not zero, that satisfies $AY=\lambda YA$, where $\lambda^n=1$, $\lambda\neq 1$. Combining this with Rosenblum's theorem, we see that if the spectra of A and XA are disjoint, the commutant of A is the same as that of A^2. We also use the theorem to give a counterexample to a conjecture of Deddens concerning intertwining analytic Toeplitz operators.

If A, B, and X are bounded operators on $\mathscr{B}$, we say X *commutes* with A if $XA = AX$, and we say X *intertwines* A and B if $XA = BX$. The set of operators that commute with A, the commutant of A, will be denoted $\{A\}'$.

LEMMA. *Suppose A is an operator such that A or A^* is one-to-one, and λ is a primitive nth root of 1. If X commutes with A^n, the operators $Y_i = \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j$, for $i = 0, 1, \cdots, n-1$, are the unique operators such that $AY_i = Y_i(\lambda^i A)$ and $nA^{n-1}X = \sum_{i=0}^{n-1} Y_i$.*

Proof. Let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j$.
Then

$$\begin{aligned}AY_i &= \sum_{j=0}^{n-1}\lambda^{ij}A^{n-j}XA^j = A^nX + \sum_{j=1}^{n-1}\lambda^{ij}A^{n-j}XA^j\\ &= XA^n + \sum_{j=1}^{n-1}\lambda^{ij}A^{n-j}XA^j\\ &= \sum_{k=0}^{n-1}\lambda^{i(k+1)}A^{n-k-1}XA^{k+1}\\ &= \left(\sum_{k=0}^{n-1}\lambda^{ik}A^{n-k-1}XA^k\right)(\lambda^iA) = Y_i(\lambda^iA)\ .\end{aligned}$$

Since $\sum_{i=0}^{n-1}\lambda^{ij} = 0$ when $j \neq 0$, and the sum is n when $j = 0$,

$$\begin{aligned}\sum_{i=0}^{n-1} Y_i &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\lambda^{ij}A^{n-j-1}XA^j\\ &= \sum_{j=0}^{n-1}A^{n-j-1}XA^j\sum_{i=0}^{n-1}\lambda^{ij} = nA^{n-1}X\ .\end{aligned}$$

Now suppose $Z_0, Z_1, \cdots, Z_{n-1}$ are operators such that $nA^{n-1}X = \sum_{i=0}^{n-1} Z_i$ and $AZ_i = Z_i(\lambda^i A)$ for each i. We have

$$\begin{aligned} nA^{n-1}Y_i &= \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}(nA^{n-1}X)A^j \\ &= \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}\left(\sum_{k=0}^{n-1} Z_k\right)A^j = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \lambda^{ij}A^{n-j-1}A^j\lambda^{-kj}Z_k \\ &= \sum_{k=0}^{n-1} A^{n-1}Z_k \sum_{j=0}^{n-1} \lambda^{(i-k)j} = nA^{n-1}Z_i \, . \end{aligned}$$

If A is one-to-one, then $A^{n-1}Y_i = A^{n-1}Z_i$ implies $Y_i = Z_i$. If A^* is one-to-one, then A^{n-1} has dense range and $Y_iA^{n-1} = \lambda^{-i(n-1)}A^{n-1}Y_i = \lambda^{-i(n-1)}A^{n-1}Z_i = Z_iA^{n-1}$, which implies $Y_i = Z_i$.

THEOREM. *Suppose A or A^* is one-to-one and n is a positive integer. Then $\{A\}' = \{A^n\}'$ if and only if $AY = Y(\lambda A)$ for $\lambda^n = 1$ implies $\lambda = 1$ or $Y = 0$.*

Proof. ($\Rightarrow$) Suppose $\{A\}' = \{A^n\}'$ and for some Y we have $AY = Y(\lambda A)$ where $\lambda^n = 1$. Then $A^nY = Y(\lambda^n A^n) = YA^n$, so $Y \in \{A^n\}' = \{A\}'$ and $AY = YA$ as well. Thus $\lambda YA = AY = YA$ and $(1-\lambda)YA = 0$. Since A or A^* is one-to-one, this means that $(1-\lambda)Y = 0$, so $\lambda = 1$ or $Y = 0$.

($\Leftarrow$) Suppose $AY = Y(\lambda A)$ for $\lambda^n = 1$ implies $Y = 0$ or $\lambda = 1$. Let X be in $\{A^n\}'$, and let λ be a primitive nth root of 1. For $i = 1, 2, \cdots, n-1$ let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j$. By the lemma, $AY_i = Y_i(\lambda^i A)$ so, since $\lambda^i \neq 1$, our hypothesis says $Y_i = 0$. Thus, we have the $n-1$ equations $\sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j = 0$, $(i = 1, 2, \cdots, n-1)$.

Consider the equations $\sum_{j=0}^{n-1} \lambda^{ji}w_j = 0$, $(i = 1, 2, \cdots, n-1)$, in the indeterminates $w_0, w_1, w_2, \cdots, w_{n-1}$. We notice that $w_0 = w_1 = w_2 = \cdots = w_{n-1}$ is a solution of these equations, and since the $(n-1)\times n$ coefficient matrix $(\lambda^{ji})_{j=0}^{n-1}{}_{i=1}^{n-1}$ has rank $n-1$, this is the only solution. In our case, we conclude $A^{n-1}X = A^{n-2}XA = \cdots = XA^{n-1}$. If A is one-to-one, $A^{n-1}X = A^{n-2}XA$ implies $AX = XA$, whereas if A^* is one-to-one, $AXA^{n-2} = XA^{n-1}$ implies $AX = XA$.

We have shown that X is in $\{A\}'$ if it is in $\{A^n\}'$. Since the reverse inclusion is automatic, we have $\{A^n\}' = \{A\}'$.

As illustrations, we prove the following corollaries.

COROLLARY 1. *If the spectrum of A and the spectrum of $-A$ are disjoint, then $\{A\}' = \{A^2\}'$.*

Proof. Since the spectra of A and $-A$ are disjoint, Rosenblum's theorem, [3], implies that the only solution of $AX = X(-A)$ is $X = 0$. Zero is not in the spectrum of A, so A is one-to-one and we apply the theorem to conclude $\{A\}' = \{A^2\}'$.

COROLLARY 2. *If the spectrum of A is contained in the quarter plane $\{z \mid \mathrm{Re}\,(z) > 0$ and $\mathrm{Im}\,(z) > 0\}$, then $\{A\}' = \{A^4\}'$.*

Proof. The spectra of A, iA, i^2A, and i^3A are disjoint, so by Rosenblum's theorem, the only solution of $AX = X(i^kA)$, for $k = 1$, 2, or 3, is $X = 0$. Zero is not in the spectrum of A, so A is one-to-one, and we apply the theorem to conclude that $\{A\}' = \{A^4\}'$.

The theorem may also be used to refute a conjecture of Deddens concerning intertwining analytic Toeplitz operators [2, page 244]. We recall that if ϕ is a bounded analytic function on the unit disk D, the analytic Toeplitz operator, T_ϕ, is the operator on the Hardy space H^2 of multiplication by ϕ. Deddens conjectured that when ϕ and ψ are bounded analytic functions on D and 0 is the only solution of $XT_\phi = T_\psi X$, then the complex conjugate of the range of ψ is not contained in the point spectrum of T_ϕ^*. To see that this is false, let f be a Riemann map of D onto the slit disk $D\backslash(-1, 0]$. Then the corollary of Theorem 5 of [1] implies that $\{(T_{f^2})^2\}' = \{T_f\}' = \{T_{f^2}\}'$. If f^2 and $-f^2$ are the ϕ and ψ of the conjecture, we note that range $\psi = D\backslash\{0\} =$ point spectrum T_ϕ^*. But since $\{(T_{f^2})^2\}' = \{T_{f^2}\}'$ and T_{f^2} is one-to-one, the theorem implies that 0 is the only operator which intertwines T_{f^2} and $-T_{f^2} = T_{-f^2}$. The difficulty seems to be associated with the fact that the multiplicities of f^2 and $-f^2$ are different on the real axis.

The unfortunate presence of A^{n-1} in the formula $nA^{n-1}X = \sum_{i=0}^{n-1} Y_i$ of the lemma is essential when A is not invertible; it is easy to give examples of operators X in $\{T_z^2\}'$ so that $2T_zX = Y_0 + Y_1$, as above, but $Y_0 \neq T_zB$ for any bounded operator B. On the other hand, if A is invertible, we may solve the equation for X and obtain $X = \sum_{i=0}^{n-1} \hat{Y}_i$, where $\hat{Y}_i = (nA^{n-1})^{-1} Y_i$. These operators are the unique operators that satisfy $A\hat{Y}_i = \hat{Y}_i(\lambda^i A)$ and $X = \sum_{i=0}^{n-1} \hat{Y}_i$.

In the above, we have found a relation between the commutants of A and $p(A)$ for the polynomials $p(z) = z^n$. Of course, there is an analogous result for polynomials of the form $p(z) = (z - \alpha)^n + \beta$. It would be interesting (and apparently more difficult) to obtain information about the relation between $\{A\}'$ and $\{p(A)\}'$ for more complicated polynomials.

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