

POLYNOMIAL COVERING SPACES AND HOMOMORPHISMS INTO THE BRAID GROUPS

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A polynomial covering space is the zero set for a parametrized family of complex polynomials. Such covering spaces arise naturally in several contexts in the study of holomorphic mappings. There are fruitful connections between polynomial covering spaces and certain homomorphisms into the Artin braid groups. We shall establish these connections and use them to obtain an algebraic classification of the polynomial covering spaces.

1. Introduction. Covering spaces defined by parametrized families of complex polynomials arise naturally in several contexts in the study of holomorphic mappings, e.g., in connection with the Weierstrass preparation theorem. A study of the topology of such covering spaces within a proper topological framework was suggested in [6] by the introduction of polynomial covering spaces. Roughly speaking, an n -fold polynomial covering space over a connected, topological space X is the zero set for a continuously varying family of simple, normed, complex polynomials of degree n parametrized by X . It is well known that an n -fold covering space over X is classified by a conjugacy class of homomorphisms of the fundamental group of X into the symmetric group on n letters. The purpose of this paper is to examine the relationship between n -fold polynomial covering spaces over X and homomorphisms of the fundamental group of X into the Artin braid group on n strings. What we are looking for are necessary and sufficient conditions, for a finite covering space over X to be equivalent to a polynomial covering space over X , and for two polynomial covering spaces over X to be equivalent as coverings.

In [4] and later papers, see e.g., the references cited in [5], Gorin and Lin also study parametrized families of simple, normed complex polynomials, but not from the point of view of examining the topological types of the associated covering spaces. The purpose of Gorin and Lin is to investigate, when a continuous family of polynomials admits a continuous factorization in polynomials of degree 1 over the parameter space. The existence of such a factorization is equivalent to the associated polynomial covering space being trivial. Of other papers somehow related to the work in this paper, we note in particular the papers by Arnol'd [1] and Zjuzin and Lin [9].

Now a short description of the contents of the paper. The link from n -fold polynomial covering spaces over X to homomorphisms of the fundamental group of X into the braid group on n strings is established in §2 via the introduction of root maps and their induced homomorphisms. In §3 we describe certain characteristic homomorphisms for an n -fold polynomial covering space over X , with Lemma 3.1 as the key result. Section 4 contains a geometric criterion for a polynomial covering space to be trivial, stated as Theorem 4.1. This is the point, where contact is made with the work of Gorin and Lin [4]. We comment on that in Remark 4.2. Section 4 closes with Example 4.3, which shows among others that a nontrivial homomorphism into the braid group on n strings can determine a trivial n -fold covering space. The main results in the final §5 are Theorem 5.1, where we give an algebraic criterion for a covering space to be equivalent to a polynomial covering space, and Theorem 5.7, where we give a solution to the problem, when two homomorphisms of the fundamental group of X into the braid group on n strings determine equivalent n -fold polynomial covering spaces over X . Theorem 5.7 provides thereby an algebraic classification of the n -fold polynomial covering spaces over X .

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2. The link from polynomial covering spaces to homomorphisms into the braid groups. Throughout this paper X denotes a connected topological space with the homotopy type of a CW -complex and with a nondegenerate base point $x_0 \in X$. The fundamental group of X with base point $x_0 \in X$ will be denoted by $\pi_1(X)$.

First we recall the basic definitions from [6]. A *simple Weierstrass polynomial* $P(x, z)$ over X is a polynomial function $P: X \times C \rightarrow C$ of the form

$$P(x, z) = z^n + \sum_{i=1}^n a_i(x)z^{n-i},$$

where $a_1, \dots, a_n: X \rightarrow C$ are complex valued, continuous functions, such that for any fixed $x \in X$, $P(x, z)$ has no multiple complex roots. Associated with $P(x, z)$ there is an n -fold covering map $\pi: E \rightarrow X$ defined by the diagram,

$$\begin{array}{ccc} E & \hookrightarrow & X \times C \\ \pi \searrow & & \swarrow \text{proj}_1 \\ & X & \end{array},$$

where $E = \{(x, z) \in X \times C \mid P(x, z) = 0\}$ and proj_1 is projection onto the first factor. $\pi: E \rightarrow X$ is called the n -fold *polynomial covering map*, and E the n -fold *polynomial covering space* over X , associated with $P(x, z)$.

The continuous functions $a_i: X \rightarrow C$, $i = 1, \dots, n$, define a (continuous) map

$$a = (a_1, \dots, a_n): X \longrightarrow B^n = C^n \setminus \Delta$$

into the complement of the discriminant set $\Delta \subseteq C^n$ in complex n -space C^n . $a: X \rightarrow B^n$ is called the *coefficient map* for the polynomial covering map $\pi: E \rightarrow X$ associated with $P(x, z)$. We know from ([6], §3) that $\pi: E \rightarrow X$ is equivalent to the pull-back of a canonical n -fold polynomial covering map $\pi^n: E^n \rightarrow B^n$ along a . Therefore n -fold polynomial covering spaces over X , which have homotopic coefficient maps, are equivalent as coverings.

In order to put polynomial covering spaces into connection with conjugacy classes of homomorphisms from the fundamental group of X into the braid groups we need some preparation.

Let Σ_n denote the symmetric group on n letters. The action of Σ_n on the space of complex n -tuples C^n by permutation of coordinates induced a free Σ_n -action on

$$F_n(C) = \{(z_1, \dots, z_n) \in C^n \mid z_i \neq z_j, \text{ all } i \neq j\}$$

with the configuration space $C_n(C)$ for an (unordered) set of n points in the plane C as orbit space. The canonical projection map

$$p_n: F_n(C) \longrightarrow C_n(C)$$

defines therefore a principal Σ_n -bundle, in particular an $n!$ -fold covering map. It is well known, see e.g., ([6], §4), that there is a homeomorphism

$$\Phi: C_n(C) \longrightarrow B^n,$$

which to the element $\text{cls}(\alpha_1, \dots, \alpha_n) \in C_n(C)$ associates the n -tuple $(a_1, \dots, a_n) \in B^n$ defined by

$$z^n + \sum_{i=1}^n a_i z^{n-i} = (z - \alpha_1) \cdots (z - \alpha_n).$$

By a theorem of Fadell and Neuwirth [2] we know that $C_n(C)$, and hence also B^n , is an Eilenberg-MacLane space of type $(B(n), 1)$, where $B(n)$ denotes the Artin braid group on n strings. Take $c_0 = (1, \dots, n) \in F_n(C)$, $\bar{c}_0 = p_n(c_0) \in C_n(C)$ and $b_0 = \Phi(c_0) \in B^n$ as base points in $F_n(C)$, $C_n(C)$ and B^n , respectively. Then the group of braids on n strings can be defined as the fundamental group $B(n) = \pi_1(C_n(C), \bar{c}_0)$.

The group $H(n) = \pi_1(F_n(C), c_0)$ is called the group of colored braids on n strings. For later reference we note that the homotopy sequence for the covering map $p_n: F_n(C) \rightarrow C_n(C)$ reduces to the following short exact sequence, which we will call *the braid group sequence*,

$$1 \longrightarrow H(n) \xrightarrow{\rho_n} B(n) \xrightarrow{\tau_n} \Sigma_n \longrightarrow 1.$$

ρ_n is induced by the projection map p_n and τ_n is a boundary operator, called the *permutation homomorphism*.

Let now $\pi: E \rightarrow X$ be an n -fold polynomial covering map with coefficient map $a: X \rightarrow B^n$. Using the homeomorphism $\Phi: C_n(C) \rightarrow B^n$ we can then define a unique map $\alpha: X \rightarrow C_n(C)$ such that the following diagram is commutative,

$$\begin{array}{ccc} & C_n(C) & \\ \alpha \nearrow & \downarrow \Phi & \\ X & & B^n \\ \searrow a & & \end{array}$$

For obvious reasons $\alpha: X \rightarrow C_n(C)$ is called the *root map* for the polynomial covering map $\pi: E \rightarrow X$. By definition of Φ , the coefficient map a and the root map α are related by the formula

$$z^n + \sum_{i=1}^n a_i(x) z^{n-i} = \prod_{z_x \in \alpha(x)} (z - z_x)$$

for all $x \in X$.

The root map $\alpha: X \rightarrow C_n(C)$ induces a homomorphism

$$\alpha_*: \pi_1(X, x_0) \longrightarrow \pi_1(C_n(C), \alpha(x_0))$$

between fundamental groups. Since $x_0 \in X$ is a nondegenerate base point and $C_n(C)$ is connected, this homomorphism determines a unique conjugacy class of homomorphisms of $\pi_1(X)$ into $B(n)$. Any one of the homomorphisms in this conjugacy class will also be denoted by

$$\alpha_*: \pi_1(X) \longrightarrow B(n).$$

Since $C_n(C)$ is an Eilenberg-MacLane space of type $(B(n), 1)$ and X has the homotopy type of a CW -complex it is well known, see e.g., Spanier ([7], Theorem 11, p. 428), that there is a bijective correspondence between the free homotopy classes of maps of X into $C_n(C)$ and the conjugacy classes of homomorphisms of $\pi_1(X)$ into $B(n)$ defined by mapping the free homotopy class of $\alpha: X \rightarrow C_n(C)$

into the conjugacy class of $\alpha_*: \pi_1(X) \rightarrow B(n)$. The homeomorphism $\Phi: C_n(C) \rightarrow B^n$ induces a bijective correspondence Φ_* from the free homotopy classes of maps of X into $C_n(C)$ to those of X into B^n . Finally, we get a surjective map by mapping the free homotopy class of $a: X \rightarrow B^n$ into the equivalence class of the n -fold polynomial covering map $\pi_a: E_a \rightarrow X$ with coefficient map a .

The above remarks are summarized in the following diagram

$$\begin{array}{ccccc}
 \text{Conjugacy class of} & & \text{Hom}(\pi_1(X), B(n))^{\text{conj.}} & & \\
 \alpha_*: \pi_1(X) \longrightarrow B(n) & & & & \\
 & \uparrow \text{bijective correspondence} & & & \\
 \text{Free homotopy class} & & [X, C_n(C)] & & \\
 \text{of } \alpha: X \longrightarrow C_n(C) & & & & \\
 & \downarrow \Phi_* \text{ bijective correspondence} & & & \\
 \text{Free homotopy class} & & [X, B^n] & \xrightarrow{\text{surjective map}} & PC_n(X) & \text{Equivalence class} \\
 \text{of } a: X \longrightarrow B^n & & & & & \text{of } \pi_a: E_a \longrightarrow X.
 \end{array}$$

In this diagram $\text{Hom}(\pi_1(X), B(n))^{\text{conj.}}$ denotes the set of conjugacy classes of homomorphisms of $\pi_1(X)$ into $B(n)$, $PC_n(X)$ denotes the set of equivalence classes of n -fold polynomial covering spaces over X , and $[X, Y]$ denotes the set of free homotopy classes of maps of X into Y for an arbitrary space Y .

Using the diagram we can freely switch between presenting an equivalence class of n -fold polynomial covering spaces over X by a free homotopy class of maps of X into B^n , or a free homotopy class of maps of X into $C_n(C)$, or a conjugacy class of homomorphisms of $\pi_1(X)$ into $B(n)$.

As we shall see in §4, the surjective map $[X, B^n] \rightarrow PC_n(X)$ is not in general injective.

3. Characteristic homomorphisms for polynomial covering maps. Consider an arbitrary n -fold covering map $\pi: E \rightarrow X$. For any ordering of the n points in the fibre over $x_0 \in X$, $\pi^{-1}(x_0)$, path lifting induces in an obvious way a homomorphism $\chi(\pi): \pi_1(X) \rightarrow \Sigma_n$. If we change base point in X or change the order of the points in the fibre over the base point, this changes $\chi(\pi)$ by an inner automorphism in Σ_n . Hence $\chi(\pi)$ is well defined by π up to conjugacy. The conjugacy class of $\chi(\pi)$ is exactly the *characteristic class* of the n -fold covering map π as defined in Steenrod ([8], 13.5, p. 61). Any choice of a homomorphism $\chi(\pi)$ as above will be called a *characteristic homomorphism* for π . It is well known that two n -fold covering maps $\pi_i: E_i \rightarrow X$, $i = 1, 2$, are equivalent if and only if $\chi(\pi_1)$, $\chi(\pi_2): \pi_1(X) \rightarrow \Sigma_n$ are conjugate. See e.g., Steenrod ([8], 13.7, p. 62).

For polynomial covering maps we can give an alternative, and

in this connection more useful, description of the characteristic homomorphisms.

Let $\pi: E \rightarrow X$ be an n -fold polynomial covering map with root map $\alpha: X \rightarrow C_n(C)$. As explained in §2, α induces a conjugacy class of homomorphisms $\alpha_*: \pi_1(X) \rightarrow B(n)$. Composing a homomorphism α_* in this conjugacy class with the permutation homomorphism $\tau_n: B(n) \rightarrow \Sigma_n$ from the braid group sequence, we get a homomorphism $\tau_n \circ \alpha_*: \pi_1(X) \rightarrow \Sigma_n$. With notation as above we have

LEMMA 3.1. *The homomorphism $\tau_n \circ \alpha_*: \pi_1(X) \rightarrow \Sigma_n$ is a characteristic homomorphism for the n -fold polynomial covering map $\pi: E \rightarrow X$, i.e., $\chi(\pi) = \tau_n \circ \alpha_*$.*

Proof. Since it is only the conjugacy class of α_* that matters, we can assume w.l.o.g. that α is base point preserving, i.e., $\alpha: (X, x_0) \rightarrow (C_n(C), \bar{c}_0)$.

Let then $f: [0, 1] \rightarrow X$ with $f(0) = f(1) = x_0$ be a representative for the element $[f] \in \pi_1(X)$, and let $(\alpha \circ f)': [0, 1] \rightarrow F_n(C)$ be the unique lifting of $\alpha \circ f: [0, 1] \rightarrow C_n(C)$ over $p_n: F_n(C) \rightarrow C_n(C)$ with initial value $(\alpha \circ f)'(0) = c_0 \in F_n(C)$. Then $(\alpha \circ f)'(1)$ defines a permutation $\sigma_f \in \Sigma_n$ of the n coordinates in $c_0 = (1, \dots, n)$. By the construction of τ_n as a boundary operator in the homotopy sequence for p_n , it is clear that $\tau_n \circ \alpha_*([f]) = \sigma_f$.

If we identify the circle S^1 with the space obtained by identifying end points in $[0, 1]$, then $f: [0, 1] \rightarrow X$ induces a map $\bar{f}: S^1 \rightarrow X$, since $f(0) = f(1) = x_0$. Let $\bar{f}^*(\pi): \bar{f}^*(E) \rightarrow S^1$ denote the n -fold polynomial covering map defined as the pull-back of π along \bar{f} . From the definition of the characteristic homomorphism $\chi(\pi)$ by path lifting, it follows easily that $\chi(\pi)([f])$ is the clutching permutation, see ([6], §8), for $\bar{f}^*(\pi)$. On the other hand it is clear that the clutching permutation for $\bar{f}^*(\pi)$ is $\sigma_f \in \Sigma_n$. Hence $\chi([f]) = \sigma_f = \tau_n \circ \alpha_*([f])$. Since $[f] \in \pi_1(X)$ was arbitrary this proves Lemma 3.1.

4. A criterion for a polynomial covering map to be trivial. It is intuitively clear that an n -fold polynomial covering map $\pi: E \rightarrow X$ is equivalent to the trivial n -fold covering map if and only if we globally can order the set of roots in the underlying simple Weierstrass polynomial in a continuous fashion. This is made precise in the following

THEOREM 4.1. *Let $\pi: E \rightarrow X$ be an n -fold polynomial covering map with root map $\alpha: X \rightarrow C_n(C)$. Then $\pi: E \rightarrow X$ is a trivial covering map if and only if the root map α admits a (continuous) lifting $\alpha': X \rightarrow F_n(C)$ over $p_n: F_n(C) \rightarrow C_n(C)$,*

$$\begin{array}{ccc}
 & F_n(C) & \\
 \nearrow \alpha' & \downarrow p_n & \\
 X & \xrightarrow{\alpha} & C_n(C) .
 \end{array}$$

Proof. Assume first, that $\pi: E \rightarrow X$ is a trivial n -fold polynomial covering map and let $h: X \times \{1, \dots, n\} \rightarrow E$ be a homeomorphism commuting with projections onto X , which defines an equivalence of π with the trivial n -fold covering map. Since E is a polynomial covering space, there is an inclusion map $E \hookrightarrow X \times C$ commuting with projections onto X . Let $h_c: X \times \{1, \dots, n\} \rightarrow C$ denote the coordinate map for h into C , when we consider h as a map into $X \times C$ via this inclusion. Obviously, the root map $\alpha: X \rightarrow C_n(C)$ is the map, which to $x \in X$ associates the configuration $\alpha(x) \in C_n(C)$ defined by the set of coordinates in C for the n points in $\pi^{-1}(x) \subset X \times C$. The map $\alpha': X \rightarrow F_n(C)$, which to $x \in X$ associates the ordered n -tuple $\alpha'(x) = (h_c(x, 1), \dots, h_c(x, n)) \in F_n(C)$ is then clearly a lifting of α as required, i.e., $\alpha = p_n \circ \alpha'$.

Assume next, that the root map $\alpha: X \rightarrow C_n(C)$ can be lifted to a map $\alpha' = (\alpha_1, \dots, \alpha_n): X \rightarrow F_n(C)$ over p_n . The map $h: X \times \{1, \dots, n\} \rightarrow X \times C$, defined by $h(x, i) = (x, \alpha_i(x))$ for $x \in X$ and $i = 1, \dots, n$, is then an embedding commuting with projections onto X , which maps $X \times \{1, \dots, n\}$ homeomorphically onto E . This proves that $\pi: E \rightarrow X$ is trivial, and hence Theorem 4.1 is proved.

REMARK 4.2. In [4], Gorin and Lin study complete solvability of polynomial equations of the type, which we in this paper call simple Weierstrass polynomials. Complete solvability of a simple Weierstrass polynomial with root map $\alpha: X \rightarrow C_n(C)$ is equivalent to the existence of a lifting $\alpha': X \rightarrow F_n(C)$ as in Theorem 4.1. See the section preceeding ([4], Lemma 1.1). The results in [4] can therefore immediately be translated into results on triviality of polynomial covering maps.

The author is much indepted to Karsten Grove for coversations on the following

EXAMPLE 4.3. Consider S^1 as the space of complex numbers of modulus 1.

For the simple Weierstrass polynomial $P(x, z) = z^2 - x^2$ over S^1 , the coefficient map $a = (a_1, a_2): S^1 \rightarrow B^2 = C^2 \setminus \Delta$ is given by $a(x) = (0, -x^2)$ and the root map $\alpha: S^1 \rightarrow C_2(C)$ is given by $\alpha(x) = \text{cls}(x, -x)$, for $x \in S^1$.

The root map $\alpha: S^1 \rightarrow C_2(C)$ clearly admits a lifting $\alpha': S^1 \rightarrow F_2(C)$ over $p_2: F_2(C) \rightarrow C_2(C)$, namely $\alpha'(x) = (x, -x)$ for $x \in S^1$. Hence the 2-fold polynomial covering map $\pi: E \rightarrow S^1$ associated with $P(x, z)$ is trivial by Theorem 4.1. This is also easy to prove directly.

We shall now prove that the coefficient map $a: S^1 \rightarrow B^2$, or equivalently, the root map $\alpha: S^1 \rightarrow C_2(C)$, is not freely homotopic to a constant map. This will show that the surjective map $[X, B^n] \rightarrow PC_n(X)$ constructed in §2 is not in general injective, in particular not for $X = S^1$ and $n = 2$.

First observe that $F_2(C) = \{(z_1, z_2) \in C^2 \mid z_1 \neq z_2\}$. Hence there is a homeomorphism $F_2(C) \rightarrow C \times (C \setminus \{0\})$, which maps $(z_1, z_2) \in F_2(C)$ into $(\zeta_1, \zeta_2) = (z_1 + z_2, z_1 - z_2) \in C \times (C \setminus \{0\})$. Under this homeomorphism, the lifting of the root map $\alpha': S^1 \rightarrow F_2(C)$, $x \mapsto (x, -x)$, obviously corresponds to the map $\tilde{\alpha}': S^1 \rightarrow C \times (C \setminus \{0\})$, $x \mapsto (0, 2x)$.

There is an obvious homotopy equivalence $C \times (C \setminus \{0\}) \simeq S^1$, defined in the second argument by radial projection. Since S^1 is a simple space, we can work with based homotopy classes in stead of free homotopy classes of maps.

Using the above remarks it follows that there is a sequence of isomorphisms

$$\pi_1(F_2(C)) \simeq \pi_1(C \times (C \setminus \{0\})) \simeq \pi_1(S^1) \simeq \mathbb{Z}$$

to the integers \mathbb{Z} , and that $\alpha': S^1 \rightarrow F_2(C)$ through these isomorphisms corresponds to $1 \in \mathbb{Z}$. Hence $\alpha': S^1 \rightarrow F_2(C)$ is not freely homotopic to a constant map. Since $p_2: F_2(C) \rightarrow C_2(C)$ is a covering map, this implies that the root map $\alpha: S^1 \rightarrow C_2(C)$ is not freely homotopic to a constant map. This finishes Example 4.3.

5. An algebraic classification of the polynomial covering maps. In the main results in this final section we provide necessary and sufficient conditions, for a homomorphism of $\pi_1(X)$ into Σ_n to represent the equivalence class of an n -fold polynomial covering space over X , and for two homomorphisms of $\pi_1(X)$ into $B(n)$ to define equivalent polynomial covering spaces over X .

Most of our investigations are related to the spaces and maps, which will be defined subsequently, in the following commutative diagram,

$$\begin{array}{ccc} \text{Hom}(\pi_1(X), H(n)) & & \\ \rho_n \circ \downarrow & & \\ \text{Hom}(\pi_1(X), B(n)) & \longrightarrow & PC_n(X) \\ \tau_n \circ \downarrow & & \downarrow \\ \text{Hom}(\pi_1(X), \Sigma_n) & \longrightarrow & C_n(X) . \end{array}$$

$C_n(X)$ and $PC_n(X)$ denote respectively the set of equivalence classes of n -fold covering spaces over X and n -fold polynomial covering spaces over X . There is an inclusion map $PC_n(X) \hookrightarrow C_n(X)$.

For any pair of groups G and H , we denote by $\text{Hom}(G, H)$ the set of homomorphisms of G into H . The maps $\rho_n \circ -$ and $\tau_n \circ -$ in the diagram are induced by composition of homomorphisms from the homomorphisms in the braid group sequence

$$1 \longrightarrow H(n) \xrightarrow{\rho_n} B(n) \xrightarrow{\tau_n} \Sigma_n \longrightarrow 1.$$

Finally, we explain now the horizontal maps in the diagram. The map

$$\text{Hom}(\pi_1(X), \Sigma_n) \longrightarrow C_n(X)$$

is the surjective map, which to the homomorphism $\varphi: \pi_1(X) \rightarrow \Sigma_n$ associates the equivalence class of the n -fold covering map $\pi: E \rightarrow X$ having φ as a characteristic homomorphism. The covering map π can be constructed as an associated bundle to the universal covering space over X viewed as a principal $\pi_1(X)$ -bundle. Compare also Steenrod ([8], 13.8, p. 64). From the classification theorem for covering maps we know, that two homomorphisms $\varphi, \psi \in \text{Hom}(\pi_1(X), \Sigma_n)$ determine the same equivalence class of n -fold covering spaces over X if and only if they are conjugate. In particular we note, that the constant homomorphism of $\pi_1(X)$ into the identity permutation in Σ_n is the unique representative for the equivalence class of the trivial n -fold covering space over X .

The map

$$\text{Hom}(\pi_1(X), B(n)) \longrightarrow PC_n(X)$$

is the surjective map, which to the homomorphism $\varphi: \pi_1(X) \rightarrow B(n)$ associates the equivalence class of the n -fold polynomial covering map $\pi: E \rightarrow X$ having $\chi(\pi) = \tau_n \circ \varphi$ as a characteristic homomorphism. That such a map exists follows by the constructions in §2 combined with Lemma 3.1.

This finishes the definitions of the spaces and maps in the above diagram.

First we make use of the diagram to prove

THEOREM 5.1. *The equivalence class of an n -fold covering map $\pi: E \rightarrow X$ represented by the homomorphism $\varphi: \pi_1(X) \rightarrow \Sigma_n$ contains a polynomial covering map if and only if there exists a homomorphism $\varphi': \pi_1(X) \rightarrow B(n)$ such that $\varphi = \tau_n \circ \varphi'$.*

Proof. An n -fold polynomial covering space over X is re-

presented by a homomorphism $\psi: \pi_1(X) \rightarrow B(n)$. The homomorphisms $\varphi, \tau_n \circ \psi \in \text{Hom}(\pi_1(X), \Sigma_n)$ represent equivalent coverings if and only if they are conjugate. Since τ_n is an epimorphism, it follows then easily, that φ and $\tau_n \circ \psi$ represent equivalent coverings if and only if there exists a homomorphism $\varphi': \pi_1(X) \rightarrow B(n)$, conjugate to $\psi: \pi_1(X) \rightarrow B(n)$, such that $\varphi = \tau_n \circ \varphi'$. Theorem 5.1 follows now immediately using the main diagram.

COROLLARY 5.2. *In order that the space X has the property that all n -fold covering spaces over X are equivalent to polynomial covering spaces, it is necessary and sufficient that*

$$\tau_n \circ -: \text{Hom}(\pi_1(X), B(n)) \longrightarrow \text{Hom}(\pi_1(X), \Sigma_n)$$

is surjective.

EXAMPLE 5.3. Suppose that $\pi_1(X)$ is a free nonabelian group. Then the condition in Corollary 5.2 is clearly satisfied. This is the case, since given a homomorphism $\varphi: \pi_1(X) \rightarrow \Sigma_n$, we can just lift the images in Σ_n of a set of generators for $\pi_1(X)$ and then extend to a homomorphism $\varphi': \pi_1(X) \rightarrow B(n)$. We can lift the images of the generators, since $\tau_n: B(n) \rightarrow \Sigma_n$ is an epimorphism. For a space X with a free nonabelian fundamental group, any finite covering space over X is therefore equivalent to a polynomial covering space. Specific examples of such spaces X are provided e.g., by punctured surfaces.

Next we prove the following algebraic analogue to Theorem 4.1.

THEOREM 5.4. *A homomorphism $\varphi': \pi_1(X) \rightarrow B(n)$ represents the trivial n -fold covering space over X if and only if there exists a homomorphism $\varphi'': \pi_1(X) \rightarrow H(n)$ such that $\varphi' = \rho_n \circ \varphi''$.*

Proof. $\varphi': \pi_1(X) \rightarrow B(n)$ represents the trivial n -fold covering space over X if and only if $\varphi = \tau_n \circ \varphi'$ is conjugate to, and therefore equal to, the constant homomorphism of $\pi_1(X)$ into the identity permutation in Σ_n . By exactness of the braid group sequence, it follows easily, that this is the case if and only if there exists a homomorphism $\varphi'': \pi_1(X) \rightarrow H(n)$ such that $\varphi' = \rho_n \circ \varphi''$.

EXAMPLE 5.5. Suppose that $\pi_1(X)$ is a torsion group. Since $B(n)$ by a theorem of Fox and Neuwirth [3] contains no torsion, the constant homomorphism into the identity element is the only homomorphism from $\pi_1(X)$ into $B(n)$. Since this homomorphism lifts to the constant homomorphism of $\pi_1(X)$ into $H(n)$, it follows

therefore by Theorem 5.4, that every polynomial covering space over X is trivial. This gives a slightly different proof of Theorem 4.3 in [6].

Finally, we turn now to the problem when two homomorphisms $\varphi, \psi \in \text{Hom}(\pi_1(X), B(n))$ determine the same equivalence class of n -fold polynomial covering spaces over X .

By the classification theorem for covering spaces, we know that two homomorphisms $\varphi, \psi \in \text{Hom}(\pi_1(X), B(n))$ determine the same equivalence class of n -fold polynomial covering spaces over X if and only if $\tau_n \circ \varphi, \tau_n \circ \psi \in \text{Hom}(\pi_1(X), \Sigma_n)$ are conjugate homomorphisms. Since τ_n is an epimorphism, this condition is satisfied if and only if there exists an element $b \in B(n)$ such that $\tau_n \circ \psi = \tau_n \circ \varphi^b$. Here $\varphi^b \in \text{Hom}(\pi_1(X), B(n))$ denotes the conjugation of φ by b , i.e., for every $\gamma \in \pi_1(X)$ we have $\varphi^b(\gamma) = b \cdot \varphi(\gamma) \cdot b^{-1}$. If $\tau_n \circ \psi = \tau_n \circ \varphi^b$, then exactness of the braid group sequence shows that there exists a unique map (not necessarily a homomorphism) $\theta: \pi_1(X) \rightarrow H(n)$ such that $(\rho_n \circ \theta) \cdot \psi = \varphi^b$. The multiplication on the left in this equation is pointwise multiplication of maps of $\pi_1(X)$ into $B(n)$.

Since $(\rho_n \circ \theta) \cdot \psi$ has to be a homomorphism of $\pi_1(X)$ into $B(n)$, θ has to satisfy a certain condition. To describe this condition, we note that the homomorphism $\psi: \pi_1(X) \rightarrow B(n)$ induces an action of $\pi_1(X)$ on $H(n)$ as follows. For $\gamma \in \pi_1(X)$ and $c \in H(n)$ we let $c^{\psi(\gamma)} \in H(n)$ denote the unique element, which exists by exactness of the braid group sequence, such that

$$\rho_n(c^{\psi(\gamma)}) = \psi(\gamma) \cdot \rho_n(c) \cdot \psi(\gamma)^{-1}.$$

The condition, the map $\theta: \pi_1(X) \rightarrow H(n)$ has to satisfy in order that $(\rho_n \circ \theta) \cdot \psi$ is a homomorphism of $\pi_1(X)$ into $B(n)$, is then easily seen to be expressed in the following

DEFINITION 5.6. Let $\psi: \pi_1(X) \rightarrow B(n)$ be a homomorphism. A map $\theta: \pi_1(X) \rightarrow H(n)$ is then called a *ψ -crossed homomorphism* if

$$\theta(\gamma_1 \cdot \gamma_2) = \theta(\gamma_1) \cdot \theta(\gamma_2)^{\psi(\gamma_1)}$$

for all $\gamma_1, \gamma_2 \in \pi_1(X)$.

Using the notation above, the arguments preceeding Definition 5.6 proves the following

THEOREM 5.7. Two homomorphisms $\varphi, \psi \in \text{Hom}(\pi_1(X), B(n))$ determine the same equivalence class of n -fold polynomial covering spaces over X if and only if there exists an element $b \in B(n)$ and a

ψ -crossed homomorphism $\theta: \pi_1(X) \rightarrow H(n)$ such that $(\rho_n \circ \theta) \cdot \psi = \varphi^b$.

Theorem 5.1 and Theorem 5.7 taken together provide a satisfactory algebraic description of the equivalence classes of polynomial covering spaces over X . It would be nice to have a geometric description, say in terms of a classifying space for the n -fold polynomial covering spaces over X .

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